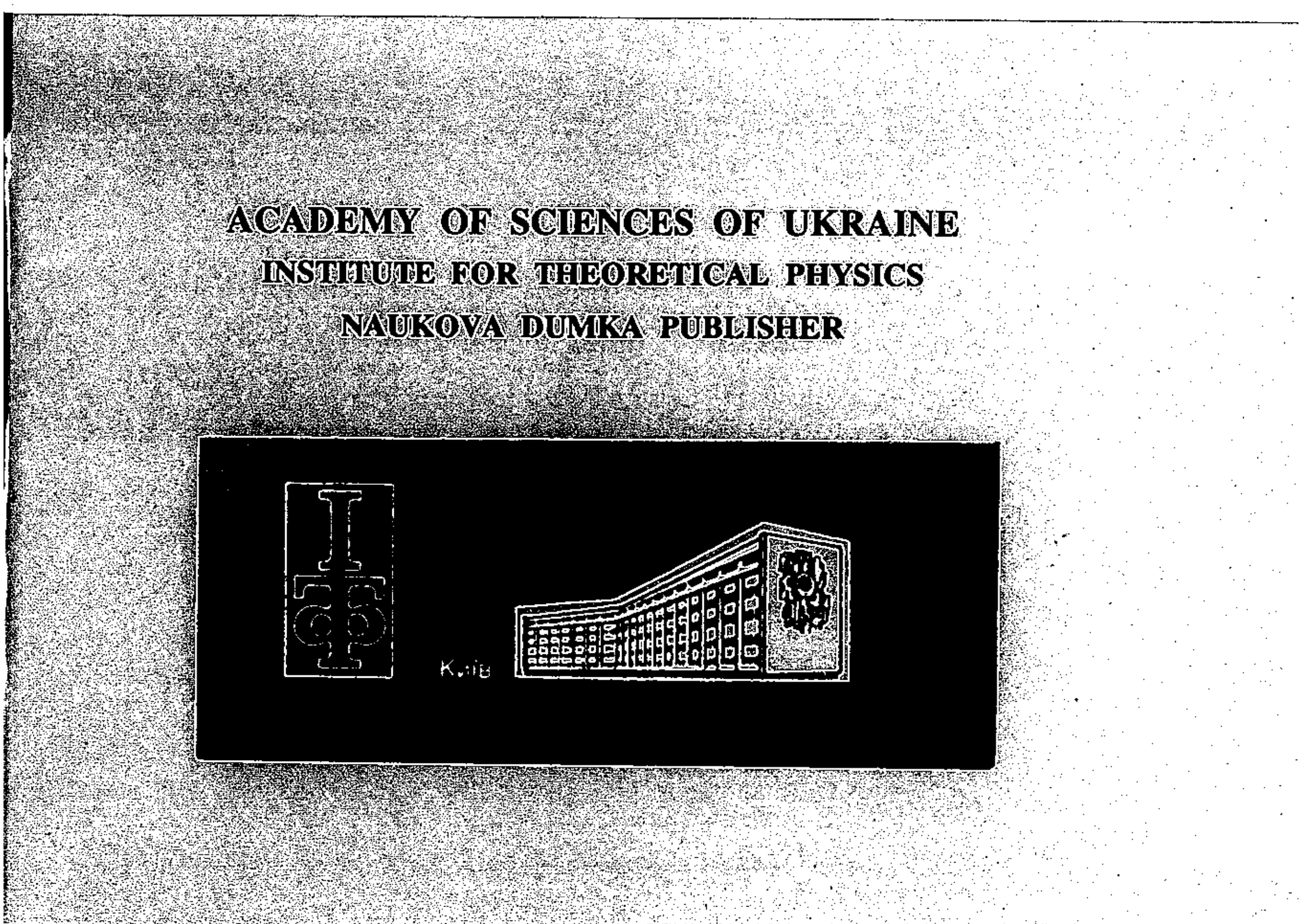
ELEMENTS OF HADRONIC MECHANICS

Vol. I:

MATHEMATICAL FOUNDATIONS

Ruggero Maria Santilli



ACADEMY OF SCIENCES OF UKRAINE
INSTITUTE FOR THEORETICAL PHYSICS

ELEMENTS OF HADRONIC MECHANICS

Vol. I:

MATHEMATICAL FOUNDATIONS

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Second Edition

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EXCERPTS FROM THE REVIEWS

A. Jannussis (Univ. of Patras, Greece): *"Hadronic Mechanics supersedes all theories to date."*
(opening address of the International Conference on the Frontiers of Physics. Olympia, Greece, 1993)

H. P. Leipholz (Univ. of Waterloo, Canada): *"Santilli's studies are truly epoch making."*

J. V. Kadeisvili (Intern. Center of Phys., Kazakhstan): *"Santilli's Lie-isotopic and Lie-admissible generalizations of the algebraic, geometric and analytic foundations of Lie's theory are of clear historical proportions."*

A. U. Klimyk (Inst. for Theor. Phys., Ukraine): *"The three books on Hadronic Mechanics are the most authoritative for a study of the Lie-isotopic and Lie-admissible generalizations of Lie's theory and their many applications."*

D. F. Lopez (Univ. of Campinas, Brasil): *"Santilli succeeded, first, in reaching a structural generalization of the available mathematics as a pre-requisite for his generalization of current physical theories. These achievements are unprecedented in the history of physics."*

A. O. E. Animalu (Univ. of Nsukka, Nigeria): *"Because of its beauty, mathematical consistency and range of applicability vastly beyond quantum mechanics, if we deny the historical character of Hadronic Mechanics we exit the boundaries of science."*

T. L. Gill (Howard Univ., Washington, D. C.): *"The three volumes on Hadronic Mechanics represent the most important contribution to physics in the last fifty years."*

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dedicated to the memory of

ENRICO FERMI,

because of his inspiring doubts on the exact validity of quantum mechanics for the nuclear structure.

See, e.g., E. Fermi, *Nuclear Physics*, Univ. of Chicago Press (1950), the beginning of Chapter VI, page 111, when referring to the applicability of quantum mechanics for the treatment of nuclear forces:

"..... there are some doubts as to whether the usual concepts of geometry hold for such small region of space."

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FOREWORD

These three volumes are devoted to a structural generalization of contemporary theoretical physics known under the name of *Hadronic Mechanics* (HM). Volume I presents a generalization of contemporary mathematical structures, including the theory of numbers, vector spaces, Lie algebras and groups, contemporary geometries, functional analysis, etc.

Volume II begins with a generalization of classical Lagrangian and Hamiltonian mechanics and then, after a suitable lifting of conventional quantization procedures, presents a step-by-step generalization of nonrelativistic and relativistic quantum mechanics capable of representing the most general known systems, while admitting of traditional mechanics and systems as particular cases.

Finally, Volume III presents a variety of novel and refreshing physical applications and experimental verifications in nuclear physics, particle physics, astrophysics, superconductivity and other unexpected fields such as conchology.

In short, Hadronic Mechanics concerns such a wide class of phenomena, that we can use for brevity the word *Nature*.

The Author's main idea consists of a generalization of the fundamental *constants* of contemporary physics into *variables* of the most general possible form representing their dependence on local physical conditions of the so-called *interior dynamical problem*. Mathematical and theoretical structures are then reconstructed in such a way to treat consistently these generalized notions.

The motivations for the consideration of these variable "constants" is rather natural. For instance, the speed of light in a physical medium is variable. Additional considerations then lead to the variable character of other "constants" in interior conditions, such as in the interior of a star. For instance, the coupling constant of quantum electrodynamics depends on quantum corrections and changes with the scale [1]. Contributions of integral character or the possible fractal structure of space-time then lead to a locally variable Planck's "constant".

The transition from contemporary theoretical physics to the covering theories presented in these volumes can be expressed via a nice concept of M. P. Bronstein

(1906–1938) on the so-called three-dimensional *Space of Physical Theories* (SPT) with axes characterized by Planck's constant $\hbar$, the gravitational constant G an the inverse of the speed of light 1/c (see ref. [3]). Conventional theories are characterized by the following *points* in this SPT:

($\hbar$, 0, 0) = nonrelativistic quantum mechanics;
(0, G, 0) = Newtonian mechanics;
(0, 0, 1/c) = special relativity;
($\hbar$, 0, 1/c) = relativistic quantum mechanics; and
(0, G, 1/c) = general relativity.

Because of the local dependence of the "constants" on density, temperature, pressure, etc., Santilli's covering theories fill up Bronstein's entire space.

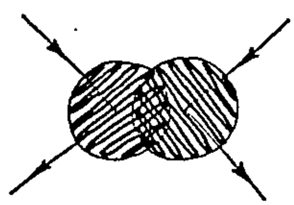
Nugzar V. Makhaldiani
Joint Institute for Nuclear Research
Dubna, Russia
October, 1993

1. N. N. BOGOLIUBOV and D. V. SHIRKOV, *Introduction to the Theory of Quantized Fields,* Wiley and Sons, New York (1980)
2. N. V. MAKHALDIANI, Number field dynamics and compactification problem in the theory of fields and strings, JINR Communication No. P2-88-916 (1988)
3. J. ELLIS, N. E. MAVROMATOS and D. V. NANOPOULOS, Non-critical string approach to black holes, time and quantum dynamics, in *From Superstring to the Origin of Space-Time,* Erice Summer School, 31-st Course, World Scientific, Singapore (1994)

PREFACE

These volumes are the first books written on a nonlinear, nonlocal and noncanonical, axiom-preserving generalization of quantum mechanics called *hadronic mechanics*, proposed by the author back in 1978 when at Harvard University under support from the U. S. Department of Energy, and subsequently studied by a number of mathematicians, theoreticians and experimentalists.

The main objective is a systematic and quantitative study of the historical, open legacy of the nonlocality of the strong interactions at large, and of the structure of hadrons in particular, due to mutual overlapping of the wavepackets/wavelength/charge-distributions of hadrons,



in such a way as to preserve causality, measurement theory, and other basic features of quantum mechanics.

The scope of this first volume is the study of the mathematical foundations of the new mechanics. The main working hypothesis is *the generalization of Planck's constant into an integro-differential operator*

$$\hbar = 1 \rightarrow \hat{\hbar} = \hbar \hat{1}$$

under the condition of verifying the needed smoothness, boundedness and regularity properties. The lifting of the unit then requires the following corresponding generalization of the associative product AB among generic quantum mechanical quantities A, B

$$A\,B \rightarrow A * B = A\,T\,B, \quad T = \text{fixed}, \quad \hat{1} = T^{-1},$$

in which case $\hat{1}$ is the correct right and left generalized unit of the theory, $\hat{1}*A = A*\hat{1} = A$.

The main idea is that the exchanges of energy are indeed discrete for particles moving in vacuum under action–at–a–distance interactions, such as for an electron in an atomic cloud. However, when the same particle is immersed within a hyperdense medium, such as for an electron in the core of a collapsing star, we expect integral contributions in the exchanges of energy due to the total immersion of the wavepacket of the particle within those of the surrounding particles.

The need for the generalization of the unit, and of the corresponding associative product, originates from the fact that the nonlocal interactions due to wave–overlappings, whether in electron pairing in superconductivity, or in deep inelastic scattering, or in other events, are of "contact" type; that is, of a type which does not admit a potential energy. Conventional Hamiltonians $H = K + V$ can therefore represent the kinetic energy K and all possible action–at–a–distance interactions with potential V. However, the contact interactions due to mutual wave–penetration, by conception, cannot be represented with the Hamiltonian H and, in this sense, they are called "nonhamiltonian". The alternative studied in these books is then their representation via the generalized unit of the theory for certain algebraic, geometric and analytic reasons presented in the text.

These preliminary ideas are sufficient to indicate the axiomatic structure of hadronic mechanics, and its connection with existing generalizations of quantum mechanics. In fact, in Ch. 7 of this volume we show that hadronic mechanics is *directly universal;* that is, capable of representing all possible nonlinear, nonlocal, nonhamiltonian, continuous or discrete, inhomogeneous and anisotropic *systems* (universality), directly in the frame of the experimenter (direct universality). However, hadronic mechanics *is not* equivalent to other generalizations treating the same systems because defined on inequivalent fields, metric and Hilbert spaces.

Consider, the generalizations of quantum mechanics known under the name of *q–deformations*, e.g., of the type

$$A\,B \rightarrow A * B = q\,A\,B, \quad q = \text{fixed} \neq 0,$$

(where q is a number). As we shall see, hadronic mechanics can be interpreted as an axiomatic *reformulation* of q–deformations which is invariant under its own time evolution and holds for arbitrary integro–differential deformations. This is essentially achieved via the redefinition of the unit

$$I \rightarrow \hat{1} = q^{-1}.$$

and consequential reformulation of the entire structure of the theory (numbers, fields, metric spaces, Lie's theory, etc.). By keeping in mind the *mathematical* consistency of the current treatment of the q-deformations (that at a fixed time in which the basic unit is not generalized), the above reformulation also resolves some of the *physical* problematic aspects emerging under time evolution, such as lack of the basic unit, inapplicability of the measurement theory, general loss of Hermiticity of the Hamiltonian, and others.

Similarly, numerous *nonlinear generalizations* of Schrödinger's equation (those with a nonlinearity in the wavefunctions ψ) have been proposed in the literature. As it is the case for the q-deformations, they are mathematically correct, but are afflicted by a number of problematic aspects of physical consistency, such as the general lack of exponentiation of an algebra to the corresponding group, the inequivalence of the Heisenberg-type and Schrödinger-type equations (due to the so-called *Okubo's No Quantization Theorem),* and others. Hadronic mechanics can be interpreted as an axiomatic *reformulation* of these studies into a form admitting nonlinearity in the wavefunctions ψ *and* their derivatives of arbitrary order $\partial\psi$, $\partial\partial\psi$, ... This axiomatization also permits a quantitative identification suitable for tests of the *deviations* from quantum mechanical formalisms implied by the nonlinearity itself.

Also, *nonlocal generalizations* of quantum mechanics for the study of wave-overlappings can be traced back to the very inception of that discipline. They were also treated via conventional quantum mechanical methods, thus leading to a number of problematic aspects still under study, such as causality. Hadronic mechanics preserves the abstract axioms of quantum mechanics and realizes them in a more general way, by therefore ensuring the preservation of causality *ab initio.* Hadronic mechanics is therefore ideally suited for an axiomatic *reformulation* of these studies into a causal description admitting all possible nonlocal-integral generalizations of quantum mechanics.

A number of *discrete generalizations* of quantum mechanics, such as those with a discrete structure in time, have been proposed in the literature although the elaborations continue to be based on conventional units and methods. These theories too are deeply linked to hadronic mechanics because the discreteness of time implies the alteration of the basic unit of time, thus requiring generalized methods for their treatment. Hadronic mechanics can be interpreted as providing an axiomatization of these generalizations by embedding the discrete structure of time in the generalized unit $\hat{1}$ of the theory. Intriguingly, hadronic mechanics shows that such discrete structure is ultimately compatible with the abstract axioms of quantum mechanics itself, when properly realized. Finally, discrete theories emerge as being compatible with conventional experimental data because (as shown in Vol. II) the appropriate expectation value of a discrete unit

recovers the conventional unit, $<\hat{1}> = \hbar = 1$.

Numerous additional generalizations of quantum mechanics exist in the literature, some of which will be studied in the appendix of Ch. 7, and others in the subsequent Volumes II and III. All these theories are independent from hadronic mechanics, yet exhibit intriguing connections with the latter whose study is beneficial to both theories.

It is evident that, in a scientific horizon of this type, I could not provide a comprehensive review of all existing generalizations without avoiding a prohibitive length. In these volumes I shall therefore limit myself to a review and re-elaboration of only some representative generalizations for each of the above classes. Nevertheless, I would be grateful to colleagues who care to bring to my attention (at the address below) studies directly or indirectly related to hadronic mechanics which I should quote in a possible future edition.

Judging from discussions and correspondence with various colleagues over the years, the primary difficulty for a first inspection of the field is of *mathematical* nature. The nonlinear-nonlocal-noncanonical generalization of the basic unit of quantum mechanics demands, for various technical reasons, a suitable generalization of the *totality* of the mathematical structure of quantum mechanics, beginning with a generalization of the contemporary notion of *number*, such as $\hbar = 1$, into a structurally more generalized notion called *isonumbers*, such as $\hat{\hbar} = \hbar\hat{1}$. In turn, generalized units, products and numbers demand a suitable generalization of the notions of field, vector spaces, transformation theory, enveloping algebras, Lie algebras, Lie groups, symmetries, symplectic, affine and Riemannian geometries, Lagrange and Hamilton mechanics, etc.

In short, the studies reported in these volumes indicate that, in the same way as the full understanding of the structure of atoms required a revision of the mathematical foundations of classical mechanics, further basic advances in the structure of hadrons require a similar revision, this time, of the mathematical foundations of quantum mechanics.

Difficulties in communicating with colleagues therefore emerge whenever hadronic mechanics is approached (and appraised) via the use of old quantum mechanical knowledge, without the awareness of numerous ensuing inconsistencies which generally remain undetected.

The author has therefore no words to recommend that colleagues seriously interested in inspecting the advances reported herein acquire a technical knowledge of the novel mathematical methods prior to any judgment and, above all, prior to setting up the mind along old lines. After all, the new mathematical methods are quite easy to understand, as one can see.

Technically, the topic of these books is in the field of the *isotopies* and *genotopies* of contemporary mathematical and physical theories proposed by

the Author back in 1978, which essentially are nonlinear, nonlocal-integral and nonpotential-nonhamiltonian liftings of given mathematical or physical structures capable of preserving the original axioms at the abstract, realization-free level (isotopies), or induce new covering axioms (genotopies).

As we shall see, the study of the fundamental hypothesis on the integral generalization of Planck's unit requires suitable nonlinear-nonlocal-nonhamiltonian isotopies and genotopies of the totality of mathematical methods used in quantum mechanics, including Hilbert spaces and all that.

The physical relevance of isotopic and genotopic methods is well established and consists in permitting quantitative studies of the transition:

a) from the *exterior dynamical problem*, characterized by motion of point-like particles within the homogeneous and isotropic vacuum;

b) to the *interior dynamical problem*, characterized by motion of extended and therefore deformable particles within inhomogeneous and anisotropic physical media, resulting in the most general known dynamical equations.

In particular, the isotopies preserve the original, abstract, algebraic, geometric and analytic axioms, thus achieving a unity of physical and mathematical thought in the treatment of both problems.

The isotopies are used when interior structural problems are studied as a whole with conserved conventional total quantities under a generalized interior structure. The genotopies are instead used to characterize one individual constituent while considering the rest of the system as external, thus resulting in the nonconservation of its physical quantities, of course, in a way compatible with total conservation laws.

The *classical isotopies and genotopies* are the classical realizations of the isotopies and genotopies of contemporary algebras, geometries, mechanics, symmetries and relativities. They have been sufficiently well identified in preceding monographs (quoted in the text), with a number of applications to Newtonian, relativistic and gravitational systems of our interior classical reality.

These volumes are the first books on the corresponding *operator isotopies and genotopies*, that is, the *axiom-preserving isotopies and axiom-inducing genotopies of quantum mechanics* originally proposed under the name of *hadronic generalization of quantum mechanics*, or *hadronic mechanics* for short, and today also known as *isotopic completion of quantum mechanics*, *isolocal realism*, and similar terms.

The operator isotopies and genotopies are far from being as developed as the corresponding classical counterparts. Despite that, I decided to write these first books for the following reasons:

1) the mathematical consistency of hadronic mechanics is now established,

thus allowing rigorous quantitative treatments of interior particle problems in a form suitable for experimental tests;

2) we have today a number of experimental verifications which, even though evidently preliminary, nevertheless confirm the predictions of the covering mechanics quite clearly; and

3) hadronic mechanics suggests a number of *novel* experiments that is, experiments on internal nonlinear-nonlocal-nonhamiltonian effects simply beyond the descriptive and predictive capacities of conventional theories, which deserve a serious consideration by the experimental community owing to their seemingly fundamental character.

Above all, a primary reason for writing these books is to point out for young minds of all ages that hadronic mechanics identifies the apparent existence of a new technology I tentatively called *hadronic technology*, because emerging from mechanisms in the structure of individual hadrons, while the current technologies emerge from mechanisms in the structure of molecules, atoms and nuclei. The societal implications of these possibilities, e.g., for possible new forms of energy, new approaches to cold fusion, new computer modeling, new medical applications, etc., have warranted this first identification of the state of the art in the conceptual, mathematical, theoretical and experimental foundations of hadronic mechanics.

Ruggero Maria Santilli
Dubna, Russia,
Kiev, Ukraine and
Palm Harbor, U.S.A.
Summer of 1993

PREFACE TO THE SECOND EDITION

In this second edition I have corrected a number of misprints and errors of the first edition which were kindly brought to my attention by a number of readers. Among them, I have corrected the old App. 6.A on the isotrigonometric functions because not compatible with the form used in Vol. II and moved the study to App. 5.C.

I have also added a number of aspects, such as: an outline of the theory of isonumbers and its application to the isotopies of cryptograms; a study of isogeometries with nondiagonal isotopic elements; the curved character of the Euclidean and Minkowskian geometry; a presentation of the so-called "geometric propulsion"; the addition of a number of aspects in isofunctional analysis; the latest updates in the construction of the Lie-admissible theory as a genotopy of the various branches of Lie's theory; and other topics.

Any additional comment by interested colleagues would be sincerely appreciated.

Ruggero Maria Santilli
Palm Harbor, FL
Fall 1995

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as well as several other colleagues for various correspondence and consultations.

Finally, I would like to thank

K. J. Roche, Boston Latin School,
for review of the English style.

However, I am solely responsible for these volumes owing to the numerous changes and expansions of the final version.

1: INTRODUCTION

1.1: STATEMENT OF THE PROBLEM

The discipline today known as *quantum mechanics* (see, e.g., ref. [1] for a historical account and ref. [2] for a contemporary account) was originally conceived for the structure of the atoms and the electromagnetic interactions at large, for which it subsequently emerged as being exact according to an overwhelming amount of experimental evidence.

Whether in its nonrelativistic, relativistic or field theoretical versions, quantum mechanics was subsequently applied to the study of the nuclear structure (see, e.g., ref.s [3,4]), to the strong interactions at large (see, e.g., ref.s [5]), as well as, more recently, to the unified gauge theories (see, e.g., ref. [6]), with equally impressive results and experimental verifications (see the recent experimental review [7]).

But physics is a discipline that will never admit final theories. No matter how effective and fundamental a theory is, the construction of a more general theory for a deeper understanding of physical reality is only a matter of time.

Despite its historical achievements and experimental verifications, quantum mechanics possesses well identified limitations essentially given by the characteristics of its original conception, those of the atomic structure, consisting [1]:

> physically, of *particles at sufficiently large mutual distances which can be well approximated as being point-like when moving in the homogeneous and isotropic vacuum (empty space) under action-at-a-distance, potential interactions*, and

> mathematically, of a theory which is *linear* in the sense of depending on the first power of the wavefunction, *local-differential* in the sense of being solely defined on a finite set of isolated points, and *potential-Hamiltonian-unitary* in the sense that all interactions are representable with a potential, thus

admitting a Hamiltonian with consequential unitary time evolution.

There are ample theoretical reasons and experimental evidence indicating that particles, their wavepackets, and/or their charge distributions can be well approximated as being point-like, not only for the atomic structure, but also for all electromagnetic and weak interactions at large, with consequential *exactly valid* of quantum mechanics.

However, there also exist theoretical reasons and experimental evidence according to which a linear, local and potential theory simply cannot describe the totality of the physical conditions existing in the Universe.

The first open problem is the achievement of a generalization of quantum mechanics for a more appropriate treatment of the strong interactions, with the understanding that the *approximate validity* of the conventional theory is beyond scientific doubts. For instance, an issue still open in the literature, at times known as *the legacy of Fermi,*[1] *Blochintsev and others* (see Efimov's monographs on nonlocality [8] and historical references therein), is that *strong interactions have a nonlocal-integral component* due to the deep mutual penetration and overlapping of the wavepackets and/or charge distributions of the interacting hadrons.

In fact, unlike the electromagnetic and weak interactions, the strong interactions have a range which is of the same order as the charge radius of all hadrons (about 1 fm = 10^{-13} cm). A necessary condition to activate the strong interactions is that hadrons enter into conditions of mutual penetration and overlapping of their wavepackets/wavelengths/charge distributions, resulting precisely in the historical legacy indicated earlier (see Fig. 1.1.1 for more details). It is known that nonlocal interactions are beyond the descriptive capacities of quantum mechanics on numerous, independent, topologic, geometric, analytic and other grounds.

Moreover, the above nonlocal interactions are necessarily of "contact" type, that is, they originate from the *actual physical contact among hadrons* and, as such, they are *nonpotential,* i.e., conceptually, they do not admit any potential of any kind and, mathematically, they are *variationally nonselfadjoint* [9], thus being generally *nonhamiltonian.* At any rate, "contact interactions" are of "zero range" by assumption, that is, they simply cannot be mediated by conventional particle-exchanges. Moreover, the inability of a Hamiltonian to represent all interactions implies that the time evolution is generally *nonunitary* (when expressed in a conventional Hilbert space, not so in suitably generalized spaces as we shall see in Vol. II). The insufficiencies of conventional quantum mechanics then again emerges from numerous, additional, independent topological, geometric, analytic, and other grounds.

Finally, deeper studies of the conditions of mutual penetration of hadrons have indicated the emergence of *internal effects which are nonlinear not only the wavefunctions, but also on their derivatives,* as typically the case for all

[1] See the Dedication of this first volume

resistive interactions caused by a physical medium. These interactions simply cannot be represented with the traditional addition of terms in the Hamiltonian and require instead a structurally novel mechanics, again, on numerous independent counts (such as the need to preserve the superposition principle under arbitrarily nonlinear interactions as necessary for consistent treatments of bound states).

THE FUNDAMENTAL INTERACTIONS OF HADRONIC MECHANICS

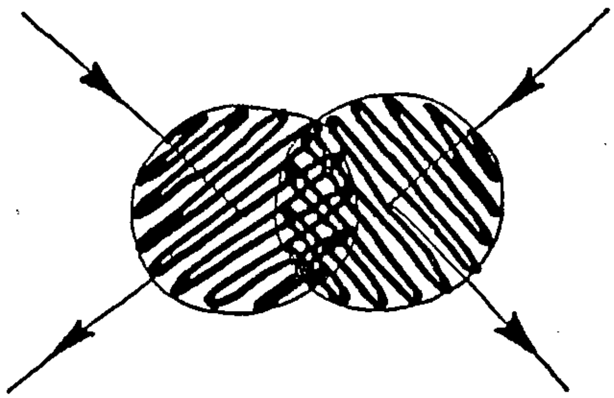

FIGURE 1.1.1: Quantum mechanics was conceived for the study of action-at-a-distance interactions among particles at large mutual distances which, as such, are representable by a potential. The interactions are therefore *local-differential;* that is, representable with differential equations defined over a finite set of isolated points. Hadronic mechanics was conceived for the study of the additional *nonlocal-integral* interactions due to mutual wave-overlapping as schematically depicted in this figure. We are here referring to interactions which, by conception, are defined over an *entire volume* and, as such, cannot be effectively approximated via their abstraction into a finite number of isolated points. The same interactions have emerged as not being derivable from a Hamiltonian as well as nonlinear in the wavefunctions and their derivatives. The name of "hadronic mechanics" were suggested by this author back in 1978 [20] as a mechanics primarily built for hadrons under nonlinear, nonlocal and nonpotential interactions (studied in Vol. II), with the understanding that the discipline is also applicable to other fields, such as superconductivity, astrophysics, and theoretical biology (see Vol. III).

In these three volumes we shall present the mathematical, theoretical and experimental foundations of the generalization of quantum mechanics proposed by this author in 1978 under the name of *hadronic mechanics* [20] and subsequently developed by mathematicians, theoreticians and experimentalists as a discipline conceived for the representation of the most general known

interactions which are:

1) *nonlinear* in the coordinate x and wavefunctions ψ, $\psi\dagger$ as well as their derivatives of arbitrarily needed order $\dot{x}$, $\ddot{x}$, $\partial\psi$, $\partial\psi\dagger$, ...;

2) *nonlocal-integral* in all needed variables;

3) *nonpotential-nonhamiltonian-nonunitary*, i.e., violating the integrability conditions for the existence of a potential as well as of a Hamiltonian, the so-called *conditions of variational selfadjointness* [9], with consequential generally nonunitary character;

4) *inhomogeneous* (e.g., because of a local variation of physical quantities such as density μ, temperature τ, index of refraction n, etc.); and

5) *anisotropic* (e.g., because of the presence of an intrinsic angular momentum which, as such, creates a preferred direction in the interior physical medium, with the understanding that the background space is and remains homogeneous and isotropic).

The representation of the above conditions will be primarily studied to attempt a deeper understanding of the *strong interactions among hadrons* from which the terms "hadronic mechanics" were originally suggested [20]. However, the same methods emerge as applicable to a variety of other fields outside hadron physics, including conditions with the exact applicability of quantum mechanics, such as a deeper understanding of the interactions among Fermions which are responsible for Pauli's exclusion principle which evidently cannot possess a potential, thus resulting to be precisely of the type under consideration.

Since the above interactions cannot be represented with a Hamiltonian by central assumption, the fundamental hypothesis studied in these volumes is to represent them via a suitable, nonlinear, nonlocal and noncanonical generalization of the fundamental unit of quantum mechanics, Planck's constant $\hbar = 1$, into an integro-differential operator $\hat{1}$ with the indicated most general possible functional dependence[2]

$$\hbar = 1 \quad \rightarrow \quad \hat{\hbar} = \hbar\,\hat{1}(t, x, \dot{x}, \ddot{x}, \psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \mu, \tau, n, ...), \qquad (1.1.1)$$

verifying certain smoothness, boundedness and regularity conditions identified later on.

It should be noted that there exist several different definitions of "nonlocality" in the current literature, although they are often referred to individual particles and/or waves and, when interactions are included, they are generally conceived to be compatible with conventional local-differential

[2] The dependence on the accelerations is absent for exterior problems, but it is a peculiarity of interior problems identified by a number of authors. As we shall see in Vol.s II and III, hadronic bound states clearly exhibit acceleration dependent forces which illustrates in part their novelty. We assume the reader is familiar with the fact that *acceleration-dependent forces are non-Newtonian* and, therefore, they are, alone, beyond the descriptive capacities of quantum mechanics.

topologies, symmetries and relativities. These definitions are ultimately reducible to point–particles in a variety of configurations, e.g., with faster than light interactions.

In these volumes we assume a definition of "nonlocality" which requires, by conception, an enlargement of the basic topology into a suitable integro–differential form (see later on Fig. I.1.4.1), with consequential necessary enlargement of space–time symmetries and relativities, which can be empirically defined as *particles and/or waves and their interactions defined over a finite volume which cannot be exactly reduced to a finite set of isolated points.*

A simple illustration of the latter type of nonlocality, referred to as of "nonlocal–integral" type, is the interaction resulting from the overlapping of the wavepackets, say, of two electrons in singlet coupling with wavefunctions $\psi_\uparrow(r)$ and $\psi_\downarrow(r)$ characterized by the *volume integral* $\int dv \psi^\dagger_\uparrow(r)\psi_\downarrow(r)$ which, as we shall see in Vol. III, occurs in the electrons of the Cooper pair in superconductivity and numerous other cases. It is evident that this type of interactions is not exactly reducible to interactions among a finite set of isolated points.

The latter "nonlocal–integral" interactions are also of "contact" type, that is, occurring because of the actual physical contact between the wavepackets for which the notion of potential and related action–at–a–distance have no mathematical or physical meaning (in fact, the interactions are variationally nonselfadjoint and cannot be consistently represented with a Hamiltonian). The insufficiency of the Hamiltonian to represent the entire system, then implies that the time evolution of a system with nonlocal–integral interactions is *nonunitary.* Numerous other consequences then follow.

Therefore, the adopted definition of nonlocal–integral interactions is sufficient, alone, to require a suitable broadening of quantum mechanics beginning with a necessary broadening of its foundations.

This first volume is devoted to the *mathematical* foundations of hadronic mechanics presented in their simplest possible form *accessible to graduate students in theoretical and experimental physics* Their presentation in the needed mathematical rigor will be left to interested mathematicians. *Theoretical* profiles are presented in Vol. II, while *applications and experimental verifications* are presented in Vol. III.

1.2: LIMITATIONS OF QUANTUM MECHANICS

For the "young minds of all ages" indicated in the Preface, there is no need to conduct experiments in order to identify the limitations of quantum mechanics, but simply observe (and, most importantly, admit) physical evidence.

First, let us observe a classical event of the interior problem of the Preface,

such as a space-ship during re-entry into our atmosphere. The *No-Reduction Theorems* [11,12] establish that such a classical system *with monotonically decaying angular momentum* cannot be consistently reduced to a finite collection of ideal elementary particles verifying the quantum mechanical rotational symmetry, thus being all in stable orbits *with conserved angular momentum* (a similar situation holds for the Lorentz symmetry as we shall see in Volume II).

Viceversa, an ensemble of quantum mechanical particles each with conserved angular momentum simply cannot yield, under any rigorous limiting procedure, a macroscopic object whose center of mass has a continuously decaying angular momentum.

Since the macroscopic object is a concrete, visual evidence, while the reduction to elementary constituents is an academic abstraction, we must expect insufficiencies in the *quantum* theory and certainly not in the macroscopic physical reality. In fact, at a deeper analysis the space-ship during re-entry experiences classical interactions precisely of the nonlinear, nonlocal and nonhamiltonian type[3] which are absent in quantum mechanics.

The studies presented in these volumes can therefore be first seen as identifying a certain generalization of *classical* mechanics for the representation of all possible classical systems of our macroscopic reality, which include:

a) the *conservative Newtonian systems* of point-particles moving in the homogeneous and isotropic vacuum which are admitted by quantum mechanics (the *exterior dynamical problem* of the Preface),

b) all *nonconservative Newtonian systems* characterized by local-differential, variationally nonselfadjoint systems with forces depending at most on the velocities, and

c) all *non-Newtonian systems* consisting of extended and therefore deformable objects under nonlocal-integral and/or acceleration dependent interactions moving within inhomogeneous and anisotropic physical media.[4]

After identifying such classical representation, we shall then identify unique and unambiguous operator maps into hadronic mechanics, so as to achieve the currently lacking mutual consistency between the classical systems of our physical reality and their operator counterparts.

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[3] Missiles in atmosphere have nowadays drag forces depending up to the *tenth* power of the speed and more, thus being manifestly nonselfadjoint [9], In addition, they have forces characterized by *integrals* over their surface σ because their shape directly affects the trajectory. This provides a primitive classical example of the type of interactions studied in these volumes, intentionally selected to void any hope of "finding a Lagrangian or a Hamiltonian" in favor of structurally more general theories (see later on Fig. 1.4.1 for a first explicit example)

[4] For historical accounts on the classical distinction between exterior and interior problems beginning with Lagrange, Hamilton, Jacobi and other founders of analytic dynamics, the reader may inspect ref.s [9,11].

Moreover, whether classical or operator, all representations will be studied under the condition of being "directly universal", that is, capable of representing all systems of the class admitted (universality), directly in the frame of the experimenter without any use of coordinate transformations (direct universality).

It should be indicated that this author already achieved such a "direct universality" in *Newtonian* mechanics via the use of the *Birkhoffian mechanics* (see ref. [10], in particular, Theorem 4.5.1, p. 54, of "direct universality"). The reader should be however aware that *the Birkhoffian mechanics is not the classical foundation of hadronic mechanics* because, even though nonhamiltonian, the former discipline is strictly local differential (owing to the use of the conventional symplectic geometry). A task of this volume is therefore that of identifying a further generalization of Birkhoffian mechanics which achieved "direct universality" for a much broader class of systems, including those of nonlocal-integral type, which results to be the unique and unambiguous classical image of hadronic mechanics.

The need for the "direct universality" is dictated by a number of rather insidious physical aspects. In fact, the *Lie-Koening theorem* [loc. cit.] does indeed ensure the possibility of constructing Hamiltonian representations for all systems that are local-differential, regular and analyticity in a star-shaped region of the local variables. However, the transformations are *noncanonical and nonlinear* because the original systems are nonhamiltonian by assumption. As a result, the transformed frames are not generally realizable with experiments owing to their nonlinearity, besides implying the loss of contemporary relativities owing to the highly noninertial character of the transformed frames.

The above occurrences illustrate the emphasis throughout these volumes of studying methods which are "universal" (rather than representing only a subclass of possible systems), and then "direct", that is, admitting of representations in the frame of the experimenter *prior* to any use of the transformation theory.

An inspection of the physical reality at the particle level without a preset mental attitude to preserve as much as possible current knowledge (which would not be scientific anyhow) reveals the existence of clear insufficiencies of quantum mechanics also at the particle level. This is due to the experimentally established existence of particle systems which simply cannot be derived from the strict implementation of first quantum mechanical axioms.

The first case that comes to mind is the *Cooper pair in superconductivity* (see, e.g., ref.s [13,14]). Clear experimental evidence establishes that ordinary electrons with negative elementary charge $-e$ can bound to each other in a singlet state at small distances in high T_c superconductivity under the mediation of cuprate ions. Even though the validity of currently available Bardeen-Cooper-Schrieffer model [loc. cit.] is out of question, recent studied reviewed in Vol. III have shown that the model is a linear-local-potential *approximation* of a nonlinear-nonlocal-nonpotential; structure.

In fact, the conditions in electron pairing are analytically equivalent to those of the classical space-ship during re-entry. Experimental evidence establishes that, in the above pairing, electrons are in conditions of mutual penetration of their wavepackets, that is, in condition which can only be quantitatively treated via *integral* representations (Fig. 1.1.1). Moreover, these are contact interactions for which the notion of potential has no physical or mathematical meaning, thus implying their *nonhamiltonian* character. The "inapplicability" (and not the "violation") of quantum mechanics[5] for quantitative treatments of electron pairs in superconductivity is then expected. Similar occurrences can be seen in numerous other cases implying short range superposition of electrons, including the familiar notion of *valence*, n which the undeniable validity of current views certainly do not prevent more accurate theories.

In reality, the insufficiencies of quantum mechanics are much deeper than the above because they are of *geometric* nature much along Fermi's vision.[1] A predominant experimental evidence in electron pairing is their *anisotropy* [13,14]. The derivation of the event from first axioms therefore requires a theory which is structurally anisotropic. The insufficiencies of quantum mechanics are then clear also from a geometric viewpoint owing to the fact that isotropy is a fundamental pillar of all its structures, from the Euclidean and Minkowski spaces, to the Galilean and Poincaré symmetries. Similar geometric insufficiencies, studied in details later on, emerge for a quantitative and direct representation of the physical medium inside hadrons which is precisely inhomogeneous and anisotropic.

Thus, the studies presented in these volumes can be seen as efforts to construct a generalization of quantum mechanics capable of a quantitative derivation of the attractive interaction of electron pairs in superconductivity from first axioms. Such a pairing will then be assumed as the origin of new models on unstable hadrons based on the synthesis of ligther massive particles, such as the synthesis of neutrons as occurring in the core of stars which, being originally made up solely of hydrogen, synthesize the neutrons from the sole use of protons and electrons. As we shall see in Vol. III, a quantitative treatment of such syntheses is beyond any realistic capability of quantum mechanics on numerous independent counts.

The reader should be aware that issues we are addresses here are not not marginal esoteric relevance, because they all have practical and technological implications. For instance, one of the primary reason which have stimulated the construction of a nonlinear–nonlocal–nonhamiltonian model of the Cooper pair is to reach a theory with specific predictive capacities to increase the temperature

[5] Quantum mechanics was strictly conceived for the exterior particle problem in vacuum in which has resulted to be exact. Quantum mechanics is therefore "inapplicable" for fundamentally different physical conditions and the use of the term "violation" would be scientifically inappropriate.

for superconductivity. A deeper understanding of the familiar notion of valence has direct applications in theoretical biology, such as for the prediction of new drugs. Similarly, the quantitative representation of the synthesis of the neutron from protons and electrons only implies a new, clean, subnuclear form of energy. The assumption of quantum mechanics as being of "final character", therefore causes serious problems on much needed basic advances in all these issues and numerous others.

Yet another particle event identifying the limitations of quantum mechanics is the so-called *Bose-Einstein correlation* (see, e.g., review [15] and references quoted therein), e.g., as occurring for the proton-antiproton annihilation at high energy, in which the particles coalesce into a state called a *fireball* which then decays in a variety of modes whose final products are correlated bosons (see Fig. 1.2.1).

In the opinion of this author, the Bose-Einstein correlation has a particularly fundamental value for the studies here considered inasmuch as it constitutes the most representative, complex and diversified manifestation of the strong interactions. As such, it touches the very foundation of the historical legacy recalled earlier.

Scientific objectivity requires the admission that the Bose-Einstein correlation is not *exactly* derivable from the strict implementation of first quantum mechanical axioms, whether nonrelativistic, relativistic or field theoretical. Needless to say, there are numerous phenomenological models providing a sufficient representation of experimental data via apparent quantum mechanical techniques [15]. The issue is that, at a closer scrutiny, these models do imply a departure from one or the other of quantum mechanical axioms.

This is due to a variety of conceptual and technical reasons studied in details in Vol. III. At this point we merely indicate that the fireball of the $p-\bar{p}$ annihilation is composed of two hadrons in conditions of total mutual penetration. But hadrons are not "ideal empty spheres" with "points" in them. Instead, hadrons are some of the densest objects measured in the laboratory until now with well defined and experimentally measured wavepackets-wavelengths-charge distributions of the order of 1 fm. The total mutual penetration of these particles, one inside the other then demands, for scientific objectivity, the expectation that the fireball includes the most general conceivable interactions of nonlinear, nonlocal, nonhamiltonian, inhomogeneous and anisotropic type.

At any rate, there is a rather general consensus that, *no correlation exists for strictly local-differential conditions.* The insufficiency of quantum mechanics is then evident, with the understanding again that its approximate character remains undeniable.

Thus, the methods presented in this volume are aimed at constructing a covering of quantum mechanics capable of a direct representation of the Bose-Einstein correlation from nonlinear-nonlocal-noncanonical first axioms. This

will then provide similar representational capabilities for strong interactions at large, and the structure of hadrons in particular.

Numerous additional limitations can be identified under the open mind indicated earlier. The best way to see them is to *identify physical conditions as <different> as possible than those of the original conception of quantum mechanics*. As another example, quantum mechanics was conceived for the characterization of particles in stable orbits under generally long range interactions verifying conventional conservation laws. To identify the limitations of the theory, one should then consider particles under *interactions which <maximize> the instability of the orbit and/or the nonlinear-nonlocal-nonhamiltonian effects.* If we consider instead physical conditions approaching as much as possible those of the original conception of the theory, no deviation should be expected, and, in fact, no deviation has been measured until now under these premises [7].

A further class of phenomena in which the limitations of quantum mechanics are also clear, is given by effects expected from the *inhomogeneity and anisotropy of physical media in which particles and/or electromagnetic waves move*. Consider an electron when a member of the atomic structure. Then, the particle moves in the homogeneous and isotropic vacuum, in which case quantum mechanics is exact.

Consider now the same electron when moving in the medium inside a collapsing star or, for that matter, the medium inside a hadron, called *hadronic medium* [20]. Then, the particle moves within a medium which is manifestly *inhomogeneous* and *anisotropic.*

Theoretical and experimental questions then arise as to whether such inhomogeneity and anisotropy have any measurable effect in the dynamical evolution of the particle considered. We are here referring to measurable effects in the intrinsic characteristics of particles such as their rest energy, the behaviour of their meanlife with speed, the behaviour of their Doppler frequencies, etc.

Customarily, these quantities are treated by Minkowskian methods. But their geometric pillars are the homogeneity and isotropy of space. The insufficiency of Minkowskian methods for inhomogeneous and anisotropic physical media must then be admitted in order not to exit the boundaries of science. The open nature of the problem herein considered then follows.

It is recommendable to identify the *origin* of the limitations of quantum mechanics in some detail so as to have a guideline during the subsequent analysis.

First, quantum mechanics is strictly *local-differential* in its topological structure, which prevents a mathematically consistent treatment of nonlocal interactions, whether in electron pairing in superconductivity, or in the Bose-Einstein correlation, or in the strong interactions at large.[6]

[6] The reader should be warned against the (not unusual) simplistic attitude of adding a

Second, quantum mechanics is structurally of *potential-Hamiltonian* type; namely, it can only represent in an established way action-at-distance interactions described by a potential. On the contrary, as indicated earlier, nonlocal effects due to mutual penetration of wavepackets are well known to be of contact type without any potential. As such, contact-nonlocal interactions are conceptually, topologically and analytically outside the representational capabilities of quantum mechanics.[7]

Third, the basic axioms of quantum mechanics require linearity in the wavefunction, as in the basic eigenvalue equations $H(x, p)\psi(x) = E\psi(x)$ and are generally violated by nonlinear generalizations, such as the use of Hamiltonians with a dependence on the wavefunction, $H(x, p, \psi)\psi(x) = E\psi(x)$.

In order to understand better these insufficiencies, let us review the essential structural lines of quantum mechanics [1,2]. The central notion is *Planck's quantum of energy*

$$\hbar = h/2\pi = 1.054589 \times 10^{-34} \text{ joule second} \tag{1.2.1}$$

The primary mathematical structure of the theory is given by:

A) The *universal, enveloping, associative, operator algebra* ξ with elements A, B, ... (say, matrices or local-differential operators) and product given by the familiar multiplication of matrices or operators AB, verifying the familiar *associativity law*

$$(A\,B)\,C \equiv A\,(B\,C), \tag{1.2.2}$$

under which Planck's constant in the form

$$\hbar = I = \text{diag.}\,(1, 1, \ldots\ldots, 1), \tag{1.2.3}$$

<nonlocal-integral potential> to the Hamiltonian because it can be proved to be mathematically and physically inconsistent on various grounds. To begin, such an addition is in violation of the local-differential topology of quantum mechanics and carries rather serious consequences, such as the inapplicability of *Mackey imprimivitivy theorem* [16] with consequential loss of conventional relativities [17]. Additional inconsistencies will be pointed out shortly.

[7] As discussed more technically in subsequent sections, the addition of a potential to a given Hamiltonian implies the underlying tacit assumption of granting potential energy to the interactions considered. For conventional action-at-a-distance interactions this is evidently correct. However, the granting of a potential energy to contact interactions due to the mutual penetration of wavepackets, has no physical sense, and results in a dynamical evolution which has no connection with that in the physical reality. As we shall see later, this is a motivation for representing internal nonlocal effects of strong interactions with the generalized unit of the theory; that is, *with a quantity other than the Hamiltonian.*

assumes the meaning of the *left and right unit* of the theory

$$\xi : \quad A\,B = \text{assoc.}, \qquad I\,A \equiv A\,I \equiv A \qquad \forall\ A \in \xi; \tag{1.2.4}$$

B) The *field* F of real numbers R or of complex numbers C.
C) The *Hilbert space* $\mathcal{H}$ with states $|\psi>$, $|\phi>$,..., and inner product

$$\mathcal{H}: \qquad <\psi|\phi> = \int dr'\,\psi^{\dagger}(t, r')\,\phi(t, r') \in C; \tag{1.2.5}$$

The remaining formulations can be derived from the above primitive structures. As an example, the fundamental *Heisenberg's equation* for the time evolution of a quantity Q in terms of a (Hermitean) Hamiltonian H

$$i\,\dot{Q} = [Q\,, H]_{\xi} = Q\,H - H\,Q\,, \tag{1.2.6}$$

is characterized by the antisymmetric brackets $[...,...]_{\xi}$ attached to the enveloping algebra ξ.

Similarly, *Schrödinger's equation*

$$i\,\frac{\partial}{\partial t}|\psi> = H|\psi> = E|\psi>, \tag{1.2.7}$$

is a consequence of the original associativity of the envelope ξ which results in the action $H|\psi>$ of the operator H on the state $|\psi>$ as being *right, modular and associative*, i.e., such that

$$A\,B\,C|\psi> = A\,(B\,C|\psi>) = (A\,B\,)\,C|\psi> = (A\,B\,C\,)|\psi>. \tag{1.2.8}$$

Finally, we recall that the *exponentiation* of Eq.s (1.2.6) into a finite Lie group is a power series expansion in the envelope ξ, namely, it is technically permitted by the infinite-dimensional basis in ξ with familiar expansion

$$e_{|\xi}^{i\alpha X} = I + i\,\alpha\,X\,/\,1! + (i\,\alpha\,X)\,(i\,\alpha\,X)\,/\,2! + \,.....,\ \alpha \in F,\ X = X^{\dagger} \in \xi, \tag{1.2.9}$$

under which the infinitesimal form (1.2.6) can be exponentiated to the Lie group of finite time-evolution

$$Q(t) = e_{|\xi}^{iHt}\,Q(0)\,e_{|\xi}^{-itH}. \tag{1.2.10}$$

This means that the above group too is fundamentally dependent on the assumption of the unit $\hbar = 1$.

The ultimate essence of quantum mechanics is embodied in the celebrated

Dirac's δ–function

$$\delta(r) \;=\; (2\pi)^{-1} \int_{-\infty}^{+\infty} dz\; e|_{\xi}^{\,i\,z\,r} \;, \tag{1.2.11}$$

verifying the familiar properties under sufficient smoothness conditions

$$\delta(r) = \delta(-r)\,, \qquad \delta(r - r') \;=\; \int_{-\infty}^{-\infty} dz\; \delta(r - z)\, \delta(z - r')\,, \tag{1.2.12a}$$

$$f(r') \;=\; \int_{-\infty}^{-\infty} \delta(r - r')\, f(r)\, dr \;, \tag{1.2.12b}$$

In fact, the δ–function characterizes the point–like structure of particles and their inherent local–differential topology.

All other aspects of quantum mechanics, such as linear operations on $\mathcal{H}$, Heisenberg's uncertainty principle, Pauli exclusion principle, transformation theory, etc., can be constructed via a judicious use of formulations derivable from or compatible with the above fundamental structures ξ, R (or C) and $\mathcal{H}$.

We are now in a position to identify in more details the following representative limitations of quantum mechanics.

Limitation 1: Lack of direct representation of extended nonspherical charge distributions of individual hadrons. The above structural lines show that the topological, geometrical and algebraic structures underlying quantum mechanics are strictly *local–differential.* As a result, *quantum mechanics cannot effectively represent the actual charge distributions of hadrons which are extended as well as nonspherical (e.g., oblate spheroidal).*

Admittedly, the *extended* character of the hadrons can be represented via the so–called *second quantization* [5,6]. However, such approach provides only the remnants of the actual shape via the so–called *form factors.* The insufficiency here considered then becomes evident by nothing that an effective theory must represent the *actual generally nonspherical shape* of the charge distributions of hadrons. In fact, assuming that form factors can represent the shape, considered, that shape must be perfectly spherical in order not to violate a pillar of the discipline: the rotational symmetry.

At any rate, the basic unit of the Euclidean space is the trivial unit matrix I = diag. (1, 1, 1) which, as such, can only geometrize the perfect sphere (or the homogeneous and isotropic vacuum).

As an example, there are indications that, as it is the case for all spinning objects in nature, the shape of the charge distribution of a nucleon is not perfectly spherical, but is instead an oblate spheroidal ellipsoid along, say, the z–axis with values for the semiaxes for the proton [18]

$$b^2{}_x \;=\; b^2{}_y \;=\; 1, \quad b^2{}_z \;=\; 0.60, \tag{1.2.13}$$

which provides one (not necessarily unique) explanation of the anomalous magnetic moments of the nucleons based on the shape alone, that is, without any consideration of any nature on the structure and its constituents. It seems evident that quantum mechanics cannot represent a realistic shape of nucleons of type (1.2.13) whether in first or second quantization. In fact, as we shall study in detail later on, extended nonspherical shapes are structurally outside the representational capabilities of a Hamiltonian and require more general theories.

In regard to the Bose–Einstein correlation, there is clear experimental evidence that the fireball is not perfectly spherical, but a highly prolate spheroidal ellipsoid oriented along the direction of the original $p-\bar{p}$ collision (see Fig. 1.2.1). The above limitation then implies *the inability of quantum mechanics to represent the highly prolate shape of the correlation fireball* ,with evident limitations in the quantitative description of the phenomenon considered.

Limitation 2: Lack of representation of the deformation of extended charge distributions. Once the need of representing the actual shape of a charge distribution is understood, one can see that *quantum mechanics is intrinsically unable to represent all possible deformations of given charge distributions, whether spherical or not, under sufficient external forces or collisions* . This is again prohibited by the underlying rotational symmetry.

The actual and direct representation of possible deformations of the charge distribution of hadrons is evidently needed for basic advances, e.g., for the synthesis of the neutron from protons and electrons alone. In quantum mechanics, all hadrons are points and, consequently, the problem of their deformability cannot be even formulated. The admission of the extended character of hadrons implies their deformability under sufficient conditions, because perfectly rigid objects, even though admitted in academic abstractions, do not exist in the physical reality. Thus, the only scientific issue is the *amount* of deformation of the charge distribution of a given hadron under given conditions, but its *existence* is beyond credible doubts.

A first illustration is given by the inability of quantum mechanics to provide an exact representation of the total magnetic moments of few-body nuclei despite tensorial and relativistic corrections, which is still lacking at this writing despite studies conducted over three-quarter of a century. The origin of this occurrence is precisely the apparent deformability of nonspherical nucleons when members of a nuclear structure with consequential alteration of their intrinsic magnetic moments. Quantum mechanics can only represent nucleons as undeformable points in first approximation and as perfectly spherical and rigid spheres in second quantization. The inability to represent total nuclear magnetic moments (and other nuclear characteristics) is then consequential.

As another example, it is known that the fireball of the Bose–Einstein

correlation expands immediately after its formation. and alters its shape under sufficiently intense external fields. The above limitation therefore implies *the inability of quantum mechanics to represent the evolution and deformations of the fireball* (see Fig. 1.2.1).

Equivalently, we can say that *quantum mechanics can only represent fireballs which, besides being perfectly spherical, are also perfectly rigid*. The ensuing limitations of the theory are then evident. In the final analysis, the rotational symmetry is taught since undergraduate courses in physics to be solely applicable to *rigid bodies*.

Limitation 3: Lack of representation of nonlocal-nonpotential interactions. Above all, a most basic limitation of the theory is *the inherent inability of quantum mechanics to represent nonlocal internal effects expected in strong interactions at large, as well as any appreciable overlappings of the wavepackets of particles (including leptons as in the Cooper pair in superconductivity).*

In regard to the Bose-Einstein correlation, this implies *the inability of quantum mechanics to reach a quantitative representation of the expected very origin of the correlation, the nonlocal interactions.* As recalled earlier, interactions of particles which can be effectively approximated as being point-like show no known correlation, while the correlation appears to be due precisely to the nonlocality of the interactions in the interior of the fireball, as we shall see in details in Vol. III.

The experimental data on the Bose-Einstein correlation therefore have fundamental significance because, in the final analysis, they can result to be the first experimental evidence on the historical legacy of the ultimate nonlocal structure of matter.

Limitation 4: Lack of representation of a number of physical systems from first principles. To illustrate the case for the Bose-Einstein correlation, consider a system of n particles represented with the symbol $k = 1, 2, ..., n$, each one possessing correlated and uncorrelated components represented with the symbols a and b, respectively. Let the states be given by $|k,a> \times |k,b>$, $k = 1, 2, ..., n$.

According to quantum mechanics the axiomatic characterization of the correlation probability is that based on the conventional expectation values, and can be written

$$C_n = \widehat{\underbrace{<1,a|<1,b| .... <n,a|<n,b|}} \begin{pmatrix} |1,a> \\ |1,b> \\ .... \\ |n,a> \\ |n,b> \end{pmatrix} =$$

$$= \sum_k (\langle k,a \mid k,a\rangle + \langle k,b \mid k,b\rangle), \qquad (1.2.14)$$

The above expression lacks exactly the cross terms $\langle k, a \mid k, b \rangle$ representing the correlation. In current "semiphenomenological models", these cross terms are introduced via a number of artificial expedients (see review [15]). However, for scientific objectivity we must admit that these models are, strictly speaking, beyond the capability of the axiom of expectation value, thus confirming the inability of quantum mechanics to derive the event from first principles.

This is for instance the case of the one or two "caoticity parameters" [15] introduced *ad hoc* "to adjust the fit". In this case an acceptable representation of the experimental data is indeed achieved, but one cannot claim that it is compatible with quantum mechanics. A similar situation occurs for various other cases, as we shall see during the course of our analysis.

Limitation 5: Loss of basic space-time symmetries under nonlinear, nonlocal and nonhamiltonian interactions. The historical open legacy of Fermi, Blochintsev and others on the ultimate nonlocality of the strong interactions has profound epistemological, theoretical and mathematical implications, because it implies the inapplicability of all conventional space–time symmetries for a number of independent reasons studied in details in volumes [9–12], such as:

a) the homogeneous and isotropic character of the basic medium of conventional relativities, empty space, is replaced by the generally inhomogeneous and anisotropic character of physical media of interior problems, whether of classical or operator type;

b) the Lie–Hamiltonian character of the conventional relativities is replaced with the nonhamiltonian structure of the interactions considered;

c) the local–differential character of the underlying topology (e.g., the Zeeman topology of the special relativity) si replaced with the nonlocal–integral nature of the events considered; and others.

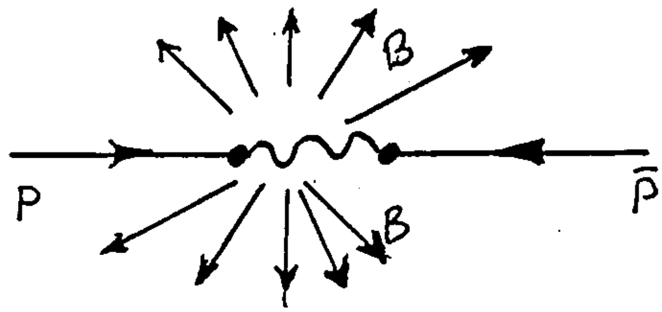
To be more specific in this important point, the admission of nonlocal interactions in physical systems such as the Cooper pair, the synthesis of the neutron from protons and electrons, the Bose–Einstein correlation, etc. implies the necessary abandonment of the conventional Poincaré symmetry owing to its notorious linear, local and canonical structure.

Yet another objective of the studies presented in these volumes is to show that, under appropriate generalized methods, the basic space–time symmetries can indeed be reconstructed as exact under nonlinear, nonlocal and nonpotential interactions. As a matter of fact, hadronic mechanics can be alternatively conceived as a mechanics capable of reconstructing as exact space–time and internal symmetries when believed to be conventionally broken. This includes the

reconstruction of the exact parity under weak interactions, the isospin symmetry under in nuclear physics, etc.

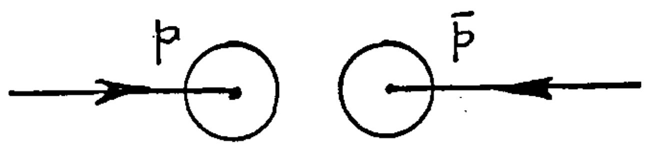
QUANTUM MECHANICAL APPROXIMATION OF THE BOSE-EINSTEIN CORRELATION

(1.1)

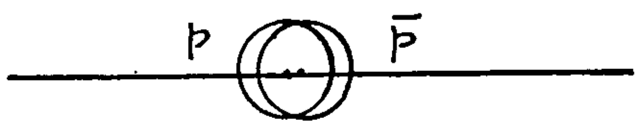


A MORE REALISTIC DESCRIPTION OF THE CORRELATION

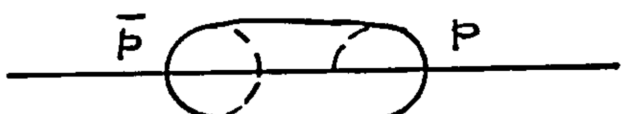
(1.2)



(1.3)



(1.4)



(1.5)

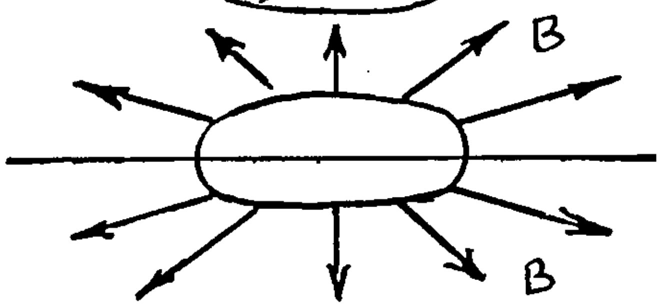


FIGURE 1.2.1: A schematic view of the quantum mechanical approximation of the Bose-Einstein correlation (Diagram 1.1), and a more realistic description suggested by available

experimental information (Diagrams 1.2–1.5). In the quantum mechanical case the original proton and antiproton are represented as points. The correlation and production of the emitted bosons B is then reducible to virtual, action-at-a-distance exchanges, resulting in Limitations 1–5 pointed out in the text. In the physical reality, the proton and antiproton are extended charge distributions of radius ≈ 1 fm (Diagram 1.2). Under very high energy, they annihilate in conditions of total mutual penetration and compression of their wavepackets (Diagram 1.3). This creates the fireball which is a highly prolate spheroidal ellipsoid oriented toward the original $p-\bar{p}$ direction (Diagram 1.4). This fireball rapidly expands and decays into particles whose final product is the set of correlated bosons B (Diagram 1.5). A satisfactory representation of the Bose–Einstein correlation must therefore be in a position to provide a quantitative representation of phases 1.2–1.5, as well as resolve Limitations 1–5 of the text from basic axioms.

As a matter of fact, we shall see that the identification of broken Lie symmetries is, in general, an indication precisely of the presence of nonlinear and/or nonlocal and/or nonpotential effects outside the capabilities of Lie theory.

In conclusion, the viewpoints studied in these volumes are the following:

I) Quantum mechanics does indeed provide an exact description of the physical conditions for which it was conceived, that of particles admitting an effective point-like approximation while moving in the homogeneous and isotropic vacuum. This includes electromagnetic and weak interactions as well as a large class of additional conditions, such as the approaching phase of the $p-\bar{p}$ constituents of the Bose–Einstein correlation;

II) Quantum mechanics is only approximately valid for particles in condition of deep mutual overlapping (Fig. 1.1.1), such as the Boson correlation, and

III) A more accurate, quantitative description of nonlinear, nonlocal, nonhamiltonian, inhomogeneous and anisotropic interactions, as expected in the Cooper pair, the synthesis of the neutron, the Bose–Einstein fireball, the structure of hadrons, and the strong interactions at large, requires a structural generalization of quantum mechanics itself, perhaps similar to the generalization of classical mechanics that resulted to be necessary for the final understanding of the atomic structure [1].

1.3: CONCEPTUAL FOUNDATIONS OF HADRONIC MECHANICS

In an attempt to resolve the limitations of quantum mechanics, this author submitted in a memoir [19] [8] of 1978 the proposal to construct the so-called

isotopies and genotopies of the conventional Lie's theory, under the name of *Lie-isotopic and Lie-admissible theories* , respectively. In the subsequent memoir [20] of the same year, this author proposed the construction of the *isotopies and genotopies of quantum mechanics* under the name of *hadronic mechanics.*

The term <isotopy> was suggested from the Greek "ισοζ τοποζ", meaning "preserving configuration" and interpreted as "axiom preserving".

The basic *isotopic equations* proposed in ref. [20], p. 752 for the time evolution of a physical quantity Q in terms of a (Hermitean) Hamiltonian H on a conventional Hilbert space are given by the following generalization of Heisenberg's equation (1.2.6)

$$i\, dQ/dt = [Q \hat{,} H] = Q \hat{T} H - H \hat{T} Q = \tag{1.3.1}$$

$$= Q\hat{T}(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, ..)H - H\hat{T}(t, x, \dot{x},. \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n,..)Q,$$

where T is a Hermitean operator, with exponentiated form

$$Q(t) = e^{i H \hat{T} t}\, Q(0)\, e^{-i t \hat{T} H}, \tag{1.3.2}$$

which admit the conventional Heisenberg's equation as a particular case, because

$$[A \hat{,} B]_{\hat{T}=1} \equiv [A , B] = A B - B A . \tag{1.3.3}$$

The following realization of the generalized unit (1.1.1),

$$\hat{1} = \hat{T}^{-1} = \hat{1}^\dagger, \tag{1.3.4}$$

is then called *isounit,* and results to be the correct left and right unit of the new theory, as we shall see in detail in the next chapter. The above formulations were called "Lie-isotopic" [19] because the brackets $[A \hat{,} B] = A T B - B T A$ preserve the original Lie axioms and, in this sense, the lifting $[A , B] \rightarrow [A \hat{,} B]$ is an isotopy.

The term <genotopy> was proposed by the author in ref. [19] from the Greek "γενναω τοποζ" meaning "inducing configuration" and interpreted as "axiom inducing", that is, an alteration of the original axioms in favor of covering axioms admitting of the original one as particular case.

The basic *genotopic equations* proposed in ref. [20], p. 746 are given by the following generalization of Eq.s (1.2.6) *and* (1.3.1)[9]

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[8] When at Harvard University under support from the U.S.Department of Energy, contract numbers ER-78-S-02-4742, AS02-78ER-4742, and DE-AC02-8-ER10651.

[9] The notation $\hat{T}$, $\hat{R}$, $\hat{S}$ etc. will result to be useful later on to denote quantities which are defined on isospaces over isofields. The symbols T, R, S, H etc. therefore means that

$$i\, dQ/dt = (Q \hat{,} H) = Q \hat{R} H - H \hat{S} Q = \qquad (1.3.5)$$

$$= Q \hat{R}(t, x, \dot{x}, \ddot{x}, \psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \mu, \tau, n, ..) H - H \hat{S}(t, x, \dot{x}, \ddot{x}, \psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \mu, \tau, n, ..) Q$$

where the operators R and S are now no longer Hermitean, but interconnected by the conjugation $R = S^{\dagger}$, with exponentiated form

$$Q(t) = e^{i H \hat{S} t} Q(0)\, e^{-t \hat{R} H}, \qquad (1.3.6)$$

which admits first the isotopic equations for $R = S^{\dagger} = T$ and the conventional equations for

$$(A \hat{,} B)_{\hat{R} = \hat{S} = 1} \equiv [A, B] = A B - B A. \qquad (1.3.7)$$

The most dominant aspect of the latter formulations is the existence of *two generalized units*, called *genounits*, one for motion forward in time, denoted with $\hat{1}^{>}$, and the second for motion backward in time, denoted $^{<}\hat{1}$ which can be identified with the inverse of two operators $\hat{R}$ and $\hat{S}$ of Eq. (1.3.5)

$$\hat{1}^{>} = \hat{S}^{-1}, \quad {}^{<}\hat{1}^{\dagger} = \hat{R}^{-1}. \qquad (1.3.8)$$

The above more general theory was called by this author "genotopic" because the generalized brackets $(A, B) = A \hat{R} B - B \hat{S} A$ *violate* this time the Lie algebra axioms in favor of covering algebras called *Lie-admissible algebras* first proposed by Albert [21] back in 1948 at the abstract level.

In fact, the brackets (A, B) characterize an explicit realization of the Lie-admissible algebras (although in a form more general than Albert's original conception we shall study later on), because their attached antisymmetric algebras are Lie-isotopic

$$(A, B) - (B, A) = [A \hat{,} B] = A \hat{T} B - B \hat{T} A, \quad \hat{T} = \hat{R} - \hat{S}. \quad (1.3.9)$$

The Lie-isotopic formulations were then studied by the author in monographs [9-12], while the more general Lie-admissible formulations were studied in monographs [22,23].

From the completely unrestricted functional dependence of the generalized units, it is evident that the above formulations have a clear capability to represent nonlinear, nonlocal, nonhamiltonian, inhomogeneous and anisotropic systems. In effect, equations (1.3.1) and (1.3.5) were subsequently proved to be

---

the quantities are computed on a conventional space over a conventional field.

"directly universal" for the systems considered, as we shall see in Ch. 7 of this volume.

The physical differences of the isotopies and genotopies were also identified in the original proposal [20] and can be summarized as follows:,

The ***isotopic formulations*** *characterize closed-isolated systems with conserved total Hamiltonian* H *and other total physical quantities under the most general possible nonlinear, nonlocal and nonhamiltonian internal forces represented by the operator* T *because, from the totally antisymmetric character of the brackets, we have*

$$i\, dH/dt \;=\; [H \hat{,} H] \;=\; H \hat{T} H \;-\; H \hat{T} H \;\equiv\; 0\,. \qquad (1.3.10)$$

The ***genotopic formulations*** *characterize open-nonconservative systems, such as one particle under conventional interactions represented by the Hamiltonian* H *and the most general known nonlinear, nonlocal and nonhamiltonian external interactions represented by the operators* R *and* S *because, from the lack of anticommutativity of the brackets, we have*

$$i\, dH/dt \;=\; (H, H) \;=\; H(\hat{R} - \hat{S})H \;\neq\; 0\,. \qquad (1.3.11)$$

The physical differences between the isotopic and genotopic formulations can also be effectively seen from the viewpoint of time-reversal invariance. In fact, one can see from the Hermiticity of the T operator T that *isotopic formulations are structurally reversible, that is, they are reversible for a time-reversal invariant Hamiltonian.*

On the contrary, it is equally easy to see from the lack of Hermiticity of the R and S operators that *genotopic formulations are structurally irreversible; that is, they are irreversible even for all time-reversal invariant Hamiltonians*.

The above occurrences suggested the characterization of the genotopic formulations with the arrow of time, the operator S characterizing motion forward in time, while the operator R characterizes the motion backward in time. By looking in retrospective, we can say that

The basic conceptual structure of hadronic mechanics has essentially remained that of the original proposal [19,20]: *the integral generalization of Planck's unit, Eq. (1.1.1), of two primary types:*

A) a first type of Hermitean-reversible character for motion in both forward and backward direction in time, which characterizes axiom-preserving generalizations of quantum mechanics, and

B) a second type requiring two different generalized units, one for motion forward in time and another for the motion backward in time, which require a generalization of the axiomatic structure of quantum mechanics when formulated in conventional Hilbert spaces over conventional fields (not so when

formulated over suitable spaces and fields, as we shall see.

THE TWO BRANCHES OF HADRONIC MECHANICS

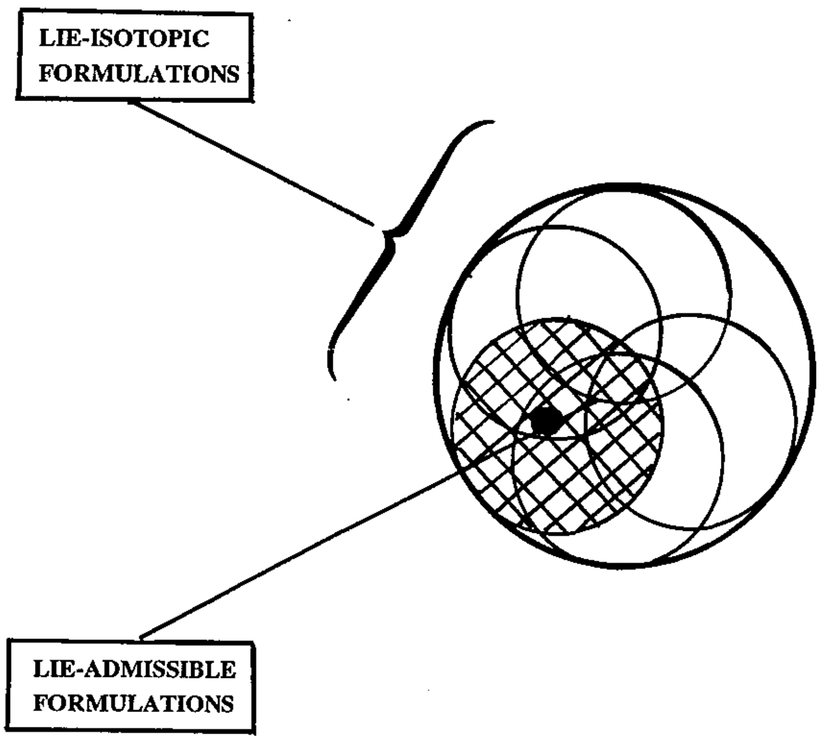

FIGURE 1.3.1: A schematic view of the main branches of hadronic mechanics, the Lie-isotopic branch describing closed-isolated systems verifying conventional total conservation laws under nonlinear-nonlocal-nonhamiltonian internal effects, and the more general Lie-admissible branch describing the most general possible open-nonconservative systems or, more specifically, one component of a Lie-isotopic system when considering the rest as external. It is generally believed that the conservation of the total energy, $dH/dt = 0$, can only occur under conservative internal forces or, more technically, for systems called *closed variationally selfadjoint* [11], such as planetary or atomic systems. This belief was disproved in memoir [20] by showing that the total energy can also be conserved under contact, nonhamiltonian internal forces. In the latter case we merely have internal exchanges of energy and other physical quantities but always such to balance each-other and result in conserved total quantities. These studies identified a new class of physical systems called *closed variationally nonselfadjoint* studied in detail at the classical level in monograph [12] of 1983, and at the operator level in memoirs [35] of 1989. A classical example is provided by Jupiter in which one can visually see in telescopes its global stability in a way compatible with irreversible, unstable interior processes, such as vortices with continuously varying angular momenta. A particle example is given by a neutron star, which is also manifestly stable as a whole, yet the orbits of the individual neutrons in its interior are generally unstable precisely because of

the interactions studied in these volumes. We can therefore say that:
A) Quantum mechanics is an operator formulation of closed variationally selfadjoint systems, i.e., isolated systems with only local-differential-potential internal forces, in which case one formulation only of Lie type is sufficient for the characterization of both the system as a whole as well as its individual constituents. Global stability is achieved in this case via the stability of each constituent; while
B) Hadronic mechanics is an operator formulation of closed variationally nonselfadjoint systems, i.e., isolated systems with local-differential-potential, as well as nonlocal-nonpotential internal effects, in which case two mutually compatible formulations are generally needed, one for the description of a stable system as a whole (in which case the isotopic brackets of the time evolution are adopted), and one for the description of the individual constituents in unstable orbits (in which case the genotopic brackets of the time evolution are generally more appropriate). Global stability is achieved in this latter case under the maximal possible instability of each constituents. We merely have the highest possible internal exchanges of energy and other physical quantities, but always such to compensate each other resulting in total conservation laws.
Intermediate cases have also been identified, i.e., systems which are closed-isolated and variationally nonselfadjoint because of contact internal forces, yet the orbits of all constituents are stable. This is generally the case of strongly interacting systems with two or three constituents In this latter case the Lie-isotopic formulations are sufficient for the representation of both, the system as a whole and each individual constituent. The Lie-isotopic formulations are therefore expected to be fully sufficient for a reinspection of quark theories, as we shall see.

Proposal [20] concluded with the illustration of the <novel> capabilities of hadronic mechanics; that is, capabilities beyond the technical capacities of quantum mechanics. As an example, in Sect. 5 of ref. [20], it was shown that the isotopic formulations provide a quantitative representation of the synthesis[10] of an electron and a proton into the π° particle which is prohibited by quantum mechanics.

The reader should be aware that a main physical objective for which the entirety of the classical and operator, Lie-isotopic and Lie-admissible studies were and continue to be conducted by this author, as clearly stated in the original proposal [20] is the following:

To attempt the identification, within the context of a covering of quantum mechanics, of the hadronic (or quark) constituents with massive, physical, ordinary particles which are freely produced in spontaneous decays generally having the lowest mode (tunnel effect of the constituents).

Such a new model of the hadronic structure is manifestly prohibited by quantum mechanics, yet it becomes quantitatively possible for the covering hadronic mechanics precisely because of the novel, internal, nonlinear, nonlocal,

[10] The author would like to thank Prof. A. N. Sissakian, Deputy Director of the J. I. N. R., Dubna, Russia, for the suggestive terms "chemical synthesis".

nonhamiltonian, inhomogeneous and anisotropic effects. In turn, the capability of producing the hadronic constituents free has permitted the first practical applications of strong interactions known to this author.

1.4: GUIDE TO THE LITERATURE

The above lines of inquiry of 1978 were subsequently subjected to systematic studies by numerous authors, as indicated by the following meetings:

1) Five *Workshops on Lie-Admissible Formulations* held in Cambridge, MA, from 1978 to 1983;

2) Five *Workshops on Hadronic Mechanics* held from 1983 to 1989 in various Countries;

3) The *First International Conference on Nonpotential Interactions and their Lie-Admissible Treatment* held at the Université d'Orléans, France, in 1982; and the *First International Workshop on new Frontiers in Physics* held at the Castle Prince Pignatelli in August 1995 (see Proceedings [24] and references therein).

In a situation of this type, in this introductory section we can only indicate the most significant steps. Specialized advances will be reviewed and quoted in the subsequent chapters. This presentation, however, is and will remain incomplete in the review and quotation of all contributions in the field to avoid a prohibitive length. Also, contributions on other lines of inquiry cannot possibly be quoted (if nothing else, because of their shear number), unless they study *a structural generalization of current theories*, such as: the *quantum groups* (see, e.g., ref. [25] and quoted papers); the so-called *q-deformations* (see, e.g., ref. [26] and quoted paper) which are particular cases of the Lie-admissible formulations and, thus, particular cases of hadronic mechanics; the studies on *nonlocality* by Russian colleagues (see, e.g., monographs [8]); the *discrete formulation* of space-time; and other true generalizations.

The most salient advances in the studies of isotopies and genotopies of quantum mechanics can be summarized as follows. The original proposal [20] of 1978 suggested the formulation of Eq.s (1.3.1)-(1.3.5) on a conventional Hilbert space, a formulation which subsequently proved to be mathematically correct, yet not sufficient on physical grounds.

For this reason Myung and Santilli proposed in ref.s [27,28] the first mathematically rigorous formulation of the isotopies, that over the *isotopies* $\hat{\mathcal{H}}_{\hat{T}}$ of a Hilbert space $\mathcal{H}$ (today called *myung-Santilli isohilbert space*) with inner product $<\psi\,|\,\hat{T}\,|\,\phi>\hat{1}, \hat{1} = \hat{T}^{-1} > 0$ (defined over a generalized field reviewed in the next Chapter) where the operator $\hat{T}$ is the same as that in Eq.s (1.3.1).

As we shall see, the liftings $\mathcal{H} \rightarrow \hat{\mathcal{H}}_{T}$ have the fundamental implication of preserving Hermiticity under isotopies, as a result of which *the observable of*

quantum mechanics remain observable in hadronic mechanics.

Subsequent studies [29] by Mignani, Myung and Santilli of 1983 showed that the preceding formulation [27,28] even though correct, were themselves insufficiently general because Eq.s (1.3.1) and (1.3.5) can also be consistently defined on an isotopic Hilbert space $\mathcal{H}_{\hat{G}}$ with inner product $< \psi \,|\, \hat{G} \,|\, \phi > \hat{1}$, $\hat{1} = \hat{T}^{-1}$. where $\hat{T} > 0$ and $\hat{G} > 0$ are two generally different, positive-definite operators (also over a generalized field). The latter generalized space (today called *Mignani-Myung-Santilli isohilbert space*) is significant whenever the sole degree of freedom of the isotopic operator T is insufficient or the application at hand (e.g., reconstruction of the exact parity under weak interactions), although the preservation of the quantum mechanical observability implies predictable restrictions on both the T and G operators (as we shall see, the commutativity of $\hat{T}\hat{G}^{-1}$ with H).

As well known, a system in quantum mechanics is characterized by only one operator $H = K + V$. The corresponding system in the "isotopic branch" of hadronic mechanics is characterized by three independent operators, the Hamiltonian H, characterizing the potential forces, the isotopic operator T characterizing the nonpotential forces, and the operator G characterizing additional degrees of freedom of the underlying Hilbert space, while in the "genotopic branch" a system is characterized by four operators, H and G as well as R and $S = R^{\dagger}$.

The algebraic part of hadronic mechanics, that of Heisenberg-type based on generalized equations (1.3.1) and (1.3.5), had reached sufficient mathematical maturity by 1983. The additional advances since that time have been in the technical knowledge of Lie-isotopic algebras, Lie-admissible algebras, isotopic Hilbert spaces, and their applications.

In 1983 we already had the isotopic generalization of Wigner's theorem on unitary symmetries [30] and a structural generalization of the Lorentz symmetry O(3.1) of isotopic type [31] which, emerged to be locally isomorphic to O(3.1) (for all $T > 0$) while producing a generalization of the conventional linear-local-canonical Lorentz transformations of the desired, most general possible nonlinear, nonlocal, noncanonical, inhomogeneous and anisotropic type. Other developments and applications were merely consequential.

The studies on the Scrödinger-type formulations equivalent to the preceding Heisenberg-type ones resulted to be considerably more laborious, to such an extent to require a further generalization of the already generalized classical Hamiltonian mechanics, the Birkhoffian mechanics of ref. [10].

To outline these studies, we recall that Myung and Santilli [27] identified the following isotopic generalization of Schrödinger's equation on the isotopic Hilbert space $\mathcal{H}_T$

$$i \frac{\partial}{\partial t} | \psi > \; = \; H \, \hat{T} \, | \psi > , \tag{1.4.1}$$

which resulted to be equivalent to Eq.s (1.3.1) under the applicable unitary-isotopic transformations (except for scalar factors subsequently resulted to be important for an overall consistency of the theory). It should be mentioned that Eq.s (1.4.1) had also been independently identified by Mignani [32], although without Hilbert space treatment.

Animalu and Santilli [33] the identified the following isotopy of the naive quantization called *naive isoquantization*

$$A \rightarrow -i\, I \log|\psi> \qquad \rightarrow \qquad \hat{A} \rightarrow -i\hat{1} \log|\hat{\psi}> \tag{1.4.2}$$

under which the Birkhoffian form of the Hamilton-Jacobi equations [10] was uniquely and unambiguously mapped exactly into the hadronic equation (1.4.1).

However, subsequent studies indicated that Eq. (1.4.1) was not compatible with the relativistic isotopic formulations [31]. More specifically, the isotopic generalization of the conventional field equations characterized by the Lorentz-isotopic symmetry of ref. [31] admitted the following generalization of the plane-waves

$$\hat{\psi}(t, r) = N e^{i(p\hat{T}r - E\hat{T}_t t)}, \tag{1.4.3}$$

where T is the space- and T_t is the *time isotopic element*, which permitted a quantitative interpretation of the local variation of the speed of light within physical media, such as our atmosphere, water, glasses, etc.

Eq. (1.4.1) admitted the simpler "plane-wave"

$$\hat{\psi}(t, r) = N e^{i(p\hat{T}r - Et)}, \tag{1.4.4}$$

without the generalized element T_t in the energy term, thus resulting not to be compatible with relativistic form (1.4.3).

Also, Eq. (1.4.4) prevented the achievement of a consistent expression for the isotopic linear momentum operator, which in fact was completely lacking at that time (mid 1980's). In turn, the lack of such consistent isotopic forms literally precluded most applications, which had to be conducted at the abstract level (as done for that reason in ref.s [30,31]).

The resolution of these basic deficiencies required this author to conduct again a laborious effort at the purely classical level because of the evident need to reach the isotopic form of the linear momentum operator via isoquantization of corresponding well defined Hamilton-Jacobi equations, as a covering of conventional quantum derivations.

As recalled earlier, a step-by-step generalization of Hamiltonian mechanics of Birkhoffian type has been proposed in memoir [19] of 1978 as a first

application of the Lie–isotopic theory, and then studied in monograph [10] of 1983.

The difficulties here mentioned are due to the fact that, while the Heisenberg–type image of Birkhoff's equations has been reached since the original proposal [20] of 1978, the achievement of a Schrödinger–type version of Birkhoffian mechanics escaped all efforts for a number of technical problems, including: an excessively general "wave functions" $\psi(t, r, p)$ with an essential dependence also in the momenta p; lack of any practically usable expression for the isotopic linear momentum; the nonlinear and noncanonical, yet strictly local–differential character of Birkhoffian mechanics as compared to the general nonlocal character of hadronic mechanics; and others.

These occurrences forced this author to reinspect the classical generalized theories *ab initio,* and to conduct a second, step–by–step generalization of Hamiltonian mechanics, this time, of the so–called *isotopic type* reviewed later on in Volume II which admit the most general possible nonlinear and noncanonical, as well as nonlocal–integral systems. This novel mechanics, was presented for the first time in memoirs [34] of 1988 jointly with the corresponding, compatible isotopies of the symplectic, affine and Riemannian geometries for interior problems (only). Memoirs [34] were then expanded in monographs [11,12] for a detailed treatment at the classical level.

The form of the basic (nonrelativistic) equations of hadronic mechanics used in these volumes were reached by Santilli in memoirs [35] of 1989 and can be written

$$i\frac{\partial Q}{\partial t} = i\hat{1}_t\frac{dQ}{dt} = [\,Q\,\hat{,}\,H\,] = Q\,\hat{T}\,H - H\,\hat{T}\,Q\,, \qquad \hat{1}_t = \hat{T}_t^{-1}, \tag{1.4.5a}$$

$$i\frac{\partial}{\partial t}|\psi> = i\hat{1}_t\frac{\partial}{\partial t}|\psi> = H\,\hat{T}\,|\psi>, \tag{1.4.5b}$$

$$P_k\,\hat{T}\,|\psi> = -i\frac{\partial}{\partial x^k}|\psi> = -i\,\hat{1}_k{}^i\frac{\partial}{\partial x^i}|\psi>, \quad \hat{1} = \hat{T}^{-1}, \tag{1.4.5c}$$

which include the final expression of the operator linear momentum so vital for practical applications.

An alternative formulation of Eq.s (1.4.5) is also possible, should be kept in mind and will be studied at some later time. It is based on the isoderivatives $\hat{T}_t\partial/\partial t$ and $\hat{T}\partial/\partial x^i$ characterized by the interchange of the isounits with the isotopic elements, $\hat{1}_t \rightarrow \hat{T}_t$ and $\hat{1} \rightarrow \hat{T}$.

Relativistic equations were then achieved via isotopies of the conventional relativistic equations, as we shall see in Vol. II, and they resulted to be fully compatible with basic nonrelativistic equations (1.4.5).

Classical studies [34] also set the basis for the novel topology of hadronic mechanics (see Fig. 1.4.1).

Two further aspects deserve a mention for advance guidance in the following analysis. Theories based on a generalized unit $\hat{1}$ permit the identification the novel antiautomorphic map

$$\hat{1} \rightarrow \hat{1}^{d} = -\hat{1}, \tag{1.4.6}$$

called by this author *isoduality*, with corresponding *isodual isotopic formulations* which were identified in ref.s [31], and studied in more details in memoirs [35].

THE TOPOLOGY OF ISOTOPIC THEORIES

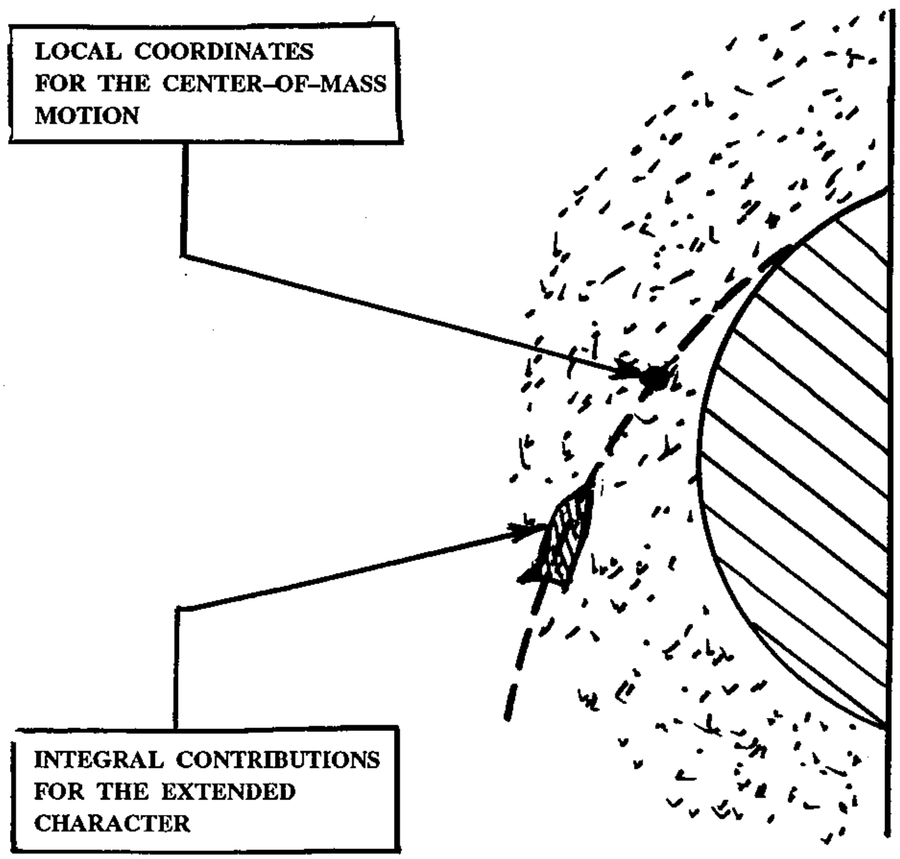

FIGURE 1.4.1. A conceptual view of the new *integro-differential topology* of hadronic mechanics. Nonlocal systems are of notoriously difficult treatment because they demand the so-called *integral topologies* which are some of the most complex mathematical constructions, particularly for physical applications. The solution proposed by this author to by-pass these difficulties is so simple to appear trivial, yet it is effective for physical applications, as we shall see. The main ideas are the following: 1) preserve the conventional local-differential variables x for the description of the trajectory of the

center-of-mass of the particle in interior conditions; 2) consider all nonlocal-integral contributions as corrections to the local-differential description; and 3) embed all nonlocal terms in the isounit of the theory. By recalling that topologies are insensitive to the functional dependence of their own unit when positive-definite, one can see that *all classical and operator isotopic theories admit the conventional local-differential topology everywhere except in the unit.* Such an integro-differential topology has important theoretical implications, such as the achievement of a fully *causal* description of *nonlocal* interactions, as well as experimental implications; e.g., the capability to test the nonlocal contribution as distinct from the conventional local ones.

An illustration is given by the space-ship during re-entry of this figure, whose shape directly affects the trajectory x(t) of the center of mass, as well known, resulting in two forces, a variationally selfadjoint (SA) force which is local-differential and derivable from a potential V(x), and a variationally nonselfadjoint (NSA) force which is generally nonlinear (in all variables), nonlocal and nonpotential. We therefore have classical equations of motion of the type

$$m\,\ddot{x} \;=\; F^{SA}(x) \;+\; F^{(NSA}(t,\, x,\, \dot{x},\, \ddot{x},\, ...), \qquad F^{NSA} = -\,\gamma\,\dot{x}^{\,2}\!\int_{\sigma} d\sigma\; \mathcal{F}(\sigma,\, ...)$$

where σ is the shape of the satellite. In these volumes we shall by-pass the notorious difficulties in the practical application of integral topologies via the representation of the local-differential part in terms of the conventional Hamiltonian $H = K(\dot{x}) + V(x)$ and the embedding of all nonlocal-NSA forces in the isounit $\hat{1}$ of the theory.Rigorous mathematical studies on this integro-differential topology have been conducted by Tsagas and Sourlas [40], and are outlined in App. 6.A.

The studies on isoduality essentially permitted *a novel interpretation of antiparticles based on theories with negative-definite generalized units.* In fact, antiparticles originate from the negative-energy solutions of relativistic equations, although these solutions behave unphysically when conventionally interpreted, that is, when the negative energies are interpreted as having the conventional unit $\hbar = I > 0$, thus forcing the conjecture of the "hole theory" with "infinite seas" of virtual antiparticles and other assumptions.

The important property is that the negative-energy solutions behave in a fully physical way when referred to negative-definite units, without any need of conjecturing infinite seas of antiparticles, or passing to second quantization.

The isotopic and isodual formulations then emerged as possessing intriguing interconnections from the finite transition probabilities existing in conventional relativistic equations between positive- and negative-energy solutions.

While the current theory of antiparticles can only be formulated at the level of *second* quantization, the isodual representation results to be fully applicable at the *classical* level. In fact, the sole antiautomorphic conjugation available in conventional theories is charge conjugation which evidently requires a Hilbert space. On the contrary, isoduality is applicable at all possible

formulations, beginning with at the classical ones, and then persisting at the operator one in which isoduality and charge conjugation results to be equivalent.

The latter occurrence has far reaching implications. It permitted the achievement of a representation of antiparticles beginning at the *Newton's* level and then passing to the *classical* Minkowskian and Riemannian formulations, the latter one permitting the first *classical* studies on scientific records as to whether a far away star or galaxy is made up of matter or antimatter.

In these volumes we shall therefore study isotopic formulations with positive–definite generalized units $\hat{1} > 0$ for the representation of *matter*, and their isodual conjugate formulations with negative–definite units $\hat{1}^d < 0$ for the representation of *antimatter*.

The last aspect deserving an advance mention regards gravitation. As we shall see, isotopic and isodual formulations, including those of the Riemannian geometries [11,12,34], permit truly remarkable and diversified advances in gravitation, including the identification of a hitherto unknown "isodual universe" for antimatter.

The aspect warranting advance notice regards the historical problem of quantization of gravity. The isotopies permit the factorization of all Riemannian metrics into the firm $g(x) = \hat{T}_{gr}(x)\,\eta$, where η is the Minkowski metric and the embedding of the isotopic part $\hat{T}_{gr}(x)$ truly representing gravitation in the generalized unit via the rule [34]

$$\hat{1}_{gr} = [\hat{T}_{gr}(x)]^{-1}, \quad g(x) = \hat{T}_{gr}(x)\,\eta. \tag{1.4.7}$$

This permits *a novel quantization of gravitation.* As a matter of fact, a quantum version of gravity has always existed. It did creek in un–noticed because embedded in the unit of relativistic quantum mechanics.

Moreover, the studies imply *a geometric unification of gravitation and relativistic quantum mechanics based on an alternative formulation of curvature via the generalization of the unit of the conventional Minkowski space* as originally proposed in ref. [31].

We see in this way that the generalized unit has a very special meaning when singular. In fact, as we shall see, *the limit* $\hat{1}_{gr}(x) \rightarrow 0$ *can represent a gravitational singularity at* x.

In addition to the two cases $\hat{1} > 0$ and $\hat{1} < 0$, the third case $\hat{1}_{gr} = 0$ also has intriguing physical interest and should be kept in mind during the analysis of these volumes. Two additional significant cases of the isounit will be identified in the next section.

Among the numerous researchers who have contributed to the development, application and test of hadronic mechanics at this writing, we mention

Animalu, Aringazin, Bartzis, Baskoutas, Borghi, Brodimas, Caldirola,

Cardone, Dall'Olio, Eder, Fronteau, Gasperini, Gill, Giori, Ioannidou, Jannussis, Kadeisvili, Kalnay, Kamiya, Karayannis, Kliros, Klimyk, Kobussen, Lin, Lopez, Mignani, Mijatovic, Myung, Nishioka, Papadoupoulos, Papaloukas, Papatheou, Rauch, Schuch, Sourlas, Skaltzas, Streclas, Tsilimigras, Veljanoski, Vlahos, Tellez Arenas, Tsagas, Weiss, Wolf,

and others we shall identify in these volumes. The understanding is that we are referring to *mathematical, theoretical or experimental contributions requiring, specifically, the generalization of the unit,* thus excluding quantum groups, q-deformations and other generalizations.

Independent reviews of the classical studies are available in monographs [36,37], while comprehensive mathematical presentations of the isotopies of Lie's theory are available in monographs [38,39].

1.5: CLASSIFICATION OF HADRONIC MECHANICS

Hadronic mechanics is nowadays a rather diversified discipline with structurally different mathematical methods in different branches. In a situation of this type, it is recommendable to assume a classification from the beginning of the studies, because it can prove to be later on a valuable guide.

First, the hadronic mechanics is divided into the two main branches identified in the preceding section:

A - The Lie-isotopic branch for closed-isolated NSA systems, which is characterized by Hermitean generalized units $\hat{1} = \hat{1}^\dagger$ for both motions forward and backward in time, and

B - The Lie-admissible branch for open-nonconservative NSA systems, which is characterized by two different generalized units $\hat{1}^>$ and $^<\hat{1}$, for motions forward and backward in time, respectively.

Next, each branch admits a classification depending on the main structural characteristics of the generalized unit. In this volume we shall assume the classification introduced by Kadeisvili [38] for the isotopies of functional analysis, here called *Kadeisvili's classification,* which divides the isotopic branch into the following five classes:

Class I: Isotopic formulations properly speaking, holding when the

generalized unit $\hat{1}$ is sufficiently smooth, bounded, nowhere degenerate, Hermitean and positive-definite. This is the class of primary interest in these volumes for the study of particles in interior conditions.

Class II: Isodual isotopic formulations, holding when the generalized unit has the same characteristics of Class I, except that it is negative-definite. This is the class of primary relevance for the study of antiparticles in interior conditions.

Class III: Indefinite isotopic formulations, i.e. the union of Classes I and II, with isounits being either positive-definite or negative-definite. This class has primary mathematical relevance, e.g., for the unified treatment of Class I and II.

Class IV: Singular isotopic formulations, holding for the union of Classes I and II plus singular generalized units. As we shall see, this class is useful for the study of gravitational singularities.

Class V: General isotopic formulations, holding for Class iv plus generalized units of arbitrary structure, thus including distributions, discontinuous functions, etc. This last class is useful for the study of fundamentally novel mathematical notions, such as a discrete group defined over a continuously varying unit (and viceversa) and, except for isolated remarks, will not be considered in these volumes for brevity.

Evidently a corresponding distinction into Classes I–V holds for the Lie-admissible/genotopic branch of hadronic mechanics with the understanding that the condition of Hermiticity and positive- or negative-definiteness are referred only to the Hermitean part of the nonhermitean operators R and S.

A **third branch of hadronic mechanics of hyperstructural type** is also conceivable via the transition from ordinary operators to the so-called *hyperoperators,* e.g., matrices whose elements are given by a *set* of conventional elements [41].

This third branch is based on the so-called *hyperstructures* [loc. cit.] which are some of the most complex known mathematical formulations, in which the product of two quantities can be a finite or infinite and ordered or non-ordered set. The latter branch is mathematically intriguing as a natural generalization of the isotopic and genotopic branches, and apparently significant for the representation of systems more complex than the physical ones, such as those in biology, but it will not be studied in these volumes to avoid a prohibitive length.

In conclusion, hadronic mechanics is a generalization-covering of quantum mechanics which possesses *ten* topologically different isotopic and genotopic classes, excluding the hyperstructural ones, and this begins to illustrate the rather vast character of the new discipline from which its "direct universality" follows (Ch. I-7).

Unless otherwise stated, the mathematical studies of this Volume I specifically treat the isotopic formulations of Class I (for particles) and II (for

antiparticles), with comments on the construction of the remaining isotopic formulations of Class III, IV and V. The genotopic formulations are studied in appendices and in Ch. I.7.

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2: ISONUMBERS AND THEIR ISODUALS

2.1: STATEMENT OF THE PROBLEM

We indicated in the Preface that the primary difficulty in addressing (and appraising) hadronic mechanics is the prior knowledge of its novel mathematical structure, because even the conventional numbers and their operations are inapplicable. The understanding is that, when inspected, such novel mathematical structure soon emerges to be simple and intriguing.

The best way to illustrate this aspect is by noting that the traditional statement "two multiplied by two equals four" is at best mathematically incomplete, because it lacks the identification of the underlying unit and of the operation of multiplication, and it is generally inapplicable under isotopies.

In fact, we shall first show in this chapter that, by assuming, say, for generalized unit $\hat{1} = 3^{-1}$, "two multiplied by two equals twelve" and then we shall show that the *numbers themselves* and their operations acquire, in general, an *integral* character as necessary from basic assumption (1.1.1).

The use under isotopies of the conventional mathematical structure of quantum mechanics therefore leads to a host of generally undetected inconsistencies.

We shall study in this chapter the generalized numbers needed for hadronic mechanics, and then study in subsequent chapters the generalized structures built on them.

As well known, the *theory of numbers* received momentous advances in the past century, thanks to the contributions of famed scholars such as Gauss [1], Abel [2], Hamilton [3], Cayley [4], Galois [5] and others (see review [6] in the early part of this century, and ref.s [7-9] for contemporary presentations).

Additional important advances in number theory were made during this century, including the axiomatic formulation, the theory of algebraic numbers, etc. (see, e.g., ref.s [10] and contributions quoted therein).

The "numbers" significant for these volumes are the *real numbers, complex numbers, quaternions* and *octonions*. The topic is therefore the

classification of all *normed algebras with identity over the reals* according to the studies, e.g., by Hurwitz [11], Albert [12] and (N.) Jacobson [13] (see also reviews [7,8]). The main properties can be expressed via the following

Theorem 2.1.1 (ref. [8], p. 122): *All possible normed algebras with multiplicative unit* 1 *over the field of real numbers are given by algebras of dimension* 1 *(real number),* 2 *(complex numbers),* 4 *(quaternions) and* 8 *(octonions).*

The fundamental notions under study in this chapter are therefore fields and normed algebras with unit.

During a talk at the conference *Differential Geometric Methods in Mathematical Physics* held in Clausthal, Germany, in 1980[11], this author submitted an axiom-preserving generalization of numbers, today known as *isotopic numbers* or *isonumbers* for short. The generalizations are induced by the so-called *isotopies* of the conventional multiplication of numbers introduced in ref.s [14,15], with consequential generalization of the basic multiplicative unit.

The isonumbers received a formal treatment in ref. [16], and first applications in ref. [17] for the isotopic lifting of unitary symmetries, in ref. [18] for the lifting of the Lorentz symmetry, and in ref.s [19.20] for the general isotopies of Lie symmetries. Subsequent studies were conducted in ref.s [21,22]. A theorem on the unification of different isonumbers (studied in Sect. 2.7] was presented in ref. [23]. The presentation of this chapter follows ref. [24] which is the first comprehensive mathematical study on the theory of isonumbers on record at this time, with the understanding that the studies are still at the beginning, and so much remains to be done.

The author also submitted in ref.s [19,20] a new conjugation, under the name of *isoduality* which yields an additional class of numbers, today known as *isodual isonumbers*. Recent presentations of isodual isonumbers can be found in ref.s [22,24].

The isonumbers were motivated by the specific physical need of a quantitative representation of the transition from the exterior to the interior dynamical problem, as discussed in Chapter 1. The isodual isonumbers were constructed for a representation of antiparticles (see Vol. II and ref. [29]).

The isonumbers and their isoduals are at the foundations of the Lie-isotopic formulations but they are inapplicable for more general theories such as the Lie-admissible formulations.

For this reason the author identified in ref. [24] an additional class of numbers under the name of *genotopic numbers*, or *genonumbers* for short. The

[11] Thanks to a kind invitation by Prof. H.-D. Doebner which is here gratefully acknowledged.

primary difference between isonumbers and genonumbers is that *isonumbers have a unique left and right generalized unit* or, equivalently, *the multiplication of the isonumbers applies to both left and right operations.*

By comparison, *genonumbers have two different generalized units, one for the multiplication from the right and a different one for the multiplication from the left,* $a<b = b<a$, $a>b = b>a$, $a<b \neq a>b$, in which case they indeed result to be at the foundation of the Lie-admissible formulations.

A third class of generalized numbers is conceivable via the so-called *hyperstructures* [32], but they will not be studies for brevity.

To avoid excessive initial complexities, we shall proceed in stages. In this and in the following chapters we shall solely study the isonumbers, their isoduals and their Lie-isotopic formulations. The primary objective of this chapter is therefore the study of the isotopies and isodualities of the notions of numbers, fields and normed algebras with unit. The isotopies of the theory of numbers [30] will be indicated in App. I.2.B.

The minimal mathematical knowledge needed for hadronic mechanics is that of isoreal and isocomplex numbers and their isoduals studied in Sect.s I.2.5 and I.2.6. The isoquaternions, isooctonions and their isoduals of Sect.s I.2.7 and I.2.8 are needed for a more technical knowledge of the topic. The more general (and complex) theory of genonumbers and related Lie-admissible formulations as well as of hypernumbers and related theory will be studied in Ch. I.7.

For a recent independent study of the field, including elements of isotopies, we suggest the monograph by Lôhmus, Paal and Sorgsepp [28]. Applications of the the generalized numbers of this chapter to classical mechanics can be found in monograph [29]. Applications to cryptology are indicated in App. I.2.C and those to particle physics and other fields in Volumes II and III.

The author would like to thank David Ring of Dunedin, FL, for bringing to his attention the fact that the Egyptians have been the firsts in recorded history to change the value of their basic unit, called *finger,* in the transition from the sides of a right triangle to the hypothenuse.

2.2: ISOUNITS AND THEIR ISODUALS

Studies [14-27] (and references quoted therein) have shown that the transition

A) from the local-canonical *exterior problem* in vacuum,

B) to the nonlocal-noncanonical *interior problem* within physical media,

can be effectively represented via an axiom-preserving *isotopic generalization* of the conventional multiplication of numbers a, b (or functions or operators).

We are here referring to the generalization of the current, simplest possible multiplication of numbers $a\times b$ (generally denoted ab), into the *isotopic multiplication,* or *isomultiplication* $a\hat{\times}b$ (later on also denoted $a*b$) introduced in

ref. [14], p. 332,

$$a \hat{\times} b \;=\; a \times \hat{T} \times b \;=\; a \hat{T} b\,, \qquad (2.2.1)$$

which will be symbolically denoted $\hat{\times} = \times\hat{T}\times$ (or $* = \times\hat{T}\times$), where $\hat{T}$ is an invertible quantity called *isotopic element* which is fixed for all possible elements a, b and generally outside the original set. The lifting $\times \to \hat{\times}$ is isotopic because (for nonsingular elements $\hat{T}$) it preserves all the original operations among ordinary numbers as seen in more details in the next section.

The conventional, right and left multiplicative unit 1 of current theories, $1\times a = a\times 1 \equiv a$, is then lifted into the form also valid for all possible elements a

$$\hat{1} \hat{\times} a \;=\; a \hat{\times} \hat{1} \;\equiv\; a\,, \qquad \hat{1} = \hat{T}^{-1}\,, \qquad (2.2.2)$$

called the *multiplicative isounit*, or *isounit* for short.

Under the condition that $\hat{1}$ preserves all the axioms of 1 the lifting $1 \to \hat{1}$ is an *isotopy*, that is, the conventional unit 1 and the isounit $\hat{1}$ (as well as the conventional product $a\times b$ and its isotopic form $a\hat{\times}b$) have the same basic axioms and coincide at the abstract level by conception.

The *isonumbers* can be first introduced as the generalization of conventional numbers when characterized by isoproduct (2.2.1) with respect to the generalized isounit $\hat{1} = \hat{T}^{-1}$.

As one can see, the isounits have a completely unrestricted functional dependence, thus admitting the most general possible integro-differential structure of type (1.1.1),

$$\hat{1} = \hat{1}(t, x, \dot{x}, \ddot{x}, \psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \mu, \tau, n, \ldots), \qquad (2.2.3)$$

A necessary condition for a quantity $\hat{1}$ to be an isounit, i.e.,a joint left and right generalized unit, is that it is Hermitean. Then, isomultiplication (2.2.1) is the same for both right and left operations.

As indicated in Sect. 1.5, in these volumes we shall use *Kadeisvili's classification* into:

Class I: Isounits properly speaking, when they are sufficiently smooth, bounded, nowhere degenerate, Hermitean and positive-definite;

Class II:Isodual isounits, when they are as in Class I, except that they are negative-definite;

Class III:Indefinite isounits, when they are as in Class I except that they have an indefinite signature with local values which can be either positive-definite or negative-definite;

Class IV:Singular isounits, when they are null or infinite at at least one given value of their variables;

Class V: General isounits, when they have an unrestricted structure, e.g., given by discrete forms, distributions, step functions, etc.

In this chapter we shall study isounits of Classes I and II with a few comments on those of Class III. The theory of isonumbers for Classes IV and V is vastly unexplored at this writing.

We should note that the most important functional dependence of the isounits is that of *integral* type. Thus, the isotopies $I \rightarrow \hat{1}$ characterize a new form of *integro-differential topology* in which all integral terms are embedded in the isounit, while the rest of the structure is conventionally local-differential (see Fig. I.4.1 and ref. [15]). As an example, in the isotopies of Minkowskian spaces, this novel structure permits the preservation of conventional topologies (e.g., the Zeeman topology) everywhere, except for the interior of the isounit itself.

The integral generalization of the unit is the conceptual, mathematical and physical foundation of hadronic mechanics, because it permits a quantitative treatment of the *integral generalization of Planck's constant* $\hbar I \rightarrow \hbar\hat{1}$ discussed in Ch. I.1.

As we shall see better in applications presented in subsequent chapters, the isounits of hadronic mechanics generally have a matrix representation with considerable degrees of freedom in their elements. As such, they permit a geometrization of inhomogeneous and anisotropic physical media, in such a way to preserve the axioms of the homogeneous and isotropic vacuum and admit the latter as a particular case.

The *isodual isounits* are given by

$$\hat{1}^d = -\hat{1}, \tag{2.2.4}$$

and are based on the following antiautomorphic conjugation of multiplication (2.2.1)

$$a \hat{\times} b \rightarrow a \hat{\times}^d b = a \hat{T}^d b = -a \hat{T} b = -a \hat{\times} b, \quad \hat{T}^d = -\hat{T}, \tag{2.2.5}$$

under which $\hat{1}^d$ (but not $\hat{1}$) is the correct left and right generalized unit of the theory,

$$\hat{1}^d \hat{\times}^d a = a \hat{\times}^d \hat{1}^d \equiv a, \tag{2.2.6}$$

The map characterized by liftings (2.2.4) and (2.2.6) was called by this author *isoduality* [20] and this terminology will be kept in these volumes. As we shall see, these liftings are significant inasmuch as they can be applied to each aspect of the Lie-isotopic formulations, yielding the *isodual Lie-isotopic theory.*

The *isodual isonumbers* were constructed via isodual multiplication (2.2.5) with respect to the the multiplicative isodual isounit $\hat{1}^d$.

Note that the notion of isoduality first applies to *conventional* numbers. In fact, the expressions

$$\hat{1}^d = -1, \quad 1^d = 1^d = -1, \tag{2.2.7}$$

characterize *isodual numbers*. This means that the conventional formulations, such as Lie's theory, Riemannian geometry, etc., admit hitherto unknown images given by *the isodual Lie theory, the isodual Riemannian geometry*, etc., which are constructed in such a way to admit everywhere the isodual unit $1^d = -1$.

One can now see the necessity of lifting the product $\times \to *$ for the very conception of isodual numbers, isodual isonumbers, and related formulations. The restriction of the studies in number theory to the conventional multiplication $\times$ may therefore be a reason why isodual numbers, isodual Lie formulations, isodual Riemannian geometry, etc. have escaped detection until recently.

The author also studied the problem whether isomultiplication (2.2.1) exhausts all isotopies of the conventional product of numbers. The issue is important because any new isotopy of the associative product characterizes a new realization of the theory of isonumbers and, therefore, a new mechanics, with new Heisenberg–type equations and all that.

Only three isotopies of the multiplication ab were found [15]:

A) The *scalar isotopy*

$$a \hat{\times} b = a\, c\, b, \quad \hat{T} = c = \text{number}, \tag{2.2.8}$$

B) The *operator isotopy*

$$a \hat{\times} b = a\, \hat{T}\, b, \quad \hat{T} = \text{operator}, \tag{2.2.9}$$

C) The *idempotent isotopy*

$$a \hat{\times} b = W\, a\, W\, b\, W, \quad W^2 = W = \text{idempotent}, \tag{2.2.10}$$

and any of their combinations which are the only known modifications of the original associative product capable of preserving, not only the original associative law, but also the scalar and distributive laws so as to preserve an algebra as commonly understood (see later Sect. I.2.4).

Other liftings are evidently possible, such as

$$a \hat{\times} b = a\, b\, \hat{T}, \qquad \text{or} \qquad a \hat{\times} b = \hat{T}\, a\, b. \tag{2.2.11}$$

However, even though preserving associativity, the latter liftings generally violate

the right or left scalar and distributive laws and, as such, they do not characterize an algebra as commonly understood, that *without* an ordering of the multiplication to the left or to the right. The latter ordering does indeed exist for the more general Lie-admissible formulations for which liftings (2.2.11) are indeed significant.

The problem whether the above liftings exhaust all possible isotopies of the multiplication is unknown at this writing.

These volumes are based on the fundamental condition that *any admitted generalization of quantum mechanics must possess a well defined unit, because necessary for measurements and other experimental applications.*

Along these lines, isotopies (2.2.8) and (2.2.9) are acceptable, while isotopies (2.2.10) are not because the product $a \hat{\times} b = WaWbW$ does not admit a consistent, left and/or right unit for all elements a, b. Similarly, liftings (2.2.11) are acceptable only for one-sided theories because they admit only one-side units.

2.3: ISOFIELDS, PSEUDOISOFIELDS AND THEIR ISODUALS

Let us introduce the following definition of isofields:

Definition 2.3.1 [24]: *Let* $F = F(a,+,\times)$ *be a "field" as conventionally understood (see, e.g., ref. [8], p. 101), here referred to a ring with elements* a, b, c, ..., *which is commutative with respect to the operation of addition* + *and associative under both the addition* + *and multiplication* $\times$ *with corresponding additive unit 0 and multiplicative unit* 1. *Then, the infinite family of "isotopic images" of* $F(a,+,\times)$, *called "isofields" and denoted* $\hat{F} = \hat{F}(\hat{a}, +, \hat{\times})$, *are given by elements* $\hat{a}$, $\hat{b}$, $\hat{c}$, ... *characterized by one-to-one and invertible maps* $a \rightarrow \hat{a}$ *of the original elements* $a \in F$ *equipped with two operations* $(+, \hat{\times})$, *the conventional addition* + *of* F *and a new multiplication* $\hat{\times}$, *called "isomultiplication", with corresponding conventional "additive unit"* 0 *and a generalized multiplicative unit* $\hat{1}$, *called "multiplicative isounit", which are such to satisfy all axioms of the original field* F, *i.e.:*

1) Axioms of addition:

1.A) The set $\hat{F}$ *is closed under addition,*

$$\hat{a} + \hat{b} \in \hat{F} \qquad \forall\, \hat{a}, \hat{b} \in \hat{F}, \tag{2.3.1}$$

1.B) The addition is commutative for all elements $\hat{a}$, $\hat{b} \in \hat{F}$

$$\hat{a} + \hat{b} = \hat{b} + \hat{a}; \tag{2.3.2}$$

1.C) The addition is associative for all $\hat{a}$, $\hat{b}$, $\hat{c}$, $\in$ F,

$$\hat{a} + (\hat{b} + \hat{c}) = (\hat{a} + \hat{b}) + \hat{c}; \tag{2.3.3}$$

1.D) There is an element 0, the"additive unit", such that for all elements $\hat{a} \in \hat{F}$

$$\hat{a} + 0 = 0 + \hat{a} \equiv \hat{a}; \tag{2.3.4}$$

1.E) For each element $\hat{a} \in \hat{F}$, *there is an element* $-\hat{a} \in \hat{F}$, *called the "opposite of* $\hat{a}$*", which is such that*

$$\hat{a} + (-\hat{a}) = 0 \tag{2.3.5}$$

2) Axioms of isomultiplication:

2.A) The set $\hat{F}$ *is closed under isomultiplication,*

$$\hat{a} \hat{\times} \hat{b} \in \hat{F}, \quad \forall \quad \hat{a}, \hat{b} \in \hat{F}, \tag{2.3.6}$$

2.B) The multiplication is generally non–isocommutive, i.e., $\hat{a} \hat{\times} \hat{b} \neq \hat{b} \hat{\times} \hat{a}$, *but "isoassociative", i.e., it satisfies the law for all elements* $\hat{a}, \hat{b}, \in \hat{F}$

$$\hat{a} \hat{\times} (\hat{b} \hat{\times} \hat{c}) = (\hat{a} \hat{\times} \hat{b}) \hat{\times} \hat{c}; \tag{2.3.7}$$

2.C) There exists a quantity $\hat{1}$, *the "multiplicative isounit", which is such that, for all elements* $\hat{a} \in \hat{F}$,

$$\hat{a} \hat{\times} \hat{1} = \hat{1} \hat{\times} \hat{a} \equiv \hat{a}, \tag{2.3.8}$$

2.D) For each element $\hat{a} \in \hat{F}$, *there is an element* $\hat{a}^{-\hat{1}} \in \hat{F}$, *called the "isoinverse", which is such that*

$$\hat{a} \hat{\times} (\hat{a}^{-\hat{1}}) = (\hat{a}^{-\hat{1}}) \hat{\times} \hat{a} = \hat{1}. \tag{2.3.9}$$

3) Properties [12] *of joint addition and isomultiplication:*

3.A) The set $\hat{F}$ *is closed under joint isomultiplication and addition,*

$$\hat{a} \hat{\times} (\hat{b} + \hat{c}) \in \hat{F}, \quad (\hat{a} + \hat{b}) \hat{\times} \hat{c} \in \hat{F}, \ \forall \ \hat{a}, \hat{b}, \hat{c} \in \hat{F} \tag{2.3.10}$$

3.B) All elements $\hat{a}, \hat{b}, \hat{c} \in \hat{F}$ *verify the right and left "isodistributive laws"*

[12] Property (2.3.10) is generally derived from axioms 1A and 2A. Nevertheless, we shall encounter in Sect. 2 (see the comments after Proposition 2.3.3) a case in which Axioms 1A and 2A are verified, but property (2.3.10) is not.

$$\hat{a}\hat{\times}(\hat{b}+\hat{c}) = \hat{a}\hat{\times}\hat{b}+\hat{a}\hat{\times}\hat{c}, (\hat{a}+\hat{b})\hat{\times}\hat{c} = \hat{a}\hat{\times}\hat{c}+\hat{b}\hat{\times}\hat{c}, \tag{2.3.11}$$

The elements $\hat{a}$ of isofields $\hat{F}(\hat{a},+,\hat{\times})$ are called "isonumbers". When there exists a least positive integer p such that the equation

$$p\hat{\times}\hat{a} = 0, \tag{2.3.12}$$

admits solution for all elements $\hat{a} \in \hat{F}$, then $\hat{F}$ is said to have "isocharacteristic p". Otherwise, $\hat{F}$ is said to have "isocharacteristic zero".

A few additional properties are needed before we can select the realization of isonumbers used in these volumes. First, we should indicate that *only isofields of isocharacteristic zero will be used throughout the our studies* . Nevertheless, we thought that an exposure of physicists to isofields of isocharacteristic p is warranted because of their potential physical relevance for a number of applications, ranging from string theory to gravitational collapse, particularly when inspected from an isotopic viewpoint.

The dominant mathematical aspect here is the isotopy. In fact, the lifting $F(a,+,\times) \to \hat{F}(\hat{a},+,\hat{\times})$ preserves all original axioms by construction. The realizations of the isonumbers must then be selected in such a way to preserve such basic isotopic character.

In this respect, we note that the liftings $a \to \hat{a}$, and $\times \to \hat{\times}$ can be used jointly or individually. The following property is then important for our analysis.

Proposition 2.3.1 [24]: *Necessary and sufficient condition for the lifting (where the multiplication is lifted but the elements are not)*

$$F(a,+,\times) \to \hat{F}(a,+,\hat{\times}) \quad \hat{\times} = \times \hat{T} \times, \quad \hat{1} = \hat{T}^{-1} \tag{2.3.13}$$

to be an isotopy is that the lifting $\times \to \hat{\times}$ is a scalar isotopy (2.2.8), i.e., $\hat{T}$ is a non-null element of the original field F.

In fact, the laws of addition are unchanged under lifting (2.3.13), while the multiplication and distributive laws can be readily verified to hold. The closure of the original set under the addition is evident because that operation is not changed. We then remain with the closure under the isomultiplication,

$$a\hat{\times}b = a\times\hat{T}\times b = a\hat{T}b \in \hat{F}, \quad \forall a,b \in \hat{F}, \tag{2.3.14}$$

which does indeed hold when $\hat{T} \in F$, by therefore establishing the sufficiency of the condition. Its necessity follows from simple contrary arguments.

Proposition 2.3.2 [24]: *The lifting (in which both the multiplication and the elements are lifted)*

$$F(a, +, \times) \;\rightarrow\; \hat{F}(\hat{a}, +, \hat{\times}), \;\; \hat{a} = a \times \hat{1} \equiv a\hat{1}, \quad \hat{\times} = \times \hat{T} \times, \quad \hat{1} = \hat{T}^{-1}, \qquad (2.3.15)$$

constitutes an isotopy even when the multiplicative isounit $\hat{1}$ *is not an element of the original field* F , *e.g., when the lifting* $\times \rightarrow \hat{\times}$ *is an operator isotopy (2.2.9).*

In fact, one can readily verify for lifting (2.3.15) the validity of all axioms of a field, and closure under addition. Closure under multiplication readily holds because

$$\hat{a} \hat{\times} \hat{b} \;=\; (a \hat{\times} \hat{1}) \times \hat{T} \times (a \hat{\times} \hat{1}) \;=\; (a \times b) \times \hat{1} \;=\; c \times \hat{1} \;=\; \hat{c} \in \hat{F},$$

$$\forall\, a, b, c = a \times b \in F, \qquad (2.3.16)$$

The above mathematically simple proposition expresses the physically fundamental capability of generalizing Planck's unit $\hbar = 1$ of quantum mechanics into an integro-differential operator $\hat{1}$ for a quantitative treatment of nonlocal interactions.

In fact, basic assumption (1.1.1) requires, by conception, an isounit which is outside the original field. The realization we shall adopt throughout these volumes is therefore form (2.3.16) with the understanding that more complex realizations are possible (see later on).

The implications of the above realization are evidently fundamental for hadronic mechanics. One implication deserving advance mention is that *the "numbers" used in hadronic mechanics have an integro-differential structure,* e.g., $\hat{2} = 2 \times \exp(\int dv \psi^\dagger \psi)$. nevertheless, *the numbers predicted by the theory for measurements are ordinary numbers.*

In fact, the above realization implies that the isomultiplication of an isonumber $\hat{a}$ by any quantity Q coincides with the conventional multiplication

$$\hat{a} \hat{\times} Q \;\equiv\; a\, Q\, . \qquad (2.3.17)$$

Thus, *the isoeigenvalues of hadronic mechanics can be made to coincide with ordinary numbers*

$$H \hat{\times} |\psi> \;=\; \hat{E} \hat{\times} |\psi> \;=\; E \times \hat{1} \times \hat{T} \times |\psi> \;\equiv\; E|\psi>, \;\; \hat{E} \in \hat{F}, \quad E \in F. \quad (2.3.18)$$

The numerical predictions of the theory are then ordinary numbers E and not isonumbers $\hat{E}$. As we shall see in Vol. II, a similar result is obtained via the use of the isotopic expectation values of the new mechanics.

It should be noted that the mathematically correct expression in hadronic mechanics is the form $H \hat{\times} | \psi > = \hat{E} \hat{\times} | \psi >$. Nevertheless, since $\hat{E} * | \psi > \equiv E | \psi >$, ordinary eigenvalues E can be used in practical calculations.[13]

Evidently, all conventional operations depending on the multiplication are altered under lifting to isofields. Let us consider the isofields $\hat{F}(a,+,\hat{\times})$ of Proposition 2.3.1 under the condition that the isounit $\hat{1}$ commutes with all elements of a. Then, the "square" $a^2 = a\, a$ is lifted into the *isosquare* $a^{\hat{2}} = a \hat{\times} a = a\, \hat{T}\, a$, with *n-th isopower*

$$a^{\hat{n}} = a\, \hat{T}\, a\, \hat{T}\, a \ldots\ldots T\, a \quad (\text{n times}) \tag{2.3.19}$$

Recall that the conventional square root can be defined as the quantity $a^{\frac{1}{2}}$ such that $(a^{\frac{1}{2}})\,(a^{\frac{1}{2}}) = a$. Then, the *isosquare root* is given by

$$a^{\hat{\frac{1}{2}}} = a^{\frac{1}{2}}\, \hat{1}^{\frac{1}{2}}, \quad a^{\hat{\frac{1}{2}}} \hat{\times} a^{\hat{\frac{1}{2}}} = a^{\hat{\frac{1}{2}}}\, \hat{T}\, a^{\hat{\frac{1}{2}}} = a\,. \tag{2.3.20}$$

The *isoinverse* is given by

$$a^{-\hat{1}} = \hat{1}\, a^{-1}\, \hat{1}, \quad a \hat{\times} a^{-\hat{1}} = \hat{1}. \tag{2.3.21}$$

The *isoquotient* can then be defined by

$$a \,\hat{/}\, b := (a / b)\hat{1} = c \qquad c\, \hat{T}\, b = a\,. \tag{2.3.22}$$

The reader can then compute all other isooperations.

In the transition to the realization of Proposition 2.3.2 we have instead

$$\hat{a}^{\hat{n}} = \hat{a} \hat{\times} \hat{a} \hat{\times} \hat{a} \hat{\times} \ldots\ldots \hat{\times} \hat{a} = a^n \hat{1}\,. \tag{2.3.23}$$

The reformulation of the remaining operations then follows, as the reader is encouraged to work out to acquire familiarity with the theory of isonumbers.

Recall that a primary objective of hadronic mechanics is the integro-differential generalization of Planck's constant $\hbar = 1 \rightarrow \hat{\hbar} = \hbar\hat{1} = \hat{1}$. It is therefore important to understand that the new unit $\hat{1}$ preserves all axiomatic properties of the original unit of quantum mechanics, $\hbar = 1$. In fact, *the isounit* $\hat{1}$ *is*

[13] It should be noted that the lifting of eigenvalues is far from being trivial. In fact, as we shall see in more details Vol. II, if an operator H has the conventional eigenvalue E°, $H | \psi > = E° | \psi >$, it admits a *different* eigenvalue E under isotopy, $H \hat{\times} | \hat{\psi} > = H\, T\, | \hat{\psi} > = E | \hat{\psi} >$, $E \neq E°$. Thus, *the isotopies of numbers imply an alteration of the eigenvalues of conventional quantum mechanical operators*. This mechanism, called *mutation* [15], is at the foundation of the capabilities of hadronic mechanics to represent the synthesis of unstable hadrons from ligther hadrons and other applications not possible with conventional eigenvalue equations.

idempotent of arbitrary (finite) order n *as the original unit* 1

$$\hat{1}^{\hat{n}} = \hat{1}\,\hat{\times}\,\hat{1}\,\hat{\times}\,.....\hat{\times}\,\hat{1} = \hat{1} \quad \text{(n times)}, \tag{2.3.24}$$

the isosquare root of the isounit is the isounit itself,

$$\hat{1}^{\hat{\frac{1}{2}}} = \hat{1}, \tag{2.3.25}$$

the isoquotient of the isounit by itself is the isounit,

$$\hat{1}\,\hat{/}\,\hat{1} \equiv \hat{1}, \tag{2.3.26}$$

the *isoinverse of the isounit is the isounit itself,*

$$\hat{1}^{-\hat{1}} = \hat{1}, \tag{2.3.27}$$

etc. This confirms the axiom-preserving character of the lifting $\hbar = 1 \rightarrow \hat{\hbar} = \hat{1}$ when realized via the isotopies.

Note that the above properties hold for the most general possible *integral* representation of $\hat{\hbar} = \hat{1}$. Note also that the number 1 is now no longer the multiplicative unit because $1\hat{\times}a \neq a$ and $1\hat{\times}\hat{a} \neq \hat{a}$.

Recall that the set of purely imaginary numbers $S = \{in\}$, $i = \sqrt{-1}$, n real, *is not* a field, evidently because it is not closed under multiplication, $(in) \times (im) = -nm \not\in S$. However, the isotopy $\hat{F}(\hat{n},+,\hat{\times})$ of real numbers n equipped with the purely imaginary isounit $\hat{1} = i$, and isoproduct $\hat{\times} = \times\hat{T}\times$, $\hat{T} = i^{-1} = -i$, does form indeed an isofield, that is, it verifies all axioms of a field. This illustrates the possibility offered by the isotopies according to which, given a set S of numbers which do not form a field, there may exist an isotopic lifting $\hat{S}$ under which $\hat{S}$ is indeed a field.

Note that, according to Hamilton [3] original conception, *the quaternions constitute a field* because their multiplication is noncommutative, but associative. On the contrary, according to Cayley [4] original conception, *octonions are not generally considered to constitute a field* because their multiplication is not associative, but verifies the weaker *right and left alternative laws*

$$(a\,b)\,b = (a\,b)\,b\ , \qquad (a\,a)\,b = a\,(a\,b)\,. \tag{2.3.28}$$

This is the reason for assuming a more general definition of field in ref. [24] which is based on the above alternative laws and, as such, it includes as "fields" the octonions. Also, in this way all "fields" coincide with all "normed algebras with unit" of Sect. 2.1.

In these volumes we shall follow for simplicity the conventional definition

of fields [8]. Nevertheless, for completeness, we shall consider the isotopy of octonions with the understanding that, according to Definition 2.3.1, they form a weaker form of fields based on the alternative law.

We now pass to the studies of a further new class of numbers, called *isodual isonumbers.* Owing to their importance for these studies as well as for clarity, it is best to present them according to the following separate definition.

Definition 2.3.2 [24]: *Let* $F(a, +, \times)$ *be a conventional field as per Definition 2.3.1. Then, the "isodual field"* $F^d(a^d, +, \times^d)$ *is constituted by elements called "isodual numbers'*

$$a^d = a \times 1^d = -a, \tag{2.3.29}$$

defined with respect to the "isodual multiplication" and related "isodual unit"

$$\times^d = \times\, 1^d \times = -\times, \quad 1^d = -1\,. \tag{2.3.30}$$

Let $\hat{F}(\hat{a}, +, \hat{\times})$ *be an isofield as per Definition 2.3.1. Then, the "isodual isofield"* $\hat{F}^d(\hat{a}^d, +, \hat{\times}^d)$ *is given by "isodual isonumbers"*

$$\hat{a}^d = a^c \times \hat{1}^d = -a^c\,\hat{1}, \tag{2.3.31}$$

where a^c *is the conventional conjugation of* F *(the identity for real numbers, the complex conjugation for complex number and the Hermitean conjugation for quaternions in matrix representation), defined in terms of the "isodual isomultiplication"*

$$\hat{\times}^d = \times\, \hat{1}^d \times = -\hat{\times}, \quad \hat{1}^d = -\hat{1}\,. \tag{2.3.32}$$

Again one can see that the isodual unit 1^d is idempotent of arbitrary degree n, that the isodual square root of 1^d is 1^d and the isodual quotient of 1^d by itself is 1^d, with similar occurrences for $\hat{1}^d$.

The reader has noted our insistence in leaving the conventional sum unchanged, and lifting only of the multiplication. The underlying reason warrants a few comments because, as indicated earlier, any generalization of conventional operations implies a new mechanics. A possible generalization of the operation of addition would therefore imply a further generalization of hadronic mechanics.

In addition to the lifting (2.2.1) of the multiplication, this author also inspected in ref. [21] (see ref. [24] for more technical studies) the following lifting of the addition

$$+ \quad \rightarrow \quad \hat{+} = +\hat{K}+, \quad \hat{K} = K \times \hat{1}, \tag{2.3.33}$$

with consequential redefinition of the conventional additive unit

$$0 \rightarrow \hat{0} = -\hat{K}. \qquad (2.3.33)$$

However, unlike the isotopy of the multiplication $\times \rightarrow \hat{\times}$, the lifting of the addition $+ \rightarrow \hat{+}$ has the following implication:

Proposition 2.3.3 [21,24]: *The liftings*

$$F(a, +, \times) \quad \rightarrow \quad \hat{F}(\hat{a}, \hat{+}, \hat{\times}), \qquad (2.3.35a)$$

$$\hat{a} = a \times \hat{1}, \hat{+} = + \hat{K} +, \hat{0} = - \hat{K} = - K \times \hat{1}, \hat{\times} = \times \hat{T} \times, \hat{1} = \hat{T}^{-1}, \qquad (2.3.35b)$$

where K $\in$ F *and* $\hat{T}$ *is invertible, is not an isotopy for all nontrivial values of the quantity* $K \neq 0$, *because it preserves all axioms of Definition 2.3.1, except the distributive law (2.3.11).*

In fact, all axioms (2.3.1)–(2.3.11) can be easily verified to be preserved under liftings (2.3.34). On the contrary, for the right distributive law we have

$$\hat{a} \hat{\times} (\hat{b} \hat{+} \hat{c}) = a \times (b + K + c) \times \hat{1} = (a \times b + a \times K + a \times c) \times \hat{1} \neq$$

$$\neq \hat{a} \hat{\times} \hat{b} \hat{+} \hat{a} \hat{\times} \hat{c} = (a \times b + K + a \times c) \times \hat{1}, \qquad (2.3.36)$$

with similar lack of identities for the left isodistributive law. Note that the set $\hat{F}$ in lifting (2.3.35) is closed under isoaddition for K $\in$ F (but not for $K \not\in \hat{F}$), and, separately, under isomultiplication for an arbitrary isounit $\hat{1}$ outside the original set F. The same results hold for the lifting $F(a, +, \times) \rightarrow \hat{F}(a, \hat{+}, \hat{\times})$, $\hat{+} = + K +$, K $\in$ F, K $\neq$ 0.

The implications of Proposition 2.3.3 are such to prevent its use in physics. A central notion of quantum mechanics is that of unitary transformations $UU^{\dagger} = U^{\dagger}U = I$, with the exponential representation in terms of a Hermitean operator X and parameter w

$$U = I + i w X / 1! + (i w X)(i w X) / 2! + \dots = e^{i w X} \qquad (2.3.37)$$

As we shall see in Chapter I.4, the isotopy of the multiplication implies a fully consistent isotopic generalization of the above notion which is convergent into a finite form

$$\hat{U} = \hat{1} + i w X / 1! + (i w X) \hat{\times} (i w X) / 2! + \dots = \hat{1} e^{i w T X} \qquad (2.3.38)$$

resulting in this way in the fundamental isotopies of these volumes, those of the time evolution, Lie's transformation groups or linear operators on a Hilbert spaces.

The point is that the isotopies of conventional unitary transformations under the lifting of the addition are divergent,

$$U = I \hat{+} i w X / 1! \hat{+} (i w X)(i w X) / 2! \hat{+} \ldots \rightarrow \pm \infty \qquad (2.3.39)$$

thus precluding the achievement of finite forms of the time evolution and other fundamental physical laws.

A property expressed by Proposition 2.3.3 is that *the lifting of the addition is not an isotopy* because one of the original axioms is not preserved. We shall then use the following notion

Definition 2.3.3 [24]: *An "isotopy" is any lifting of a given mathematical or physical structure preserving the original axioms. A "pseudoisotopies" is a lifting which preserves only part of the original axioms*.

As we shall see, the difficult task is in the identification of which property is a true axiom of a given conventional formulation and which is not. As a matter of facts, the isotopies can help precisely in the identification of true axioms and their separation from other algorithms which do not have a truly essential character.[14]

In App. I.2.C we show that, despite the shortcoming indicated earlier, pseudoisonumbers do have intriguing applications in cryptology for an increased security of the electronic or conventional information.

In this section we have studied the lifting of the multiplication $\times \rightarrow \hat{\times}$ and/or of the addition $+ \rightarrow \hat{+}$ *which do not require ordering*, that is, the action to the right is the same as that to the left (see Ch. 2.7 for the introduction of ordering and a further generalization of isonumbers for the Lie-admissible formulations). This results in the following two groups of generalized fields and related numbers:

1) Isofields $\hat{F}(\hat{a},+,\hat{\times})$, which are characterized by the lifting of the multiplication $\times \rightarrow \hat{\times}$ while keeping the conventional addition to ensure the preservation of the distributive law (2.3.11). They can be classified in the same way as the isounits resulting in:

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[14] As we shall see in Vol. II, the isotopies of the Riemannian geometry show that all familiar properties are indeed true geometric axioms because preserved under isotopies, *except Einstein's tensor* $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R$, which emerges as being "geometrically incomplete", that is, lacking a certain term to be invariant under isotopies.

Isofields properly speaking (Class I),
Isodual isofields (Class II),
Indefinite isofields (Class III),
Singular isofields (Class IV) and
General isofields (Class V).

The isotopic branch of hadronic mechanics is based on the following four fundamental types of numbers [24]:

1.a) Ordinary numbers: *real numbers* $R(n,+,\times)$, *complex numbers* $C(c,+,\times)$, *quaternions* $Q(q,+,\times)$ and *octonions* $O(o,+,\times)$ which will be used for the characterization of *particles in vacuum*;

1.b) Isodual numbers: *isodual real numbers* $R^d(n^d,+,\times^d)$, *isodual complex numbers* $C^d(c^d,+,\times^d)$, *isodual quaternions* $Q^d(q^d,+,\times^d)$, and *isodual octonions* $O^d(o^d,+,\times^d)$ which will be used for the characterization of *antiparticles in vacuum*;

1.c) Isonumbers: *isoreal numbers* $\hat{R}(\hat{n},+,\hat{\times})$, *isocomplex numbers* $\hat{C}(\hat{c},+,\hat{\times})$, *isoquaternions* $\hat{Q}(q,+,\hat{\times})$ and *isooctonions* $\hat{O}(\hat{o},+,.\hat{\times})$ which will be used for the characterization of *particles within physical media*;

1.d) Isodual isonumbers: *isodual isoreal numbers* $\hat{R}^d(\hat{n}^d,+,\hat{\times}^d)$, *isodual isocomplex numbers* $\hat{C}^d(\hat{c}^d,+,\hat{\times})$, *isodual isoquaternions* $\hat{Q}^d(\hat{q}^d,+,\hat{\times}^d)$, and *isodual isooctonions* $\hat{O}^d(\hat{o}^d,+,\hat{\times}^d)$ which will be used for the characterization of *antiparticles within physical media*.

2) Genofields, which is a generalization of isofield with the selection of an ordering of the multiplication to the left or to the right studied in Ch. I.7 and applied for the more general Lie-admissible branch of hadronic mechanics.

3) Pseudoisofields and **4) pseudogenofields,** which are further generalizations based on the lifting of the addition which relaxes at least one axiom of conventional fields, and which do not possess a known significance at this time in classical or operator physics, but which do have indeed applications in other fields (see App. I.2.C).

5) Hypermumbers, which can be constructed via the use of the hyperstructures [32], but are not studied in these volumes for brevity.

The above classification is sufficient to illustrate the rather broad and diversified character of the theory of isonumbers, genonumbers, and hypernumbers as well as the broad character of the mechanics and other formulations build on them, such as geometries, algebras and mechanics.

Except for marginal comments, in the remaining parts of this chapter we shall study the generalized numbers at the foundation of the isotopic, time-reversal invariant branch of hadronic mechanics, which are the isonumbers of types 1.a-1.d above.

2.4: ISONORMED ISOALGEBRAS AND THEIR ISODUALS

A further notion needed for the study of explicit realizations of isonumbers is the applicable definition of algebra. In fact, conventional numbers constitute normed algebras with unit, as recalled in Sect. I.2.1. It is then important to identify the corresponding notion under isotopies.

Definition 2.4.1 [8,14,24]: *Let* U *be a conventional algebra (see, e.g., ref. [8]) with elements* A, B, C *(say, matrices) and (abstract) product* A⊙B *(say, the associative product* AB *or some nonassociative form) over a field* F(a,+,×) *with elements a, b, .. operations* a + b *and* a×b *and related units* 0 *and* 1 *satisfying the basic scalar and distributive laws*

$$(a \times A) \odot B = A \odot (a \times B) = a \times (A \odot B), \tag{2.4.1a}$$

$$(A \times a) \odot B = A \odot (B \times a) = (A \odot B) \times a, \tag{24.1b}$$

$$A \odot (B + C) = A \odot B + A \odot C, (B + C) \odot A = B \odot A + C \odot A. \tag{2.4.1c}$$

The algebra U *is called a "division algebra" when the equation* A×y = B *always admit a solution in* y ∈ F *for* A ≠ 0. *The algebra* U *is said to admit a unit when there is a quantity* I *such that*

$$I \odot A = A \odot I = A, \tag{2.4.2}$$

for all A ∈ U. *Finally, the algebra* U *is "normed" when it admits a norm* | A | *satisfying the basic axioms*

$$|A \odot B| = |A| \times |B|, \quad |n \times A| = |n| \times |A|, \tag{2.4.3}$$

The infinitely possible "isotopic images" Û *of* U, *called "isoalgebras" for short, are given by the original elements* A, B, C, ... *equipped with a new isomultiplication* A$\hat{\odot}$B *over an isofield* F̂(a,+,$\hat{\times}$) *of elements* a, b, c, (without lifting) *with operations* + *and* $\hat{\times} = \times \hat{T} \times$, *and related units* 0 *nd* $\hat{1} = \hat{T}^{-1}$ *under the condition of preserving the original axioms of* U, *i.e., of verifying the following left and right "isoscalar and isodistributive laws"*

$$(a \hat{\times} A) \hat{\odot} B = A \hat{\odot} (a \hat{\times} B) = a * (A \hat{\odot} B), \tag{2.4.4a}$$

$$(A \hat{\times} a)\hat{\circ} B = A \hat{\circ}(B \hat{\times} a) = (A \hat{\circ} B)\hat{\times} a, \qquad (2.4.4b)$$

$$A \hat{\circ}(B + C) = A \hat{\circ} B + A \hat{\circ} C, \qquad (B + C)\hat{\circ} A = B \hat{\circ} A + C \hat{\circ} A, \qquad (2.4.4c)$$

for all elements A, B, C $\in \hat{U}$ *and* a, b, c $\in \hat{F}$. *The isoalgebra* $\hat{U}$ *is called an "isodivision algebra" when the equations* $A \hat{\times} y = B$ *always admit a solution* y *for* $A \neq 0$. *An isoalgebra* $\hat{U}$ *is said to admit an isounit* $\hat{1}$ *when*

$$\hat{1}\hat{\circ} A = A \hat{\circ}\hat{1} = A, \qquad (2.4.5)$$

for all $A \in \hat{U}$. *Finally, the isoalgebra* $\hat{U}$ *is said to be "isonormed", when it admits an isotopic image* $\hat{|}A\hat{|}$ *of* $|A|$ *which verifies the axioms*

$$\hat{|}A \hat{\circ} B\hat{|} = \hat{|}A\hat{|}\hat{\times}\hat{|}B\hat{|} \in \hat{F}, \quad \hat{|}n \hat{\times} A\hat{|} = \hat{|}n\hat{|}\hat{\times}\hat{|}A\hat{|} \in \hat{F} \qquad (2.4.6)$$

The "isodual algebra" U^d *is the image of* U *under the isodual field* $F^d(a^d,+,\times^d)$, *while the "isodual isoalgebra"* $\hat{U}^d$ *is the image of* $\hat{U}$ *under the isodual isofield* $\hat{F}^d(a^d,+,\hat{\times}^d)$.

Note the differentiation, in general, between the isomultiplication $A\hat{\circ}B$ of the elements of the isoalgebras say, matrices, from the isomultiplication of the elements of the isofields $a\hat{\times}b$, which can be ordinary numbers. However, one should keep in mind that, when the elements of $\hat{U}$ and $\hat{F}$ coincide, the two multiplications coincide too, $\hat{\circ} \equiv \hat{\times}$, as it is the case when isonormed algebras are realized in terms of isonumbers (see subsequent sections).

A significant property is that *the units of the algebra and that of the basic field are generally different for conventional algebras (e.g., the number 1 and the unit matrix, respectively), while the isounit of the basic isofield and that of the isoalgebras generally coincide and are given by the same quantity* $\hat{1}$.

A realization of the isonorm is the following. Let $\hat{e}_k$ be an "isobasis" of $\hat{U}$ over the isofield $\hat{F}(a,+,\hat{\times})$ of Proposition 2.3.1, i.e., such that a generic element $A \in \hat{U}$ can be written

$$A = \sum_{k=1,\dots,m} n_k \hat{\times} \hat{e}_k, \qquad n_k \in \hat{F}. \qquad (2.4.7)$$

and $\hat{e}^{\hat{2}} = \sum_k \hat{e}_k \hat{\circ} \hat{e}_k = \hat{1}$. The *isonorm* of $\hat{U}$ in the isobasis considered is then given by

$$\hat{|}A\hat{|} = (\sum_{k=1,\dots,m} n_k^{\hat{2}})^{\hat{1/2}} \times \hat{1} = (\sum_{k=1,\dots,m} n_k \hat{\times} n_k)^{\hat{1/2}} \times \hat{1} \in \hat{F}. \qquad (2.4.8)$$

The extension of the above notions to isofields $\hat{F}(\hat{a},+,\hat{\times})$ of Proposition 2.3.2 is trivial and, as such, it will be ignored.

The isoalgebra $\hat{U}$ is said to be *isoassociative* when it satisfies the (scalar and distributive laws and the) isoassociative law

$$A \hat{\circ} (B \hat{\circ} C) = (A \hat{\circ} B) \hat{\circ} C, \quad \forall A, B, C \in \hat{U}; \tag{2.4.9}$$

and it is said to be *isoalternative* when it verifies the isoalternative laws

$$A^{\hat{2}} \hat{\circ} B = A \hat{\circ} (A \hat{\circ} B), \quad A \hat{\circ} B^{\hat{2}} = (A \hat{\circ} B) \hat{\circ} B. \tag{2.4.10}$$

By recalling that ordinary numbers are associative and that they are alternative only under the inclusion of the octonions, in this chapter we are primarily interested in isoassociative isonormed isoalgebras with isounit $\hat{1}$, with the extension to isoalternative algebras when the inclusion of isooctonions is desired.

As well known, the scalar and distributive laws (2.4.1) are basic axioms for any structure to characterize an "algebra" as commonly understood [7–10]. The images of an algebra U under the isotopies over isofields here considered are then true algebras because they preserves axioms (2.4.1) by central assumption. However, the images of U (and $\hat{U}$) under the pseudoisofield $\hat{F}(a, \hat{+}, \hat{\times})$ of Proposition 2.3.3 (in which the addition is also lifted) implies the loss of the distributive laws and, for this reason, they are no longer algebras as commonly understood. We shall then call them *pseudoisoalgebras* [24].

As we shall see, the isotopies of the operations with numbers require, for mathematical consistency, corresponding compatible isotopies of all other operations on algebras.

A case deserving advance mention because needed in the subsequent sections is the notion of determinant of a (conventional) matrix A which is applicable to an isonormed isoalgebra. The conventional notion is inapplicable under isotopies and must be replaced by the *isodeterminant* [16,21]

$$\text{Isodet } A = [\, \text{Det}_F (A \times T) \,] \times \hat{1}, \tag{2.4.11}$$

where $\text{Det}_F A$ represents the conventional determinant computed in the conventional field F.

In fact, Det A violates the basic axioms under isotopies, e.g.,

$$(\text{Det } A) \hat{\times} (\text{Det } B) \neq \text{Det } AB \text{ and } \neq \text{Det } (A \hat{\circ} B), \ \text{Det } A^{\hat{-1}} \neq (\text{Det } A)^{\hat{-1}}, \text{etc.} \tag{2.4.12}$$

However, $\hat{\text{Det}}\ A$ does preserve the above axioms because

$$\text{Isodet } (A \hat{\circ} B) = (\text{Isodet } A) \hat{\times} (\text{Isodet } B), \ \ \text{Isodet } (A^{\hat{-1}}) = (\text{Isodet } A)^{\hat{-1}}. \tag{2.4.13}$$

The corresponding *isodual isodeterminant* is given by [21,24]

$$\text{Isodet}^d\, A \;=\; [\, \text{Det}_F(\, A \times T^d\,)\,] \times \hat{1}^d \;; \tag{2.4.14}$$

which is now computed in F^d.

An n×n *isomatrix* $\hat{A}$ is the ordinary n×n script in which the elements are isoreal or isocomplex isonumbers $\hat{a} = a \times \hat{1}$. In this case, all products among elements are isotopic, $\hat{a} \hat{\times} \hat{b} = (ab)\hat{1}$, and we have

$$\text{Isodet}\, \hat{A} \;=\; (\,\text{Det}\, A\,) \times \hat{1}\,. \tag{2.4.15}$$

with similar expressions for other properties.

2.5: ISOREAL NUMBERS AND THEIR ISODUALS

By following ref. [24], we shall now study in more details explicit realizations of the isoreal numbers and their isoduals.

2.5.A: Realization of ordinary real numbers. Let us recall for completeness and notational convenience (see, e.g., ref. [7]) that conventional real numbers $n \in R(n,+,\times)$ are realized on the one-dimensional real Euclidean space $E_1(x,\delta,R(n,+,\times))$, which essentially represents a straight line with origin at 0, local coordinates x, metric $\delta = 1$, additive unit 0 and multiplicative unit 1. In fact, the *dilations*

$$y' = n \times y = n\, y, \qquad n \in R(n,+,\times), \qquad y,\, y' \in E_1(x,\delta,R)\;, \tag{2.5.1}$$

characterize an isomorphism of the reals $R(n,+,\times)$ into the commutative one-dimensional *group of dilations* G(1).

The trivial basis is $e = 1$, with norm given by the familiar positive-definite expression

$$|\,n\,| \;=\; (n \times n)^{\frac{1}{2}} \;> 0\,, \tag{2.5.2}$$

verifying axioms (2.4.3),

$$|\,n \times n'\,| \;=\; |\,n\,| \times |\,n'\,|\,. \tag{2.5.3}$$

This shows that *real numbers constitute a one-dimensional normed associative and commutative algebra* U(1) [7].

2.5.B: Realization of isodual real numbers. Isodual real numbers $n^d \in R^d(n^d,+,\times^d)$ are conventional numbers n, although defined with respect to the isodual unit $1^d = -1$. The isodual conjugation for real numbers can then be written

$$n = n \times 1 \quad \rightarrow \quad n^d = n \times 1^d = -n. \tag{2.5.5}$$

Thus, *all numerical values change sign under isoduality.* One should however keep in mind that such a sign inversion occurs only when the isodual real numbers are projected in the field of conventional real numbers.

As a specific example, the negative integer number −3 referred to negative unit −1 is fully equivalent to the positive integer +3 referred to the positive unit +1.

The representation of $R^d(n^d,+,\times^d)$ constitutes the first occurrence in our analysis requiring a generalized notion of space. In fact, the one-dimensional Euclidean space is evidently inapplicable because the underlying field is now the isodual field $R^d(n^d,+,\times^d)$.

The identification of the generalized space applicable under isotopies was first done in ref. [18], as reviewed in details in the next chapter. In the simple case here considered, it is given by the one-dimensional, real, isodual, Euclidean space $E^d(x^d,\delta^d,R^d(n^d,+,\times^d))$, which is also a straight line, although with conventional additive unit 0, isodual multiplicative unit $1^d = -1$, isodual coordinates $x^d = -x$ and isodual metric $\delta^d = -\delta = -1$. The *isodual dilations* are then given by

$$y' = n^d \times^d y = n \times y. \tag{2.5.6}$$

They establish an isomorphism between $R^d(n^d,+,\times^d)$ and the *isodual group of dilations* $G^d(1)$, i.e., the conventional group $G(1)$ reformulated with respect to the multiplicative unit 1^d (see Chapter I.4 for details).

Note that $E_1(x,\delta,R)$ and $E_1{}^d(x,\delta^d,R^d)$ are anti-isomorphic and the same property holds for $G(1)$ and $G^d(1)$. Note also that isodual dilations coincide with the conventional ones, and this could be a reason for the lack of detection of isodual numbers until ref.s [19,20].

The *isodual basis* is

$$e^d = 1^d, \tag{2.5.7}$$

and the *isodual norm* becomes now *negative definite*

$$|n|^d := (n \times n)^{\frac{1}{2}} \times 1^d = |n| \times 1^d = -|n| < 0, \tag{2.5.8}$$

although preserving the basic axioms (2.4.6),

$$|n^d \times^d n'^d|^d = |n^d|^d \times^d |n'^d|^d . \qquad (2.5.9)$$

The above results show that *isodual real numbers constitute a one-dimensional isodual, associative and commutative normed algebra* $U^d(1)$ *which is anti-isomorphic to* $U(1)$ [24].

2.5.C: Realization of isoreal numbers. We consider now the isoreal numbers $\hat{n} = n \times \hat{1}$ as elements of an isofield of Class I, $\hat{R}_I(\hat{n},+,\hat{\times})$ with isomultiplication $\hat{\times} = \times T \times$, and multiplicative isounit $\hat{1} = \hat{T}^{-1} > 0$ generally outside the original set $R(n,+,\times)$, as requested for basic assumption (1.1.1). Their representation requires the lifting of the original Euclidean space into a form compatible with the basic isofield $\hat{R}_I(\hat{n},+,\hat{\times})$, which is given by the *isoeuclidean spaces* [18] of Class I, $\hat{E}_{I,1}(x,\hat{\delta},\hat{R}(\hat{n},+,\hat{\times}))$, with metric $\hat{\delta} = \hat{T}\delta$ over $\hat{R}(\hat{n},+,\hat{\times})$ (see next chapter for details).

One should keep in mind that $\hat{E}_{I,1}(x,\hat{\delta},\hat{R})$ is a simple, yet bona-fide nonlinear, nonlocal and noncanonical generalization of the original space, because the original one dimensional metric $\delta = 1$ is now lifted into the expression

$$\hat{\delta} \;=\; T(t, x, \dot{x}, \ddot{x}, \psi, \psi\dagger, \partial\psi, \partial\psi\dagger, ...)\,\delta\,; \qquad (2.5.10)$$

Thus, the one-dimensional isospace $\hat{E}_{I,1}(x,\hat{\delta},\hat{R})$ represents a generalization of the conventional straight line, here called an *isostraight line*, because of its intrinsically nonlinear, nonlocal and noncanonical metric $\hat{\delta}(t, x, \dot{x}, \ddot{x}, ...)$ with multiplicative isounit $\hat{1} = \hat{1}(t, x, \dot{x}, \ddot{x}, ...)$, yet it preserves the original axioms of the straight line as ensured by the isotopies (see Ch. I.5 for more details on this feature of isogeometries).

$\hat{R}_I(\hat{n},\hat{+},\hat{\times})$ can then be realized via the *isodilations* on $\hat{E}_{I,1}(x,\hat{\delta},\hat{R})$

$$y' \;=\; \hat{n}\,\hat{\times}\,y \;=\; n\,y\,, \qquad (2.5.11)$$

which, again, coincide with the original dilations, as it is the case for the isodual dilations, thus providing a reason for the lack of detection of the isoreal numbers until recently.

Isodilations (2.5.11) characterize an isomorphism of the isoreal numbers with the one-dimensional *group of isodilations* $\hat{G}(1)$, i.e., the group $G(1)$ realized with respect to the isounit $\hat{1}$ (see Ch. I.4 for details). The local isomorphism $E(x,\delta,R(n,+,\times)) \approx \hat{E}_{I,1}(x,\hat{\delta},\hat{R}(\hat{n},+,*)$ holds for all positive-definite isounits (see next chapter) and readily implies $\hat{G}(1) \approx G(1)$.

The *isobasis* is now given by

$$\hat{e} = \hat{1}, \qquad (2.5.12)$$

while the *isonorm* can be defined by

$$\uparrow \hat{n} \uparrow \ = (n \times n)^{\frac{1}{2}} \times \hat{1} \ = \ |n| \times \hat{1} \, , \tag{2.5.13}$$

namely, by the conventional norm, only rescaled to the new unit $\hat{1}$, which is the essence of the transition from real number n to their isotopes $\hat{n} = n \times \hat{1}$.

In particular, axioms (2.4.6) is satisfied,

$$\uparrow \hat{n} \hat{\times} \hat{n}' \uparrow \ = \ \uparrow \hat{n} \uparrow \hat{\times} \uparrow \hat{n}' \uparrow \, , \tag{2.5.14}$$

with the same product inside and out because referred to the same elements. One can see that the *isoreal numbers constitute a one-dimensional, isonormed, isoassociative and isocommutative isoalgebra* $\hat{U}(1) \approx U(1)$ [24].

2.5.D: Realization of isodual isoreal numbers. We consider now the isodual isonumbers of Class II

$$\hat{n}^d = n \times \hat{1}^d \ = \ - \hat{n} \ \in \hat{R}_{II}{}^d(\hat{n}^d,+,\hat{\times}^d) \, . \tag{2.5.15}$$

In this case we need the one-dimensional *isodual isoeuclidean space* of Class II, $\hat{E}_{II,1}{}^d(x^d,\hat{\delta}^d,\hat{R}^d)$, and the *isodual isodilations*

$$y' \ = \ \hat{n}^d \ \hat{\times}^d \ y \ = \ n \, y \, , \tag{2.5.16}$$

which also coincide with the conventional dilations, by characterizing an isomorphism of the isodual isoreal numbers with the one-dimensional *isodual group of isodilations* $\hat{G}^d(1)$, i.e., the image of $G(1)$ under the isodual isounit $\hat{1}^d = -\hat{1}$. The evident underlying isomorphism

$$E_{II,1}{}^d(x^d,\delta^d,R_{II}{}^d(n^d,+,\times^d)) \ \approx \ \hat{E}_{II,1}{}^d(x^d,\hat{\delta}^d,\hat{R}_{II}{}^d \, (\hat{n}^d,+,\hat{\times}^d)) \, , \tag{2.5.17}$$

then implies $\hat{G}^d(1) \approx G^d(1)$. The *isodual isobasis* is now given by

$$\hat{e}^d \ = \ \hat{1}^d \, , \tag{2.5.19}$$

with *isodual isonorm*

$$\uparrow \hat{n}^d \uparrow^d \ = (n \times n)^{\frac{1}{2}} \times \hat{1}^d \ = \ - \uparrow \hat{n} \uparrow < 0 \, , \tag{2.5.20}$$

which is also negative-definite, yet verifying basic axiom (2.4.6),

$$\uparrow \hat{n}^d \hat{\times}^d \hat{n}'^{\,d} \uparrow^d \ = \ \uparrow \hat{n}^d \uparrow^d \ \hat{\times}^d \ \uparrow \hat{n}'^{\,d} \uparrow^d \, , \tag{2.5.21}$$

Thus, *the isodual isoreal numbers are a realization of the one-dimensional isodual, isonormed, isoassociative and isocommutative isoalgebra* $\hat{U}^d(1) \approx U^d(1)$ [24].

The extension of the above results to the case of pseudoisoreal numbers and their isoduals is left to the interested reader.

2.6: ISOCOMPLEX NUMBERS AND THEIR ISODUALS

2.6.A: Realization of ordinary complex numbers. Let us recall for completeness (see, e.g., ref. [7]) that conventional complex numbers

$$c \;=\; n_0 + n_1 \times i \quad \in \quad C(c,+,\times), \quad n_0\,, n_1 \in R(n,+,\times), \tag{2.6.1}$$

where i is the imaginary unit are represented in a *Gauss plane* [1], which is essentially a realization of the two-dimensional Euclidean space $E_2(x,\delta,R(n,+,\times))$ with basic separation

$$x^2 \;=\; x^t\,\delta\,x \;=\; x^i\,\delta_{ij}\,x^j \;=\; x_1{}^2 + x_2{}^2 \;\in\; R(n,+,\times)\,. \tag{2.6.2}$$

Its group of isometries, the one-dimensional *orthogonal group* O(2), is the invariance of the circle (2.6.2), as well known. For this reason, complex number can be represented via the fundamental representation of O(2) (see below).

The correspondence between complex numbers $c = n_0 + n_1{\times}i$ and the Gauss plane with points $P = (x^1, x^2)$ is then made one-to-one by the *dilative rotations*

$$z' \;=\; (x^1 + x^2 \times i\,)' \;=\; c \odot z \;=\; (\,n_0 + n_1 \times i\,) \odot (\,x^1 + x^2 \times i\,)\,, \tag{2.6.3}$$

with multiplication rule

$$c \odot z \;=\; (n_0\,, n_1\,) \odot (x^1\,, x^2\,) \;=\; (\,n_0 \times x^1 \;-\; n_1\,x^2\,, n_0\,x^2 \;+\; n_1\,x^1\,)\;, \tag{2.6.4}$$

which is known to preserve all properties to characterize a field, thus establishing a one-to-one correspondence between complex numbers and points in the Gauss plane. Transformations (2.6.3) form a two-dimensional group of dilations G(2) in one to one correspondence with $C(c,+,\times)$.

Complex numbers also admit the matrix representation

$$c \;:=\; n_0 \times I_o \;+\; n_1 \times i_1 \;=\; \begin{pmatrix} n_0 & n_1{\times}i \\ n_1{\times}i & n_0 \end{pmatrix} \tag{2.6.5a}$$

$$1_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{2.6.5b}$$

which are the identity and fundamental representation of O(2), respectively, as expected.

The *norm* is then given by the familiar expression

$$|c| = |n_0 + n_1 \times i| = (\text{Det } c)^{\frac{1}{2}} = (\bar{c} \times c)^{\frac{1}{2}} = (n_0^2 + n_1^2)^{\frac{1}{2}} > 0, \tag{2.6.6}$$

which readily verifies axioms (2.4.3)

$$|c \odot c'| = |c| \times |c'| \in R, \quad c, c' \in C, \tag{2.6.7}$$

where now we have different products because referred to different elements.

Finally, the identification of the basis in terms of matrices (2.6.5b)

$$e_1 = 1_0, \quad e_2 = i_1, \tag{2.6.8}$$

implies the well known result that *complex numbers constitute a two-dimensional, normed, associative and commutative algebra* U(2) [7].

2.6.B: Realization of isodual complex numbers. We now consider the isodual complex numbers from Definition 2.3.2 [24]

$$C^d = \{ (c^d, +, \times^d) \mid \times^d = -\times;\ 1^d = -1;\ c^d = \bar{c} \times 1^d = -\bar{c},\ \bar{c} \in \bar{C} \}, \tag{2.6.9}$$

where $\bar{c}$ is the usual complex conjugation. Thus, given a complex number $c = n_0 + n_1 \times i$, its isodual is given by

$$c^d = -\bar{c} = n_0^d + n_1^d \times \bar{i} = -n_0 - n_1 \times \bar{i} = -n_0 + n_1 \times i \in C^d. \tag{2.6.10}$$

In this case we need the two-dimensional *isodual Euclidean space* $E_2^d(x^d, \delta^d, R^d(n^d, +, \times^d))$ with basic invariant

$$\begin{aligned} (x^d)^{2d} &= (x^d)^t \delta^d x^d = x^i \delta^d_{ij} x^j = (x_1^d)^{2d} + (x_2^d)^{2d} = \\ &= x_1^d \times^d x_1^d + x_2^d \times^d x_2^d = -x_1^2 - x_2^2 \in R^d(n^d, +, \times^d) \end{aligned} \tag{2.6.11}$$

whose group of isometries is the one-dimensional *isodual orthogonal group* $O^d(2)$ first proposed by this author in ref. [20], i.e., the image of O(2) under the lifting $I = \text{diag.}(1,1) \rightarrow I^d = \text{diag.}(-1, -1)$ (see Ch. I.4 for details). We then expect isodual complex numbers to be characterized by the representation of $\hat{O}^d(2)$.

diag. (1,1) → I^d = diag. (−1, −1) (see Ch. I.4 for details). We then expect isodual complex numbers to be characterized by the representation of $\hat{O}^d(2)$.

We now introduce the *isodual Gauss plane* [21] as the image of the conventional plane under isoduality. The correspondence between isodual complex numbers and the isodual Gauss plane with points $P^d = (x^{1d}, x^{2d})$ is then made one-to-one by the *isodual dilative rotations*

$$z^{d\,\prime} = (-x^1 + x^2 \times i)' = c^d \odot^d z^d = (n_0{}^d + n_1{}^d \times^d i^d) \odot^d (x^{1d} + x^{2d} \times^d i^d), \quad (2.6.12)$$

with multiplication rules

$$c^d \odot^d z^d = (-n_0, n_1) \odot^d (x^{1d}, x^{2d}) =$$
$$= (-n_0 x^1 + n_1 x^2, -n_0 x^2 + n_1 x^1), \quad (2.6.13)$$

which can be easily shown to preserve all properties to characterize a field. Also isodual transformations (2.6.13) form an isodual group $G^d(2)$ antiisomorphic to G(2). We therefore see that, as expected, *the one-to-one correspondence between complex numbers and the Gauss plane persists under isoduality*.

Isodual complex numbers also admit the matrix representation

$$c^d = n_0{}^d \times 1_0{}^d + n_1{}^d \times i_1{}^d = \begin{pmatrix} -n_0 & n_1 \times i \\ n_1 \times i & -n_0 \end{pmatrix} \quad (2.6.14a)$$

$$1_0{}^d = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i_1{}^d = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad (2.6.14b)$$

which are the isodual unit and isodual representations of $O^d(2)$, respectively.

Note that in the above representation we have used the property $i^d = \bar{i} \times 1^d = (-i) \times (-1) \equiv i$ according to which *the imaginary unit is isoselfdual*, i.e., invariant under isoduality.

The *isodual norm* is now given by

$$|c^d|^d := [\det_R(c^d \times T^d)]^{\frac{1}{2}} \times 1_0{}^d = (\bar{c}^d \times c^d)^{\frac{1}{2}} \times 1_0{}^d, \quad (2.6.15)$$

can be written

$$|c^d|^d = (c \times \bar{c})^{\frac{1}{2}} \times 1_0{}^d = (n_0{}^2 + n_1{}^2)^{\frac{1}{2}} \times 1_0{}^d, \quad (2.6.16).$$

and also verifies axioms (2.4.6),

$$|c^d \odot^d c'^d|^d = |c^d|^d \times^d |c'^d|^d \in R^d, \quad c^d, c'^d \in C^d. \quad (2.6.17)$$

Finally, the identification of the *isodual basis* in terms of matrices (2.6.14)

$$e_1{}^d = 1_0{}^d, \quad e_2{}^d = i_1{}^d, \tag{2.6.18}$$

implies that *isodual complex numbers constitute a two-dimensional, isodual, normed, associative and commutative algebra* $U^d(2)$ *which is anti-isomorphic to* U(2) [24]

2.6.B: Realization of isocomplex numbers. By following again ref. [24], we consider now the isofield of isocomplex numbers from Definition 2.3.1

$$\hat{C} = \{ (\hat{c},\hat{+},\hat{\times}) \mid \hat{\times} = \times T \times, \hat{1} = T^{-1}, \hat{c} = c \times \hat{1}, c \in C(c,+,\times) \}, \tag{2.6.19}$$

with generic element $\hat{c} = \hat{n}_0 + \hat{n}_1 \times i$. In this case we need the two-dimensional isoeuclidean space of Class I, $\hat{E}_{I,2}(\hat{x},\hat{\delta},\hat{R}(\hat{n},+,\hat{\times}))$, where $\hat{\delta} = \hat{T}\delta = (\hat{\delta}_{ij})$, $\hat{x}^k = x^k$, $\hat{x}_k = \hat{\delta}_{ki}\hat{x}^i \neq x_k$, i, j, k = 1, 2, where the isounit $\hat{1}$ is the same for both the isofield and the isospace.

The realization most used in the physical literature is that with diagonalized and positive-definite isotopic element and isounit as discussed in more details in the next chapter

$$\hat{T} = \text{diag.}\,(b_1{}^2, b_2{}^2), \quad \hat{1} = \text{diag.}\,(b_1{}^{-2}, b_2{}^{-2}), \quad b_k > 0, \ k = 1, 2, \tag{2.6.20}$$

with basic isoseparation

$$\hat{x}^2 = (\hat{x}^t \hat{\delta} \hat{x})\hat{1} = (x^i \hat{\delta}_{ij} x^j) \times \hat{1} = (x^1 b_1{}^2 x^1 + x^2 b_2{}^2 x^2) \times \hat{1} \in \hat{R}(\hat{n},+,\hat{\times}), \tag{2.6.21}$$

whose group of isometries is the one-dimensional *isoorthogonal group* $\hat{O}(2) \approx O(2)$ (see Ch. I.4 for details), i.e., the group O(2) constructed with respect to the multiplicative isounit $\hat{1} = \text{diag.}\,(b_1{}^{-2}, b_2{}^{-2})$, which provides the invariance of all possible ellipses with semiaxes $a = b_1{}^{-2}$, $b = b_2{}^{-2}$ as the infinitely possible deformation of the circle [20]. We then expect that isocomplex numbers are characterizable via the fundamental isorepresentation of $\hat{O}(2)$.

We now study the *isogauss plane*, first introduced by this author in ref. [21], which is the set of points $P = (\hat{x}^1, \hat{x}^2)$ on $\hat{E}_{I,2}(\hat{x},\hat{\delta},\hat{R}(\hat{n},+,\hat{\times}))$ for the characterization of isocomplex numbers $\hat{c} = (\hat{n}_0, \hat{n}_1)$.

The correspondence between the isocomplex numbers $\hat{C}(\hat{c},+,\hat{\times})$ and the isogauss plane can be made one-to-one by the *isodilative isorotations*

$$\hat{z}' = (\hat{x}^1 + \hat{x}^2 \times i)' = \hat{c} \,\hat{\circ}\, \hat{z} \tag{2.6.22}$$

with isomultiplication rule

$$\hat{c} \,\hat{\circ}\, \hat{z} = (\hat{n}_0, \hat{n}_1) \,\hat{\circ}\, (\hat{x}^1, \hat{x}^2) =$$

$$= \{ [(n_0 x^1) \hat{1} - \Delta^{\frac{1}{2}} (n_1 x^2) \times \hat{1}], [(n_0 \; x^2) \hat{1} + (n_1 x^1) \times \hat{1}] \}, \qquad (2.6.23a)$$

$$\Delta = \mathrm{Det}\, \hat{T} = b_1^2 b_2^2, \qquad (2.6.23b)$$

where the appearance of the $\Delta^{\frac{1}{2}}$ factor will be justified shortly, and confirmed later on for the case of isoquaternions and isooctonions studied in Appendices I.4.A and I.4.B.

It is easy to see that the isogauss plane preserves all axioms to characterize an isofield. In particular, isotransformations (2.6.22) form a two-dimensional *isodilation isogroup* $\hat{G}(2) \approx G(2)$. As expected, the one-to-one correspondence between complex numbers and points in the Gauss plane is preserved under isotopy.[15]

The implications are however nontrivial, as illustrated by a number of properties, such as the *lack* of existence of unitary transformations

$$c' = U \odot c \odot \hat{U}, \; U \odot U^{\dagger} = U^{\dagger} \odot U = I = \mathrm{diag.}\,(1, 1), \qquad (2.6.24)$$

mapping the matrix representation of complex numbers into their isotopic form. The understanding is that a transformation does indeed exist, but it is of the more general isotopic type

$$\hat{c} = \hat{U} \hat{\odot} c \hat{\odot} \hat{U}^{\dagger}, \qquad \hat{U} \hat{\odot} \hat{U}^{\dagger} = \hat{U}^{\dagger} \hat{\odot} \hat{U} = \hat{1}. \qquad (2.6.25)$$

Another way to see the nontriviality of the isotopy is by noting that *the conventional trigonometry is inapplicable to the isogauss plane.* In fact, conventional functions such as cos α, sen α, etc. which are well defined in the Gauss plane, have no mathematical or physical meaning in our isogauss plane, as discussed in Appendix I.2.D. The reader should be aware that, by no means, realization (2.6.23) is unique, owing to the intriguing "degrees of freedom" of the isotopic formulations studies later on.

Isocomplex numbers also admit the following two-by-two matrix representation

$$\hat{c} = \hat{n}_0 \times \hat{1}_0 \hat{+} n_1 \times \hat{1}_1 = \begin{pmatrix} n_0 \times b_1^{-2} & i \times n_1 \times b_1^2 \times \Delta^{-\frac{1}{2}} \\ i \times n_1 \times b_2^2 \times \Delta^{-\frac{1}{2}} & n_0 \times b_2^{-2} \end{pmatrix} \qquad (2.6.26a)$$

$$\hat{1} = \hat{1}_0 = \begin{pmatrix} b_1^{-2} & 0 \\ 0 & b_2^{-2} \end{pmatrix}, \; \hat{1}_1 = i\,\hat{1} = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & i \times b_1^2 \\ i \times b_1^2 & 0 \end{pmatrix}, \qquad (2.6.26b)$$

[15] Note that the notion of *point* in the isoeuclidean plane can be introduced despite its nonlocal-integral character thanks to its integro-differential topology (Fig. I.4.1). In fact, the isogauss plane is everywhere local-differential except at the isounit.

$$\Delta = \mathrm{Det.}\,\hat{T} = b_1{}^2 b_2{}^2 , \qquad (2.6.26c)$$

which satisfy rule (2.6.23) and characterize the isounit and the fundamental isorepresentation of $\hat{O}(2)$, respectively (see Ch. I.4, and subsequent confirmation via the fundamental isorepresentation of the isotopic $S\hat{U}(2)$ group for the isoquaternions and isooctonions).

Then, the set $\hat{S}(\hat{c},\hat{+},\hat{\times})$ of matrices (2.6.26a) is closed under addition and isomultiplication, while each element possesses the isoinverse

$$\hat{c}^{-\hat{1}} = c^{-1} \times \hat{1} , \qquad (2.6.27)$$

where c^{-1} is the conventional inverse. Thus, $\hat{S}(\hat{c},+,\hat{\times})$ is an isofield. The local isomorphism $\hat{S}(\hat{c},+,\hat{\times}) \approx \hat{C}(\hat{c},+,\hat{\times})$ is then consequential.

The *isonorm* is defined, from Eq.s (2.4.7) and (2.4.10) by

$$\hat{\uparrow} \hat{c} \hat{\uparrow} = [\, \mathrm{Det}_R(\hat{c} \times \hat{T}) \,]^{\frac{1}{2}} \times \hat{1}_0 = (n_0{}^2 + \Delta\, n_1{}^2)^{\frac{1}{2}} \times \hat{1}_0 , \qquad (2.6.28)$$

and satisfies axioms (2.4.6),

$$\hat{\uparrow} \hat{c} \,\hat{\circ}\, \hat{c}' \hat{\uparrow} = \hat{\uparrow} \hat{c} \hat{\uparrow} \hat{\times} \hat{\uparrow} \hat{c}' \hat{\uparrow} \in \hat{R}, \quad \hat{c}, \hat{c}' \in \hat{C} . \qquad (2.6.29)$$

Finally, the *isobasis*

$$\hat{e}_1 = \hat{1}_0 , \quad \hat{e}_2 = \hat{i} , \qquad (2.6.30)$$

shows that *isocomplex numbers constitute a two-dimensional, isonormed, isoassociative and isocommutative isoalgebras over the isoreals* $\hat{U}(2) \approx U(2)$, a result first achieved in ref. [24].

2.6.C: Realization of isodual isocomplex numbers. We consider now the isodual isocomplex numbers

$$\hat{C}^d = \{\, (\hat{c}^d,+,\hat{*}^d) \mid \hat{c}^d = - \bar{c}\, \hat{1}^d, \quad \hat{*}^d = \times\, T^d \times,\ T^d = - T,\ \hat{1}^d = T^{d\ -1},\ c \in C(c,+,\times) \,\} , \qquad (2.6.31)$$

with generic element

$$\hat{c}^d = \hat{n}^d + \hat{n}_1{}^d \hat{\times}^d \hat{i}^d = - \hat{n}_0 + \hat{n}_1\, i . \qquad (2.6.32)$$

where we have again used the isoselfduality of the imaginary unit, $i^d = i$. In this case we need the two-dimensional *isodual isoeuclidean space* of Class II, $\hat{E}_{II,2}{}^d(\hat{x}^d,\hat{\delta}^d,\hat{R}^d(n^d,+,\hat{\times}^d))$ with realization

$$\hat{T}^d = \mathrm{diag.}\,(-b_1{}^2 , -b_2{}^2) , \quad \hat{1}^d = \mathrm{diag.}\,(-b_1{}^{-2} , -b_2{}^{-2}) , \quad b_k > 0, \quad k = 1, 2, \qquad (2.6.33)$$

and basic isodual isoseparation

$$(\hat{x}^d)^{\hat{2}d} = [(\hat{x}^d)^t \hat{\delta}^d x^d) \times \hat{1}^d = (x^i \hat{\delta}^d_{ij} x^j) \times \hat{1}^d =$$

$$= (-x^1 b_1^2 x^1 - x^2 b_2^2 x^2) \hat{1}^d \in \hat{R}^d(\hat{n}^d,+,*^d), \qquad (2.6.35)$$

whose group of isometries is the *isodual isoorthogonal group* $\hat{O}^d(2) \approx O^d(2)$ [20].

The *isodual isogauss plane* (identified for the first time in ref. [21]) is then the set of points $P = (\hat{x}^{1d}, \hat{x}^{2d})$ on $\hat{E}_{I,2}{}^d(x^d,\delta^d,\hat{R}^d(\hat{n}^d,+,\hat{x}^d))$ for the characterization of isodual isocomplex numbers $\hat{c}^d = (-\hat{n}_0, \hat{n}_1)$.

The correspondence between the isodual isocomplex numbers $\hat{C}^d(\hat{c}^d,+,\times^d)$ and the isodual isogauss plane can be made one-to-one by the *isodual isodilative isorotations*

$$\hat{z}^{d\prime} = (\hat{x}^d + \hat{x}^d \times^d ii)' = \hat{c}^d \hat{\circ}^d \hat{z}^d, \qquad (2.6.36)$$

with multiplication rule

$$\hat{c}\,\hat{\circ}^d \hat{z}^d = (\hat{n}_0, \hat{n}_1)\,\hat{\circ}^d (\hat{x}^d, \hat{x}^d) =$$

$$= \{[(-n_0 x^1)\,\hat{1} + \Delta^{\frac{1}{2}}(n_1 x^2) \times \hat{1}], [(-n_0 \times x^2)\hat{1} + (n_1 x^1) \times \hat{1}]\}, \qquad (2.6.37a)$$

$$\Delta = \mathrm{Det}\,\hat{T} = b_1^2 b_2^2, \qquad (2.6.37b)$$

It is easy to see that the isodual isogauss plane preserves all axioms to characterize an isodual isofield. Also, isodual isotransformations (2.6.36) form an *isodual isodilation isogroup* $\hat{G}^d(2) \approx G^d(2)$. As expected, *the one-to-one correspondence between complex numbers and Gauss plane is also preserved under isodual isotopy.*

Isodual isocomplex numbers also admit the two-by-two matrix representation

$$\hat{c}^d = \hat{n}_0{}^d \times^d \hat{1}_0{}^d + n_1{}^d \times^d \hat{1}^d = \begin{pmatrix} -n_0 b_1^{-2} & i\, n_1 b_1^2 \Delta^{-\frac{1}{2}} \\ i\, n_1\, b_2^2 \Delta^{-\frac{1}{2}} & -n_0 b_2^{-2} \end{pmatrix} \qquad (2.6.38a)$$

$$\hat{1}_0{}^d = \begin{pmatrix} -b_1^{-2} & 0 \\ 0 & -b_2^{-2} \end{pmatrix}, \quad \hat{1}^d = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & i\, b_1^2 \\ i\, b_2^2 & 0 \end{pmatrix}, \qquad (2.6.38b)$$

which satisfy isomultiplication rule (2.6.37), and characterize the isodual isounit and fundamental representation of $\hat{O}^d(2)$, respectively.

Then, the set $\hat{S}^d(\hat{c}^d,+,\hat{\times}^d)$ of matrices (2.6.38) is closed under addition and isomultiplication, each element possesses the isodual isoinverse

$$\hat{c}^{d-\hat{1}d} = (\hat{c}^d)^{-1} \times 1^d . \tag{2.6.39}$$

Thus $\hat{S}^d(\hat{c}^d,+\times^d)$ is an isofield. The local isomorphism $\hat{S}^d(\hat{c}^d,+,\hat{\times}^d) \approx \hat{C}^d(\hat{c}^d,+,\times^d)$ is then consequential.

The *isodual isonorm* is defined by

$$\uparrow \hat{c}^d \uparrow^d = [\, \mathrm{Det}_R(\hat{c}^d \hat{T}^d)\,]^{\frac{1}{2}} \times \hat{1}_0{}^d = (\, n_0{}^2 + \Delta\, n_1{}^2)^{\frac{1}{2}} \times \hat{1}_0{}^d \tag{2.6.40}$$

and satisfies axioms (2.4.6),

$$\uparrow \hat{c}^d \,\hat{\odot}^d\, \hat{c}'^{\,d} \uparrow^d = \uparrow \hat{c}^d \uparrow^d \hat{\times}^d \uparrow \hat{c}'^{\,d} \uparrow^d \in \hat{R}^d, \quad \hat{c}^d, \hat{c}'^d \in \hat{C}^d . \tag{2.6.41}$$

Finally, the *isodual isobasis*

$$\hat{e}_1{}^d = \hat{1}_0{}^d, \quad \hat{e}_2{}^d = \hat{1}_1{}^d, \tag{2.6.42}$$

shows that *isodual isocomplex numbers constitute a two-dimensional, isodual, isonormed, isoassociative and isocommutative isoalgebras over the isodual isoreals isoreals* $\hat{O}^d(2) \approx U^d(2)$ (a result first proved in ref. [24]).

In conclusion, *the "numbers" used in hadronic mechanics are characterized by the lifting of conventional real numbers* n *or complex numbers* c *into the most general known integro-differential expressions* $\hat{n} = n \times \hat{1}$ *and* $\hat{c} = c \times \hat{1}$, *respectively, with an integral dependence on all possible local quantities and their derivatives*

$$n \rightarrow \hat{n} = \hat{n}(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, \dots), \tag{2.6.43a}$$

$$c \rightarrow \hat{c} = \hat{c}(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, \dots), \tag{2.6.43b}$$

as a direct way to represent integro-differential generalizations of Planck's unit, Eq. (1.1.1).

Moreover, the generalizations are nontrivial inasmuch as they are not unitarily equivalent to the conventional numbers. We finally note that, even under the condition

$$\Delta = b_1{}^2\, b_2{}^2 = 1, \tag{2.6.44}$$

realized for

$$b_1 = b_2{}^{-1} = \lambda, \tag{2.6.45}$$

isocomplex numbers preserve their nontrivially generalized form

$$\hat{c} = \hat{n}_0 \times \hat{1}_0 \hat{+} n_1 \times \hat{1}_1 = \begin{pmatrix} n_0 \times \lambda^{-2} & i \times n_1 \times \lambda^2 \\ i \times n_1 \times \lambda^{-2} & n_0 \times \lambda^2 \end{pmatrix} \tag{2.6.46}$$

because the "hidden quantity" $\lambda \neq 0$ has an unrestricted functional dependence, $\lambda(x, \dot{x}, \ddot{x}, ...)$. As we shall see in Vol.s II and III, a number of intriguing physical applications originate precisely from the above "hidden degree of freedom" λ.

2.7: ISOQUATERNIONS AND THEIR ISODUALS

2.7.A: Realization of quaternions. Recall (see, e.g., ref.s [7,8] and quoted literature) that quaternions $q \in Q(q,+,\times)$ admit a realization in the complex Hermitean Euclidean plane $E_2(z,\delta,C)$ with separation

$$E_2(z,\delta,C): \quad z^\dagger z = \bar{z}^i \delta_{ij} z^j = \bar{z}^1 z^1 + \bar{z}^2 z^2, \quad \delta^\dagger \equiv \delta , \tag{2.7.1}$$

whose basic (unimodular) invariant is SU(2). Thus, quaternions can be characterizable via the fundamental (adjoint) representation of SU[2), i.e., by Pauli's matrices, as reviewed below.

Quaternions can be first realized via pairs of complex numbers, $q = (c_1, c_2)$, $q \in Q$ and c_1, $c_2 \in C$ with multiplication $\odot$ (see below). A *Hermitean dilative rotation* on $E_2(z,\delta,C)$, i.e., one leaving invariant $z^\dagger z$, is given by

$$z'^1 = c_1 \odot z^1 + c_2 \odot z^2 , \qquad z'^2 = -\bar{c}_2 \odot z^1 + \bar{c}_1 \odot z^2, \tag{2.7.2}$$

where the dilation is represented by the value $\bar{c}_1 \odot c_1 + \bar{c}_2 \odot c_2 \neq 1$. Again, transformations (2.7.2) form a group G(4), this time associative but noncommutative, which is in one-to-one correspondence with quaternions.

Rule (2.7.2) characterizes the following matrix representation of quaternions $Q(q,+,\times)$ over the field of complex numbers $C(c,+,\times)$

$$q = \begin{pmatrix} c_1 & c_2 \\ -\bar{c}_2 & \bar{c}_1 \end{pmatrix} \tag{2.7.3}$$

which is also one-to-one. By assuming

$$c_1 = n_0 + n_3 \times i , \quad c_2 = n_1 + n_2 \times i, \tag{2.7.4}$$

matrix (2.7.3) admits the representation

$$q = n_o \times 1_o + n_1 \times i_1 + n_2 \times i_2 + n_3 \times i_3, \qquad (2.7.5)$$

where the i's are the celebrated two-dimensional *Pauli's matrices* plus the two-dimensional identity,

$$1_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, i_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ i_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad (2.7.6)$$

with fundamental properties

$$i_n \times i_m = -\epsilon_{nmk} \times i_k, \quad n \neq m, \qquad n, m = 1, 2, 3, \qquad (2.7.7)$$

where ϵ_{nmk} is the conventional totally antisymmetric tensor of rank three. The algebra A of Pauli's matrices is closed under commutators, and characterize the fundamental representation of the **su(2)** Lie algebra

$$[\, i_n, i_m \,] = i_n \times i_m - i_m \times i_n = -2 \times \epsilon_{nmk} \times i_k, \qquad (2.7.8)$$

with Casimir invariants 1_o and $i^2 = \sum_{k=1,2,3} i_k^{\,2}$,

$$[1_o, i_k] = [\, i^2, i_k \,] = 0, \quad k = 1, 2, 3, \qquad (2.7.9)$$

and eigenvalues on a two-dimensional basis ψ with normalization $\psi\dagger\times\psi = 1$

$$\sum_{k=1,2,3} i_k^{\,2} \times \psi = \sum_{k=1,2,3} i_k \times i_k \times \psi = -3 \times \psi. \qquad (2.7.10)$$

By noting that

$$q\dagger = n_o \times 1_o - n_1 \times i_1 - n_2 \times i_2 - n_3 \times i_3, \qquad (2.7.11)$$

the *norm* of q can be written

$$|\, q \,| = (\, q\dagger q \,)^{\frac{1}{2}} = (\, \sum_{k=0,1,2,3} n_k^{\,2} \,)^{\frac{1}{2}}, \qquad (2.7.12)$$

and also satisfies axioms (2.4.3),

$$|\, q \odot q' \,| = |\, q \,| \times |\, q \,| \in R, \quad q, q' \in Q. \qquad (2.7.13)$$

The *basis*

$$e_1 = 1_0, \qquad e_{k+1} = i_k, \quad k = 1, 2, 3, \qquad (2.7.14)$$

then establishes that *quaternions constitute a normed, associative,*

noncommutative algebra of dimensions 4 over the reals U(4) [7,8].

2.7.B: Realization of the isodual quaternions. We consider now the isodual quaternions $q^d \in Q^d\,(q^d,+,\times^d)$ [24] which can be represented via the isodual complex Hermitean Euclidean space

$$E_2^d(z^d,\delta^d,C^d(c^d,+,\times^d)):\ (\bar{z}^{id}\,\delta^d_{ij}\,z^{jd})\times 1^d = (-\bar{z}^1 z^1 - \bar{z}^2 z^2)\times 1^d \in R^d, \quad (2.7.15)$$

in which case they can be realized via pairs of isodual complex numbers (Sect. I.2.6) $q^d = (c_1{}^d, c_2{}^d)$, $q^d \in Q^d$, $c^d{}_1$, $c^d{}_2 \in C^d$. An *isodual Hermitean dilative rotation* on $E_2^d(z^d,\delta^d,C^d(c^d,+,\times^d))$, i.e., one leaving invariant $z^d{\dagger}\delta^d z^d$, is given by

$$z'^{1d} = c_1{}^d \odot^d z^{1d} - \bar{c}_2{}^d \odot^d z^{2d}, \qquad z'^{2d} = c_2{}^d \odot^d z^{1d} + \bar{c}_1{}^d \odot^d z^{2d}, \quad (2.7.16)$$

where the dilation is represented by the value $\bar{c}_1{}^d\odot^d c_1{}^d + \bar{c}_2{}^d \odot^d c_2{}^d \neq -1$. Again, transformations (2.7.16) form an associative but noncommutative isodual group $G^d(4)$, which is in one-to-one correspondence with isodual quaternions $Q^d(q^d,+,\times^d)$.

Rule (2.7.16) characterizes the following matrix representation of isodual quaternions over the field of isodual complex numbers $C^d\,(c^d,+,\times^d)$

$$q^d = \begin{pmatrix} c_1{}^d & -\bar{c}_2{}^d \\ c_2{}^d & \bar{c}^d \end{pmatrix} \quad (2.7.17)$$

By assuming

$$c_1{}^d = -n_0 + n_3 \times i, \quad c_2{}^d = -n_1 + n_2 \times i, \quad (2.7.18)$$

and by recalling that $-\bar{c}^d = c$, $i^d = i$, we have the representation

$$q^d = n_0{}^d \times^d 1_0{}^d + n_1{}^d \times^d i_1{}^d + n_2{}^d \times^d i_2{}^d + n_3{}^d \times^d i_3{}^d =$$
$$= -n_0 \times 1_0 + n_1 \times i_1 + n_2 \times i_2 + n_3 \times i_3, \quad (2.7.19)$$

where the i's are the Pauli's matrices reviewed above. We learn in this way that the Pauli's matrices change sign under isoduality although their product with isodual numbers is isoselfdual.

By using the results of Sect. 2.4, the *isodual norm* is then defined by

$$|q^d|^d = [\,\mathrm{Det}_C\,(q^d \times T^d)\,] \times 1^d = \{-\textstyle\sum_{k=0,1,2,3} n_k{}^2\}^{\frac{1}{2}} \times 1^d, \quad (2.7.20)$$

with property

$$|q^d \odot^d q'^d|^d = |q^d|^d \times^d |q'^d|^d \in R^d, \quad q^d, q'^d \in Q^d. \quad (2.7.21)$$

The use of the *isodual basis*

$$e^d{}_1 = 1^d{}_0, \quad e^d{}_{k+1} = i_k, \; k = 1, 2, 3, \tag{2.7.22}$$

then shows that *isodual quaternions constitute an isodual four-dimensional, normed, associative and noncommutative algebra over the isodual reals* $U^d(4)$, *which is anti-isomorphic to* $U(4)$ [24]

2.7.C: Realization of isoquaternions. To study the isoquaternions $\hat{q} \in \hat{Q}(\hat{q},+,\hat{\times})$ [24], we need the *two-dimensional, complex Hermitean isoeuclidean space* of Class I, $\hat{E}_{I,2}(\hat{z},\hat{\delta},\hat{C})$, $\hat{z}^k = z^k$, $\hat{z}_k = \hat{\delta}_{ki}\hat{z}^i$, $\hat{\delta} = \hat{T}\delta = (\hat{\delta}_{ij})$ on the isofield $\hat{C}(\hat{c},+,\hat{\times})$ with (real) separation (see next chapter for more details)

$$\hat{z}^\dagger \hat{\delta}\, \hat{z} = \bar{z}^i \hat{\delta}_{ij} z^j = \bar{z}^1 b_1{}^2 z^1 \hat{+} \bar{z}^2 b_2{}^2 z^2, \quad \hat{\delta}^\dagger \equiv \hat{\delta} > 0, \tag{2.7.23}$$

basic isotopic element and isounit

$$\hat{T} = \text{Diag.}(b_1{}^2, b_2{}^2), \qquad \hat{1} = \text{Diag.}(b_1{}^{-2}, b_2{}^{-2}), \; b_k > 0, \tag{2.7.24}$$

whose (unimodular) invariance group is the Lie-isotopic group $S\hat{U}(2)$ (see Ch. I.4 and II.6). Isoquaternions can therefore be characterized via the fundamental isorepresentation of the isotopic $\mathbf{s\hat{u}(2)}$ algebra.

A *Hermitean isodilative isorotation* on $\hat{E}_{I,2}(\hat{z},\hat{\delta},\hat{C}(\hat{c},+,\hat{\times}))$ is given by

$$\hat{z}'^1 = \hat{c}_1 \hat{\circ} \hat{z}^1 + \hat{c}_2 \hat{\circ} \hat{z}^2, \qquad \hat{z}'^2 = -\hat{c}_2 \hat{\circ} \hat{z}^1 + \bar{c}_1 \hat{\circ} \hat{z}^2, \tag{2.7.25}$$

where the dilation is represented by the value $\bar{c}_1 \hat{\circ} \hat{c}_1 + \bar{c}_2 \hat{\circ} \hat{c}_2 \neq \hat{1}$.

The map of isoquaternions into two-by-two matrices on $\hat{C}(\hat{c},\hat{+},\hat{\times})$ must now be characterized by the isorepresentations of the Lie-isotopic algebra $S\hat{U}(2)$ first studied in ref.s [21, 25] (see also ref. [27]), which can be expressed in terms of the basic isounit

$$\hat{1} = \hat{1}_0 = \begin{pmatrix} b_1{}^{-2} & 0 \\ 0 & b_2{}^{-2} \end{pmatrix} \tag{2.7.26}$$

and the *fundamental isorepresentation of* $\mathbf{s\hat{u}(2)}$

$$\hat{1}_1 = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & i\, b_1{}^2 \\ i\, b_2{}^2 & 0 \end{pmatrix}, \; \hat{1}_2 = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & b_1{}^2 \\ -b_2{}^2 & 0 \end{pmatrix}, \hat{1}_3 = \Delta^{-\frac{1}{2}} \begin{pmatrix} i\, b_2{}^2 & 0 \\ 0 & -i\, b_1{}^2 \end{pmatrix} \tag{2.7.27}$$

As expected, the $\hat{1}$-matrices verify the isotopic image of properties (2.7.7), i.e.,

$$\hat{\imath}_n \hat{\circ} \hat{\imath}_m = -\Delta^{\frac{1}{2}} \epsilon_{nmk} \hat{\imath}_k , \quad n \neq m , \quad n, m = 1, 2, 3 , \Delta = b_1^2 b_2^2, \tag{2.7.28}$$

and are therefore closed under isocommutators (as a necessary condition to have an isotopy), resulting the Lie-isotopic $\hat{su}(2)$ algebra

$$[\hat{\imath}_n \hat{,} \hat{\imath}_m] := \hat{\imath}_n \hat{\circ} \hat{\imath}_m - \hat{\imath}_m \hat{\circ} \hat{\imath}_n = -2 \Delta^{\frac{1}{2}} \epsilon_{nmk} \hat{\imath}_k , \tag{2.7.29}$$

with *isocasimir invariants* and generalized eigenvalues equations studied in Ch. II.6. For alternative realizations with $\Delta = 1$ see Sect. I.4.7.

Note the abstract identity of the isotopic **$\hat{su}(2)$** with the conventional **su(2)** algebra. Nevertheless, Pauli's matrices and their isotopic covering are not unitarily equivalent.

Note also that the isoinvariance $\hat{O}(2)$ of the isocomplex numbers (Sect. 2.6) is a subgroup of $S\hat{U}(2)$ characterizable by $\hat{\imath}_1$, thus confirms the matrix isorepresentation of isocomplex numbers.

Isoquaternions can therefore be written in the form (apparently presented in ref. [24] for the first time)

$$\hat{q} = n_0 \hat{1}_0 + n_1 \hat{\imath}_1 + n_2 \hat{\imath}_2 + n_3 \hat{\imath}_3 =$$

$$= \begin{pmatrix} (n_0 b_1^{-2} + \Delta^{-\frac{1}{2}} i \, n_3 b_2^2) & \Delta^{-\frac{1}{2}} (-n_2 + i \, n_1) b_1^2 \\ \Delta^{-\frac{1}{2}} (n_2 + i \, n_1) b_2^2 & (n_0 b_2^{-2} - \Delta^{-\frac{1}{2}} i \, n_3 b_1^2) \end{pmatrix} \tag{2.7.30}$$

It is straightforward to show that the set $\hat{S}(\hat{q},+,\hat{\times})$ of all possible expression (2.7.30) preserves the axioms of the original set $S(q,+,\times)$. In fact, the set $\hat{S}(\hat{q},\hat{+},\hat{\times})$ is a four-dimensional vector space over the isoreals $\hat{R}(\hat{n},+,\hat{\times})$ which is closed under the operation of conventional addition and isomultiplication, thus being an isofield. The isomorphism $\hat{S}(\hat{q},+,\hat{\times}) \approx \hat{Q}(\hat{q},+,\hat{\times})$ then follows.

The *isonorm* of the isoquaternions is given by

$$\hat{\uparrow}\hat{q}\hat{\uparrow} = [\,\mathrm{Det}_R(\hat{q} \; \hat{T})\,]^{\frac{1}{2}} \hat{1}_0 = (\hat{q}^\dagger \hat{\circ} \hat{q})^{\frac{1}{2}} \hat{1}_0 , \tag{2.7.31}$$

and can be written

$$\hat{\uparrow}\hat{q}\hat{\uparrow} = [\, n_0^2 + \Delta(n_1^2 + n_2^2 + n_3^2)\,] \hat{1}_0 \; , \tag{2.7.32}$$

which should be compared with expression (2.7.12) for the ordinary quaternions. Isonorm (2.7.31) also verifies the basic rule

$$\hat{\uparrow}\hat{q} \hat{\circ} \hat{q}'\hat{\uparrow} = \hat{\uparrow}\hat{q}\hat{\uparrow} \hat{\times} \hat{\uparrow}\hat{q}'\hat{\uparrow} \in \hat{R} , \quad \hat{q}, \hat{q}', \hat{\circ} \in \hat{Q} . \tag{2.7.33}$$

The *isobasis*

$$\hat{e}_1 = \hat{1}_0, \quad \hat{e}_{k+1} = \hat{1}_k, \quad k = 1, 2, 3, \tag{2.7.34}$$

then establishes that *isoquaternions constitute a four-dimensional, isonormed, isoassociative, non-isocommutative isoalgebras over the isoreals* $\hat{U}(4) \approx U(4)$ [24].

2.7.D: Realization of isodual isoquaternions. The *isodual isoquaternions* $\hat{q}^d \in \hat{Q}^d(\hat{q}^d,+,\hat{o}^d)$ can be characterized via the two-dimensional isodual complex Hermitean isoeuclidean space of Class II over the isodual isocomplex field,

$$\hat{E}^d_{II,2}(\hat{z}^d,\hat{\delta}^d,\hat{C}^d(c^d,+,\hat{\times}^d)): \quad \hat{z}^{d\dagger}\, \hat{\delta}^d\, z^d = \bar{z}^{1d}\, \hat{\times}^d\, z^{1d} + \bar{z}^{2d}\, \hat{\times}^d\, z^{2d} = -\bar{z}^1\, b_1{}^2 z^1 - \bar{z}^2\, b_2{}^2\, z^2, \tag{2.7.35}$$

with basic isodual isotopic element and isodual isounit

$$\hat{T}^d = \mathrm{Diag.}(-b_1{}^2, -b_2{}^2), \quad \hat{1}^d = \mathrm{Diag.}(-b_1{}^{-2}, -b_2{}^{-2}), \tag{2.7.36}$$

whose (unimodular) invariance is now that of the isodual Lie-isotopic group $S\hat{U}^d(2)$ (see Ch. I.4). An *isodual Hermitean isodilative isorotation* on $\hat{E}^d_{II,2}(\hat{z}^d,\hat{\delta}^d,\hat{C}^d(\hat{c}^d,+,\hat{\times}^d))$, is given by

$$\hat{z}'^{1d} = \hat{c}^d{}_1\, \hat{o}^d\, \hat{z}^{1d} - \bar{\hat{c}}^d{}_2\, \hat{o}^d\, \hat{z}^{2d}, \quad \hat{z}'^{2d} = \hat{c}^d{}_2\, \hat{o}^d\, \hat{z}^{1d} + \bar{\hat{c}}^d{}_1\, \hat{o}^d\, \hat{z}^{2d}, \tag{2.7.37}$$

where the dilation is represented by the value $\bar{\hat{c}}^d{}_1 \hat{o}^d \hat{c}^d{}_1 + \bar{\hat{c}}^d{}_2 \hat{o}^d \hat{c}^d{}_2 \neq \hat{1}^d$.

Isoquaternions then admit a realization in terms of the isodual isorepresentation of $\mathbf{s\hat{u}^d}$ **(2)** which can be written

$$\hat{q}^d = \hat{n}_0{}^d + \hat{n}_1{}^d \times^d \hat{1}_1{}^d + \hat{n}_2{}^d \times^d \hat{1}_2{}^d + \hat{n}_3{}^d \times^d \hat{1}_3{}^d =$$

$$= -\hat{n}_0 + \hat{n}_1\, \hat{1}_1 + \hat{n}_2\, \hat{1}_2 + \hat{n}_3 \hat{1}_3 =$$

$$= \begin{pmatrix} (-n_0\, b_1{}^{-2} + \Delta^{-\frac{1}{2}}\, i\, n_3\, b_2{}^2) & \Delta^{-\frac{1}{2}}\, (-n_2 + i\, n_1)\, b_1{}^2 \\ \Delta^{-\frac{1}{2}}\, (n_2 + i\, n_1)\, b_2{}^2 & (-n_0\, b_2{}^{-2} - \Delta^{-\frac{1}{2}}\, i\, n_3\, b_1{}^2) \end{pmatrix} \tag{2.7.38}$$

It is again easy to show that the set $\hat{S}^d(\hat{q}^d,+,\times^d)$ of all possible matrices (2.7.38) is an isofield. The isomorphism $\hat{S}^d(\hat{q}^d,+,\times^d) \approx \hat{Q}^d(\hat{q}^d,+,\hat{\times}^d)$ then follows.

The *isodual isonorm* is now given by

$$\hat{\uparrow}\hat{q}^d\hat{\uparrow}^d = [\,\mathrm{Det}_R(\hat{q}^d \times^d \hat{T}^d)\,]^{\frac{1}{2}} \times \hat{1}_0{}^d = (\hat{q}^{\dagger d} \times^d \hat{q}^d)^{\frac{1}{2}} \times \hat{1}_0{}^d =$$

$$= [\,-n_0{}^2 - \Delta(n_1{}^2 + n_2{}^2 + n_3{}^2)\,]\, \hat{1}_0{}^d, \tag{2.7.39}$$

and also verified the basic rule

$$\hat{\uparrow}\hat{q}^d\, \hat{o}^d\, \hat{q}'^d\hat{\uparrow}^d = \hat{\uparrow}\hat{q}^d\hat{\uparrow}^d\, \hat{\times}^d\, \hat{\uparrow}\hat{q}'^d\hat{\uparrow}^d \in \hat{R}^d, \quad \hat{q}^d, \hat{q}'^d, \hat{o}^d \in \hat{Q}^d. \tag{4.A.40}$$

The *isodual isobasis* is now given by

$$\hat{e}^d{}_1 = \hat{1}^d{}_0 , \quad \hat{e}^d{}_{k+1} = \hat{1}^d{}_k , \quad k = 1, 2, 3 , \tag{2.7.41}$$

and proves that *isodual isoquaternions constitute a four-dimensional, isodual, isonormed, isoassociative, non-isocommutative isoalgebra over the isodual isoreals* $\hat{U}^d(4) \approx U^d(4)$ [24].

In summary, the isotopy of the conventional quaternions permits the introduction of nontrivial degrees of freedom represented by the diagonal elements of the isotopic element $\hat{T} = \text{diag.}\,(b_1{}^2, b_2{}^2)$, owing to their unrestricted functional dependence $b_k(t, x, \dot{x}, ...) \neq 0$. The "isotopic degrees of freedom" persist even under condition (2.6.44), (2.6.45) under which the regular isopauli matrices (2.7.26)

$$\hat{1}_1 = \begin{pmatrix} 0 & i\lambda^2 \\ i\lambda^{-2} & 0 \end{pmatrix}, \hat{1}_2 = \begin{pmatrix} 0 & \lambda^2 \\ -\lambda^{-2} & 0 \end{pmatrix}, \hat{1}_3 = \begin{pmatrix} i\lambda^{-2} & 0 \\ 0 & -i\lambda^2 \end{pmatrix}, \tag{2.7.42}$$

called *standard isopauli matrices* [25–27] (see Ch. II.6 for their detailed study).

It should be indicated for completeness that in this section we have studied the isotopies and isodualities only of the fundamental form of quaternions. For additional forms for which no isotopies and isodualities have been studied until now, such as the *spit quaternions, antiquaternions and semiquaternions,* we refer the interested reader to monograph [28].

2.8: ISOOCTONIONS AND THEIR ISODUALS

For completeness, we also present realizations of octonions, isodual octonions, isooctonions and isodual isooctonions, which follow very closely the construction of isoquaternions and their isoduals from isocomplex numbers and their isoduals.

2.8.A: Realization of octonions. Recall (see, e.g., ref. [7,8] and contributions quoted therein), that the octonions $o \in O(o,+,\times)$ can be realized via two quaternions, $o = (q_1, q_2)$, with composition rules

$$o \odot o' = (q_1, q_2) \odot (q'_1, q'_2) = (q_1 \odot q'_1 + q_1 \odot q'_2, -\bar{q}_1 \odot q'_2 + \bar{q}_1 \odot q_2), \tag{2.8.1}$$

The antiautomorphic conjugation of an octonion is given by

$$\bar{o} = (\bar{q}_1, -q_2). \tag{2.8.2}$$

It is then possible to introduce the *norm*

$$|o| := (\bar{o} \odot o)^{\frac{1}{2}} = |q| + |q_2|, \tag{2.8.3}$$

which also verifies the basic axiom

$$|o \odot o'| = |o| \times |o'| \in R, \quad o, o' \in O. \tag{2.8.4}$$

We finally recall that *the octonions form an eight dimensional normed, nonassociative and noncommutative, alternative algebra* U(8) *over the field of reals* R(n,+,×) [7,8] .

2.8.B: Realization of isodual octonions. The *isodual octonions* are defined via the isoconjugation

$$o^d = (q_1{}^d, q_2{}^d) \tag{2.8.5}$$

this time, over the isodual reals $R^d(n^d,+,\times^d)$, and are therefore different than the conventional conjugate octonions $\bar{o}$, Eq. (.2.8.2). Their isodual multiplication is

$$o^d \odot^d o'^{\,d} = (q_1{}^d, q_2{}^d) \odot^d (q'_1{}^d, q'_2{}^d) =$$

$$(q_1{}^d \odot^d q'_1{}^d - \bar{q}_1{}^d \odot q'_2{}^d, \; q_1{}^d \odot^d q'_2{}^d + \bar{q}_1{}^d \odot^d q_2{}^d), \tag{2.8.6}$$

the isodual antiautomorphism is then given by

$$\bar{o}^d = (\bar{q}^d, -q_2{}^d). \tag{2.8.7}$$

It is then possible to introduce the *isodual norm*

$$|o^d|^d = (\bar{o}^d \odot^d o^d)^{\frac{1}{2}} \times 1^d = |q_1{}^d|^d + |q_2{}^d|^d, \tag{2.8.8}$$

which also verifies the basic axiom

$$|o^d \odot^d o'^{\,d}| = |o^d|^d \times^d |o'^{\,d}|^d \in R^d, \quad o^d, o'^d, \times^d \in O^d. \tag{2.8.9}$$

Thus *the isodual octonions form an eight dimensional isodual, normed, nonassociative, alternative and noncommutative algebra* $U^d(8)$ *over the isodual real numbers* $R^d(n^d,+,\times^d)$ [24].

2.8.C: Realization of isooctonions. Isooctonions [24] $\hat{o} \in \hat{O}(\hat{o},+,\hat{\times})$ can be

defined as the pair of isoquaternions , $\hat{o} = (\hat{q}_1 , \hat{q}_2)$ over the isoreals $\hat{R}(\hat{n},+,\hat{\times})$ with multiplication rules

$$\hat{o} \,\hat{\odot}\, \hat{o}' = (\hat{q}_1, \hat{q}_2)\,\hat{\odot}\,(\hat{q}'_1, \hat{q}'_2) = (\hat{q}_1 \,\hat{\odot}\, \hat{q}'_1 + \hat{q}_1 \,\hat{\odot}\, \hat{q}'_2, -\bar{q}_1 \,\hat{\odot}\, \hat{q}'_2 + \bar{q}_1 \,\hat{\odot}\, \hat{q}_2), \tag{2.8.10}$$

It is then easy to see that the lifting $o \rightarrow \hat{o}$ is an isotopy, thus preserving all original axioms of o. In fact, we have the antiautomorphic conjugation

$$\tilde{\hat{o}} = (\tilde{q}, -\hat{q}_2), \tag{2.8.11}$$

and the *isonorm*

$$\uparrow\hat{o}\uparrow = (\bar{\hat{o}} \odot \hat{o})^{\frac{1}{2}} \times 1 = \uparrow\hat{q}_1\uparrow + \uparrow\hat{q}_2\uparrow \tag{2.8.12}$$

with property

$$\uparrow\hat{o} \,\hat{\odot}\, \hat{o}'\uparrow = \uparrow\hat{o}\uparrow \,\hat{\times}\, \uparrow\hat{o}'\uparrow \in \hat{R}, \quad \hat{o}, \hat{o}' \in \hat{O}. \tag{2.8.13}$$

It is then easy to see that *isooctonions form an eight dimensional isonormed, non-isoassociative, non-isocommutative, isoalternative isoalgebra* $\hat{U}(8) \approx U(8)$ *over the isoreals* $\hat{R}(\hat{n},+,\hat{\times})$ [24].

2.8.D: Realization of isodual isooctonions. The notion of isoduality applies also to the isooctonions yielding the isodual isooctonions $\hat{o}^d = (\hat{q}_1{}^d , \hat{q}_2{}^d)$ with composition rule

$$\hat{o}^d \,\hat{\odot}^d\, \hat{o}'^{\,d} = (\hat{q}_1{}^d, \hat{q}_2{}^d)\,\hat{\odot}^d\,(\hat{q}'_1{}^d, \hat{q}'_2{}^d) =$$

$$(\hat{q}_1{}^d \,\hat{\odot}^d\, \hat{q}'_1{}^d - \bar{q}_1{}^d \,\hat{\odot}^d\, \hat{q}'_2{}^d,\; q_1{}^d \,\hat{\odot}^d\, q'_2{}^d + \bar{q}_1{}^d \,\hat{\odot}^d\, q_2{}^d), \tag{2.8.14}$$

Then we have the isodual isoantiautomorphism

$$\tilde{\hat{o}}^d = (\tilde{q}^d, -\hat{q}_2{}^d). \tag{2.8.15}$$

the *isodual isonorm*

$$\uparrow\hat{o}^d\uparrow^d := (\bar{\hat{o}}^d \,\hat{\odot}^d\, \hat{o}^d)^{\frac{1}{2}} \times 1^d = \uparrow\hat{q}_1{}^d\uparrow^d + \uparrow\hat{q}_2{}^d\uparrow^d \tag{2.8.16}$$

which also verifies the basic axiom

$$\uparrow\hat{o}^d \,\hat{\odot}^d\, \hat{o}'^{\,d}\uparrow = \uparrow\hat{o}^d\uparrow^d \,\hat{\times}^d\, \uparrow\hat{o}'^{\,d}\uparrow^d \in \hat{R}^d, \quad \hat{o}^d, \hat{o}'^d \in \hat{O}^d. \tag{2.8.17}$$

It is then possible to prove that *isodual isooctonions form an eight*

dimensional isodual, isonormed, non-isoassociative, non-isocommutative, but isoalternative isoalgebra $\hat{O}^d(8) \approx \hat{U}^d(8)$ *over the isodual isofield* $\hat{R}^d(\hat{n}^d,+,\times^d)$ [24].

In this section we have studied the isotopies and isodualities of the conventional notion of octonions. For additional forms of octonions (e.g., the *sedenions*) and the construction of their representations, we suggest the consultation of ref. [28] and literature quoted therein.

We close this section by suggesting caution in the use of octonions and their isotopies as fields because of the loss of associativity and, thus, the loss of enveloping associative algebras of Lie algebras, in favor of alternative algebras. In turn, such a loss, unless properly treated, has fundamental physical implications we shall see in Vol. II, such as: the general loss of the equivalence between Heisenberg's and Schrödinger's representations, the general loss of the exponentiation of an algebra to a corresponding group with consequential loss of the notion of symmetry, and other serious drawbacks.

Nevertheless, when properly treated, octonions do have intriguing applications. As an illustration, we here mention the approach studied by Lõhmus, Paal and Sorgsepp [28] via their "octonionization of Dirac's equation" which resolves all the above problematic aspects, resulting in one of the first (if not the only) formulation of Dirac's equations for quarks with fractional charges.

2.9: ISOTOPIC UNIFICATION OF CONVENTIONAL NUMBERS

One additional property of isonumbers which is important for the subsequent analysis, is given by their capability to unify different conventional numbers into one single, abstract notion of isonumber.

This property, called "isotopic unification" (first identified in ref. [23]) has the following three important applications.

Number theory: According to contemporary formulations (see ref.s [7–12]), real numbers, complex numbers and quaternions are considered to be different mathematical entities, possessing different properties and structures. This conception is surpassed by the isonumber theory because, as shown below in this section, one single entity, the abstract notion of isoreals, can unify all above indicated conventional numbers evidently because of the degree of freedom offered by the isounit, with intriguing mathematical and physical possibilities for novel applications.

Lie's theory: In the contemporary formulation of Lie's theory, nonisomorphic simple Lie groups of Cartan's classification of the same

dimension, such as O(3) and O(2.1), or O(4), O(3.1) and O(2.2), etc. are generally considered to be different entities possessing different structures and properties. As shown for the first time in ref. [18], this approach too is surpassed by isotopic theories which offer the possibility of unifying all simple Lie groups of the same dimensions into one single, abstract Lie-isotopic group. An evident pre-requisite for such unification is precisely the unification of all fields studied in this section.

Quantum mechanics on quaternionic fields: Even though the most dominant use of fields in contemporary quantum mechanics is restricted to real and complex fields, the generalization of quantum mechanics over a quaternionic field has been recently studied by various authors (see ref.s [7-9] and literature quoted therein). In these volumes we shall show that this approach too is superseded by isotopic techniques because quantum mechanics on a quaternionic field is a particular case of hadronic mechanics on an isoreal field.

The existence of an isotopic unification of all numbers had been conjectured by the author in various publications throughout the years, but it was proved only recently by Kadeisvili, Kamiya and Santilli in ref. [21]. The main result is the following

Theorem 2.7.1 : *Let* R, C, Q *be the fields of real numbers, complex numbers and quaternions, respectively,* R^d, C^d, Q^d *the isodual fields,* $\hat{R}$, $\hat{C}$, $\hat{Q}$ *the isofields, and* $\hat{R}^d$, $\hat{C}^d$, $\hat{Q}^d$ *the isodual isofields as defined in preceding sections. Then all these fields can be constructed with the same methods for the construction of* $\hat{R}$ *from* R, *under the relaxation of the condition of positive-definiteness of the isounit, thus achieving a unification of all fields, isofields and their isoduals into the single, abstract isofield of Class III, hereon denoted* $\hat{\mathfrak{R}}$.

Proof: The field of real numbers R is a trivial particular case of $\hat{\mathfrak{R}}$ when the isotopy is the identity, $\hat{\mathfrak{R}}_{1=\hat{1}} \equiv R$. The fact that the field of complex numbers C is a subcase of $\hat{\mathfrak{R}}$ can be proved as follows. Introduce the binary (Cayley-Dickinson) realization [28] of the elements of $\hat{\mathfrak{R}}$, $\hat{a} = (a_1, a_2)$, where $(a_1, 0)$ and $(0, a_2)$ represent the real (Re) and imaginary (Im) parts, respectively, with the following isomultiplication

$$(a_1, a_2) \hat{\times} (b_1, b_2) = (a_1 \times b_1 - b_2 \times a_2, a_1 \times b_2 + b_2 \times a_1), \qquad (2.9.1)$$

where $\times$ represents the conventional multiplication, and introduce the additional multiplication for elements of Im C

$$(0, a) \hat{\times}_2 (0, b) := (0, a) \hat{\times} (0, -1) \hat{\times} (0, b). \qquad (2.7.2)$$

Then, $\hat{\mathfrak{R}}$ can be decomposed into the tensorial product of the following two parts

$$\mathfrak{R}_1 = \{ (a, 0) \mid a \in R, \hat{1} = (1, 0) \} , \tag{2.9.3}$$

$$\mathfrak{R}_2 = \{ (0, a) \mid a \in R, \hat{1} = (0, 1) \} . \tag{2.9.4}$$

The local isomorphism $\mathfrak{R}_1 \approx \text{Re C}$ is trivial. The fact that $\mathfrak{R}_2 \approx \text{Im C}$ follows from the expressions $\hat{1} = (0, 1)$, $T = \hat{1}^{-1} = (0, -1)$. Thus, the multiplication in $\mathfrak{R}_2$ is characterized by

$$\hat{a} \hat{\times} \hat{b} = a \times (0, 1) \hat{\times}_2 b \times (0, 1) = a \times (0, 1) \hat{\times} (0, -1) \hat{\times} b \times (0, 1) . \tag{2.9.5}$$

Moreover,

$$(0, 1) \hat{\times} \hat{b} = \hat{b} \hat{\times} (0, 1) \equiv \hat{b} , \tag{2.9.6a}$$

$$(0, a) \hat{\times} (0, a^{-1}) = (0, 1), \qquad \exists \quad a \neq 0 \tag{2.9.6b}$$

and this proves that $\mathfrak{R}_2 \approx \text{Im C}$. Thus, in the above binary realization and multiplications (2.9.6a) and (2.7.6b), $\mathfrak{R}$ coincides with C.

The proof that the field of quaternions Q is a subcase of $\mathfrak{R}$ can be done via the quaternary realization $\mathfrak{R} \approx C \hat{\times} C$ with isomultiplication

$$(a_1, a_2) \hat{\times} (b_1, b_2) = (a_1 b_1 - \bar{b}_2 a_2, a_1 \bar{b}_1 + b_2 a_1) , \tag{2.9.7}$$

for all $a_1, a_2, b_1, b_2 \in C$ and $\bar{a}$ denoting conventional complex conjugation in C.

Then $\mathfrak{R}$ can be decomposed into the following parts

$$\mathfrak{R}_1 = \{ (a, 0) \mid a \in C \}, \quad \mathfrak{R}_2 = \{ (0, b) \mid b \in C \} . \tag{2.9.8}$$

The product for $\mathfrak{R}_2$ can be defined as

$$(0, a) \hat{\times}_2 (0, b) = (0, a) \hat{\times} (0, -1) * (0, b) = (0, b a) . \tag{2.9.9}$$

By making use of these products we readily obtain that $\mathfrak{R}_1 \approx C$. To identify the role of $\mathfrak{R}_2$ we note that

$$\hat{a} \hat{\times} \hat{b} = a \times (0, 1) \hat{\times} (0, -1) * (0, 1) b = (0, b a) = a b (0, 1) . \tag{2.9.10}$$

This implies that, in the above quaternary realization of the elements with multiplications (2.9.9), $\mathfrak{R}$ coincides with Q.

The inclusion in $\mathfrak{R}$ of all isotopes $\hat{R}$, $\hat{C}$ and $\hat{Q}$ readily follows from the lifting of all trivial unit 1 into isotopic form $\hat{1}$ with corresponding lifting of the

related operations. The inclusion of isodual fields and isodual isofields follows from the the assumption of Class III which includes positive-definite, as well as negative-definite isounits **q.e.d.**

The following property is also implicit in the above proof.

Corollary 2.7.1.A [23]: *If the isofield* $\hat{R}$ *is such that* $\hat{R} = \{ (0, x) \mid x \in R, \hat{1} = (0, 1) \}$, *then* $\hat{R} \approx \mathrm{Im}\, C$ *with respect to product (2.7.9) and (2.9.10).*

For completeness we point out that the octonions O are locally isomorphic to the realization $\mathfrak{K} = Q \hat{\times} Q$ essentially along the lines for $\mathfrak{K} = C \hat{\times} C \approx Q$. Consider again the binary realization of the elements, $\hat{a} = (a_1, a_2)$, although now a_1 and a_2 represent quaternions, and introduce the isomultiplication in $\mathfrak{K}$

$$(a_1, a_2) \hat{\times} (b_1, b_2) := (a_1 b_1 - \bar{b}_2 a_2,\ a_2 \bar{b}_1 + b_2 a_1), \qquad (2.9.11)$$

where $a_k, b_k \in Q$, $k = 1, 2$, with the additional multiplication for the elements (0, a)

$$(0, a) \hat{\times}_2 (0, b) = [(0, a) * (0, -1)] * (0, b). \qquad (2.9.12)$$

Then, as it was the case for quaternions, $\mathfrak{K}$ can be decomposed into the tensorial product of the following two parts

$$\mathfrak{K}_1 = \{ (a, 0) \mid a \in Q, \hat{1} = (1, 0) \}, \quad \mathfrak{K}_2 = \{ (0, a) \mid a \in Q, \hat{1} = (0, 1) \}. \qquad (2.9.13)$$

The local isomorphism $\mathfrak{K}_1 \approx Q$ is trivial. To identify the role of $\mathfrak{K}_2$ note that

$$\hat{a} \hat{*}_2 \hat{b} = a\ (0, 1) \hat{\times}_2 b\ (0, 1) = [a\ (0, 1) \hat{\times} (0, -1)] \hat{\times} b\, (0, 1). \qquad (2.9.14)$$

Moreover, also as in the case of quaternions,

$$(0, 1) \hat{\times}_2 \hat{b} = \hat{b} \hat{\times}_2 (0, 1) \equiv \hat{b},\ (0, a) \hat{\times}_2 (0, a^{-1}) = (0, 1),\ \exists\ a \neq 0 \qquad (2.9.15),$$

and

$$[(0, a) \hat{\times}_2 (0, b)] \hat{\times}_2 (0, c) = (0, a) \hat{\times}_2 [(0, b) *_2 (0, c) = (0, c\, b\, a). \qquad (2.9.16)$$

Thus, in the above considered realization with isomultiplication (2.9.15) and (2.7.16) $\mathfrak{K}$ is locally isomorphic to the octonions.

APPENDIX 2.A: "HIDDEN NUMBERS" OF DIMENSION 3, 5, 6, 7

Historically, the conventional numbers were studied via the solution of the following problem (see, e.g., ref. [8])

$$(a_1{}^2 + a_2{}^2 + ... + a_n{}^2) \times (b_1{}^2 + b_2{}^2 + ... + b_n{}^2) = A_1{}^2 + A_2{}^2 + ... + A_n{}^2, \quad (2.A.1a)$$

$$A_k = \sum_{r,s} c_{krs}\, a_r\, b_s . \quad (2.A.1b)$$

where the a's, b's and c's are elements of a conventional field $F(a,+,\times)$ with familiar operations $+$ and $\times$. As well known, the only possible solutions of problem (2.A.1) studied by Gauss [1], Abel [2], Hamilton [3], Cayley [4], Galois [5], Albert [12], Jacobson [13] and others are of dimension 1, 2, 4, 8 (Theorem 2.1.1).

The isotopies and pseudoisotopies of the theory of numbers creates the problem of the possible existence of *"hidden numbers"*, that is, new solutions of dimension different than 1, 2, 4, 8 which are hidden in the operations $\times$ and/or $+$. This problem, studied for the first time in ref. [24], essentially asks whether the classification of Theorem 2.1.1 persists under isotopies, pseudoisotopies and their isodualities, or it is incomplete.

It is easy to see that the reformulation of problem (2.A.1) under the isotopies of the multiplication $\times \rightarrow \hat{\times} = \times T \times$, $1 \rightarrow \hat{1} = \hat{T}^{-1}$, *does not* lead to new solutions. In fact, Problem (2.A.1) under lifting $\times \rightarrow \hat{\times}$ is given by

$$(a_1{}^{\hat{2}} + a_2{}^{\hat{2}} + ... + a_n{}^{\hat{2}}) \,\hat{\times}\, (b_1{}^{\hat{2}} + b_2{}^{\hat{2}} + ... + b_n{}^{\hat{2}}) = A_1{}^{\hat{2}} + A_2{}^{\hat{2}} + ... + A_n{}^{\hat{2}}, \quad (2.A.2a)$$

$$A_k = \sum_{r,s} c_{krs} \,\hat{\times}\, a_r \,\hat{\times}\, b_s , \quad (2.A.2b)$$

where the a's, b's and c's now belong to an isofield of the type $\hat{F}(a,+,\hat{\times})$, in which case $\hat{1}$ is an element of the original field F (Proposition 2.3.1). Problem (2.A.2) can then be written in conventional operations

$$(a_1{}^2 + a_2{}^2 + ... + a_n{}^2) \times (b_1{}^2 + b_2{}^2 + ... + b_n{}^2) = \hat{T}^{-2} (A_1{}^2 + A_2{}^2 + ... + A_n{}^2), \quad (2.A.3a)$$

$$A_k = \hat{T}^2 c_{krs}\, a_r\, b_s , \qquad n \leq 8 \quad (2.A.3b)$$

The substitution of the latter expression into the former, then recovers Problem (2.A.1) identically for liftings of Class I, II, and III. The reformulation in the isofield $\hat{F}(\hat{a},+,\hat{\times})$ is also equivalent to the original one. We can therefore summarize the studies of this section with the following generalization of Theorem 2.1.1:

Theorem 2.A.1 [24]: *All possible isonormed isoalgebras with multiplicative*

isounit of Kadeisvili's Class I over the isoreals are the isoalgebras of dimension 1 *(isoreals),* 2 *(isocomplex),* 4 *(isoquaternions) and* 8 *(isooctonions), and the classification persists under isoduality.*

Nevertheless, there exists a third formulation of pseudoisotopic type (Proposition 2.3.3 and Definition 2.3.3) characterized by the further lifting of the addition

$$+ \rightarrow \hat{+} = + \hat{K}, \quad 0 \rightarrow \hat{0} = -\hat{K}, \; \hat{K} = K \times \hat{1} \tag{2.A.4}$$

under which problem (2.A.2) can be rewritten over the pseudoisofield $\hat{F}(\hat{a},\hat{+},\hat{\times})$

$$(\hat{a}_1{}^{\hat{2}} \hat{+} \hat{a}_2{}^{\hat{2}} \hat{+} .... \hat{+} \hat{a}_n{}^{\hat{2}}) \hat{\times} (\hat{b}_1{}^{\hat{2}} \hat{+} \hat{b}_2{}^{\hat{2}} \hat{+} .... \hat{+} \hat{b}_n{}^{\hat{2}}) = \hat{A}_1{}^{\hat{2}} \hat{+} \hat{A}_2{}^{\hat{2}} \hat{+} .... \hat{+} \hat{A}_n{}^{\hat{2}}, \tag{2.A.5a}$$

$$\hat{A}_k = \textstyle\sum_{r,s} \hat{c}_{krs} \hat{\times} \hat{a}_r \hat{\times} \hat{b}_s = (\sum_{r,s} c_{krs}\, a_r\, b_s)\hat{1} = A_k \times \hat{1}, \tag{2.A.5b}$$

and can be rewritten in conventional operations

$$[(a_1{}^2 + a_2{}^2 + ... + a_n{}^2)\hat{1} + (n-1)K\hat{1}]\, T\, [(b_1{}^2 + b_2{}^2 + ... + b_n{}^2)\hat{1} + (n-1)K\hat{1}] =$$

$$= (A_1{}^2 + A_2{}^2 + ... + A_n{}^2)\hat{1} + (n-1)K\hat{1}, \quad \hat{A}_k = A_k\hat{1}, \tag{2.A.6}$$

where we have the cancellation of the isotopic element as in the preceding cases, but the preservation of the additive "degree of freedom" K.

The conjecture of the existence of "hidden numbers" was therefore formulated in ref. [24], specifically, *under the pseudoisofield* $\hat{F}(\hat{a},\hat{+},\hat{\times})$, *that is, under the loss of the distributive law* (Proposition 2.3.3).

We here limit ourselves to the following example of "hidden number" of dimension 3

$$(\hat{1}^{\hat{2}} \hat{+} \hat{2}^{\hat{2}} \hat{+} \hat{3}^{2}) \hat{\times} (\hat{5}^{\hat{2}} + \hat{6}^{\hat{2}} + \hat{7}^{\hat{2}}) = \hat{12}^{\hat{2}} \hat{+} \hat{24}^{\hat{2}} \hat{+} \hat{30}^{2}, \tag{2.A.7}$$

Note that the combinations for the elements in the r. h. s. do exist in terms of elements in the l. h. s.

$$12 = 2 \times 6, \quad 24 = 2 \times 5 + 2 \times 7, \quad 30 = 3 \times 3 + 3 \times 7. \tag{2.A.8}$$

Problem (2.A.7) can then be written

$$[(1^2 + 2^2 + 3^2)\hat{1} + 2K\hat{1}]\, \hat{T}\, [(5^2 + 6^2 + 7^2)\hat{1} + 2K\hat{1}] =$$

$$= (12^2 + 24^2 + 30^2)\hat{1} + 2K\hat{1}, \tag{2.A.9}$$

which reduces to the following equation in K

$$4\,K^2 + 246\,K - 80 = 0, \tag{2.A.10}$$

with solution

$$K = 0.325.......... \tag{2.A.11}$$

However, the above solution is not an integer. This implies the loss of closure under isoaddition (see the comments after Proposition 2.3.3). As a result, starting with an original set of integers, one must complete them under pseudoisotopies into the field of all real numbers. The issue left open in ref. [24] is therefore the problem whether the above solutions do indeed constitute a pseudoisofield.

To understand the example one should recall that the solution considered *does not* exist for ordinary numbers (because the dimension n = 3 is prohibited by Theorem 2.1.1), i.e.,

$$(\,1^2 + 2^2 + 3^2\,)\,(\,5^2 + 6^2 + 7^2\,) \;\neq\; 12^2 + 24^2 + 30^2. \tag{2.A.12}$$

The reader can then construct explicit examples of "hidden numbers" of dimension 5, 6, 7.

Note that Problems (2.A.2) and (2.A.5) are restricted to dimensions $n \leq 8$. This is due to the fact that algebras of dimensions higher than 8 are no longer alternative [8], and such a property is expected to persist under isotopies and pseudoisotopies.

Genonumbers will be studied in Ch. I.7. It is possible to show that the results of this appendix essentially persist in the restriction of the multiplication to be one–sided, and the differentiation of the multiplication into one to the left and one to the right.

For further generalizations of conventional numbers via ternary operations and other needs, we suggest ref. [28] and literature quoted therein.

Among endless novel problems identified by the isofields which are still open at this writing, we suggest the study of the novel notion of "number with a singular unit", i.e., the isofields of Class IV which are at the foundations of the isotopic studies of gravitational collapse and are vastly unknown at this writing; or the study of isofields of isocharacteristic $p \neq 0$, to see whether new fields, are permitted by the isotopies.

APPENDIX 2.B: THEORY OF ISONUMBERS

In the main text of this chapter we have studied the *realizations* of isonumbers

and their isoduals as needed for the applications of Vol.s II and III. The study of the *properties* of isonumber and their isoduals is the subject of a new discipline called by this author *theory of isonumbers* [24] which is given by the isotopies and isodualities of the conventional *theory of numbers* (see, e.g., [30]).

The latter theory is a notoriously vast field and its lifting via isotopies and isodualities cannot possibly be studied here. We shall therefore content ourselves with the mere indication that the covering theory of isonumbers exists and it is nontrivial. Its detailed study must be conducted elsewhere.

Consider the field $F(a,+,\times)$ of positive integers a, b, c, ... = 1, 2, 3, 4, ... An integer a is said to be *composite* when there exist two integers b and c other than 1 and a such that $a = b\times c$, otherwise the number a is said to be *prime*. The numbers b and c are also called *factors* of a.

Consider now the isofield $\hat{F}(a,+,\hat{\times})$ of positive integers a, b, c, ... of Proposition 2.3.1 with isoproduct $b\hat{\times}c = b\times\hat{T}\times c$ where $\hat{T}$ is an element of the original field F, i.e., $\hat{T}$ = 1, 2, 3, ... and, correspondingly, the isounit is $\hat{1}$ = 1, 1/2, 1/3, ... We then have the *isofactorization* of an integer a when there exist two integers b and c such that $a = b\hat{\times}c$, in which case the integers b and c are called *isofactors.* An integer a is said to be *isoprime* when there exist no integers b and c such that $a = b\hat{\times}c$.

Note that the definition of isoprime is stronger than that of prime because it excludes even the values 1 and a as factors. In fact, for the isofield $F(a,+,\hat{\times})$ we have in general $a\hat{\times}1 = a\times\hat{T} \neq a$ and $a\hat{/}a = \hat{1} \neq 1$. As a result, we have the following properties:

Proposition 2.B.1: *The number* 1 *and any positive integer* a = 1, 2, 3, ... *are not isofactors of the number* a *unless the unit is specifically assumed to have the value* 1.

For instance, assume $\hat{T} = 2$. Then $a\hat{\times}1 = 2\times a$ and $a\,\hat{/}\,a = a/2$.

Proposition 2.B.2: *The integer numbers* 2, 4, 6, ... *are necessarily composite under the specific assumption of the number* 1 *as the basic unit.*

For instance, assume $\hat{T} = 3$. Then it is easy to prove that the integer 4 is isoprime.

Proposition 2.B.3: *The integer numbers* 1, 2, 3, 5, 7, 11, ... *are necessarily prime under the specific assumption of the number* 1 *as unit.*

For instance, assume $\hat{T} = 5$, then the number 5 is no longer prime because it admits the factorization $5 = 1\hat{\times}1$, while the prime numbers are 1, 2, 4, 7, 8, 10, 11, ...

The above results indicate that the notions of factorization and prime are

not true axioms of the theory of numbers because they are are specifically dependent on the selected unit and are not invariant under isotopies. This is sufficient to illustrate the nontriviality of the theory of isonumbers and the existence of corresponding generalization of the conventional theory [30].

An intriguing antiautomorphic image of the theory of numbers, which is absent in the conventional theory, is given by the *isodual theory of numbers* which is based on the isodual field $F^d(a^d,+,\times^d)$ with isodual unit $I^d = -1$, and elements $a^d = a\times I^d = -a$. The latter theory is significant to illustrate that *the various properties of the conventional theory of numbers are isoselfdual (invariant under isoduality).* In fact, if an integer p is prime with respect to the basic unit +1, it is evident that the integer $p^d = -p$ is also prime with respect to the isodual unit -1.

The theory of isonumbers is only the first of a chain of liftings of the conventional theory of numbers which is permitted by the generalization of the basic unit. Additional generalizations are given by the *theory of genonumbers* and the *theory of hypernumbers* and their isoduals indicated in Ch. I.7, as well as the *theories of pseudoisonumbers, pseudogenonumbers and pseudohyper-numbers* and their isoduals.

In conclusion, we can say that, after its identification since biblical times, the number +1 has remained for thousands of years the true, ultimate foundation, not only of all of mathematics, beginning with the theory of numbers, but also all of quantitative sciences which are evidently defined on conventional fields of numbers. A primary objective of these volumes is to indicate that the removal of the biblical restriction of the unit to +1 and the assumption of an arbitrary quantity as unit implies a genuine broadening of all mathematical and quantitative sciences into chains of diversified new disciplines with intriguing and basically novel developments and applications.

APPENDIX 2.C: ISOCRYPTOLOGY AND PSEUDOISOCRYPTOLOGY

One of the first practical applications of the theory of numbers is in the security of Governmental, banking and industrial information via a discipline known as *cryptology* [see, e.g., [31]).

One of the first practical applications of the theory of isonumbers, genonumbers, hypernumbers and their pseudo-formulations is for the improvement of said security of information via new disciplines here called *isocryptology, genocryptology, hypercryptology, pseudoisocryptology, etc.*

As well known, all conventional cryptograms can be "broken", that is, it is possible to identify their "key" in a finite time via a sufficiently powerful computer. The main idea of the *isocryptograms* is that the use of the theory of

isonumbers, rather than that of numbers, can improve considerably the security of the information evidently because we now have *numbers with infinitely possible units.* Thus, the solution of the cryptogram now requires the identification of its "key" *as well as the identification of the selected isounit.*

The *pseudoisocryptograms* offer even better security because of the availability, this time, of *two different and independent infinite possibilities, the additive and the multiplicative isounits.*

As an illustration, consider the *circular arithmetic modulo* m on the field F(a,+,×) of integers a = 0, 1, 2, ..., with additive unit 0 and multiplicative unit 1, were addition and multiplication are first done ordinarily and then the result is subtracted by m after discarding all multiples of m (from which we have the name of '"circular arithmetics", as realized, say, in the measure of time via hours in which case m = 12, or in the measure of one hour in which m = 60, etc.). As an example, for m = 8 we have

$$6 + 4 = 2 \text{ (because } = 10 - 8), \tag{2.C.1a}$$

$$6 + 7 + 4 = 1 \text{ (because } = 17 = 2\times 8 + 1), \tag{2.C.1b}$$

$$2\times 2 = 4 \text{ (because } = 0\times 8 + 4), \tag{2.C.1c}$$

$$6\times 4 = 0 \text{ (because } = 24 = 3\times 8). \tag{2.C.1d}$$

When m is prime the circular arithmetic modulo m is that of the *Galois fields.*

Consider now the *pseudoisofield* $\hat{F}(a,\hat{+},\hat{\times})$ of integers a = 0, 1, 2, 3, ... with *isoaddition* $\hat{+} = + \hat{K} +$, *additive isounit* $\hat{0} = -\hat{K}$, $\hat{K} = 1, 2, 3, \ldots \in F$, *isomultiplication* $\hat{\times} = \times\hat{T}\times$, and *multiplicative isotopic element* $\hat{T} = 2, 3, \ldots \in F$.

The *circular isoarithmetics modulo* m defined over $F(a,\hat{+},\hat{\times})$ is then given by the *pseudoisoaddition and isomultiplications* with the result pseudoisosubtracted by m after ignoring all *isomultiples* of m, i.e., $1\hat{\times}m$, $2\hat{\times}m$, $3\hat{\times}m$, etc.

Consider for instance m = 8 as before, with additive isounit $\hat{K} = 2$ and multiplicative isotopic element $\hat{T} = 3$. Then,

$$6 \hat{+} 4 = 6 \text{ (because } = 6 + 2 + 4 = 12 + 2 + (-8)), \tag{2.C.2a}$$

$$6 \hat{+} 7 \hat{+} 4 = 15 \text{ (because } = 6 + 2 + 7 + 2 + 4 = 21 = 0\hat{\times}8 + 2 - 8), \tag{2.C.2b}$$

$$2\hat{\times}2 = 6 \text{ (because } = 2\times 3\times 2 = 12 = 0\hat{\times}8 + 2 - 8), \tag{2.C.2c}$$

$$6\hat{\times}4 = 0 \text{ (because } = 6\times 3\times 4 = 72 = 3\hat{\times}8). \tag{2.C.2d}$$

The improved security in the transition from conventional circular arithmetics modulo m to the pseudoisotopic images is then evident. In fact, the

"key" to find in the former case is *only one number*, the number m, while in the latter case one has to find *three independent numbers*, m, $\hat{K}$ and $\hat{T}$.

The application of the above results to *alphabetic isocryptograms* is straightforward. The English alphabet is composed of 26 letters, and can therefore represented with the circular isoarithmetics A = 0, B = 1, C = 2, ... modulo 26 on the pseudoisofield $F(a,\hat{+},\hat{\times})$ with a transparent increase of security over conventional cryptograms, such as the *Vigenère Cryptogram* [31].

Conventional *binary codes* are circular arithmetics modulo m = 2 with numbers 0 and 1 over the conventional field of integers $F(a,+,\times)$ with additive unit 0 and multiplicative unit 1. They too can be evidently lifted into *pseudoisobinary codes* constituted by the circular arithmetics modulo m = 2 with two numbers 0 and 1 defined over the pseudoisofield $\hat{F}(a,\hat{+},\hat{\times})$ and therefore possessing *two infinite degrees of freedom*, the additive isounit $\hat{0} = + \hat{K} \in F$ and the multiplicative isotopic element $\hat{T} \in F$. The pseudoisotopies are applicable to current codes, such as the *data encryption standard* (DES) with an increase in security such to warrant their study, which evidently has to be conducted elsewhere.

Further, more complex cryptograms even more difficult to break are given by the *pseudogenocryptograms,* in which we have the additional, hidden selection of the *ordering of the addition and multiplication to the left and those to the right whose results are generally different among themselves,* and the yet more complex *pseudoypercryptograms* in which, besides all the above, the result of each individual operations of addition and multiplications is given by *sets of numbers* [32].

APPENDIX 2.D: INAPPLICABILITY OF TRIGONOMETRY

Trigonometry is a basic tool of quantum mechanics, e.g., because trigonometric functions are fundamental for the characterization of spherical harmonics and, thus, for the study of angular momentum in vacuum (e.g., that of electrons in atomic orbits).

It is important to see that conventional trigonometry is inapplicable in hadronic mechanics, so as to prevent a host demonstrable, yet generally undetected inconsistencies and misjudgments.

To state it differently, because of protracted use, noninitiated researchers often approach the problem of the *interior angular momentum* (e.g., the angular momentum of an electron when in the core of a collapsing star) via the use of conventional trigonometry, related spherical harmonics, and corresponding conventional local-differential formulations (say, in Euclidean space). In so doing, however, they completely ignore the effect to be described caused by the

hyperdense medium in the orbital motion, thus *de facto* ignoring the presence of the interior of the star, without any real departure from the original motion of the atomic electron in empty space.

Let us consider the conventional Gauss plane [1] with x- and y-axis (the conventional two-dimensional Cartesian plane). Its trigonometric quantities can be defined via the distance D of a point $P_1(x_1, y_1)$ from the origin

$$D = (x_1^2 + y_1^2)^{\frac{1}{2}}, \qquad (2.D.1)$$

the related Pythagorean theorem

$$x_1^2 + y_2^2 = D^2, \qquad (2.D.2)$$

and the cosine of the angle α between two vectors leading from the origin to two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

$$\text{Cos}\ \alpha = \frac{x_1 x_2 + y_1 y_2}{(x_1^2 + y_1^2)^{\frac{1}{2}} (x_2^2 + y_2^2)^{\frac{1}{2}}}. \qquad (2.D.3)$$

The above elementary and familiar notions are inapplicable under isotopies. To begin, we have the loss of straight lines in favor of the most general possible curvature, that dependent also in velocities and acceleration. Second, the notion of conventional distance is inapplicable, e.g., because the conventional product "x y" now has no mathematical or physical meaning under isotopies. Third, the conventional Pythagorean theorem has no mathematical or geometric sense under isotopies. Thus, the very notion of "angle" between two intersecting "straight lines" in the Gauss plane cannot be preserved for curved lines in our isogauss plane.

The reconstruction of trigonometry under isotopy shall be studied in Ch. I.5 as parte of the study of the isogeometries.

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4: ISOSPACES AND THEIR ISODUALS

3.1: STATEMENT OF THE PROBLEM

The fundamental representation spaces of contemporary physics are the 3–dimensional *Euclidean space* E, the (3+1)–dimensional *Minkowski space* M, the (3+1)–dimensional *Riemannian spaces* $\Re$, and others well known spaces.

All these spaces are dependent on the field in which they are defined, the field of real numbers R = R(n,+,×). The Euclidean space can then be written

$$E = E(r,\delta,R) : r = (x^1, x^2, x^3), \quad \delta = \text{diag.}\,(1,\ 1,\ 1)\ , \tag{3.1.1a}$$

$$r^2 \ = \ x^i\, \delta_{ij}\, x^j = \ x^1\, x^1 + \ x^2 x^2 \ + \ x^3\, x^3 \ \in \ R = R(n,+,\times)\,, \tag{3.1.1.b}$$

where i, j = 1, 2, 3 and δ is the *Euclidean metric*; the Minkowski space can be written

$$M = M(x,\eta,R), \quad x = (\mathbf{r}\,, x^4\,)\ , \quad x^4 = c_o t, \qquad \eta = \text{Diag.}\,(1,\ 1,\ 1,\ -1)\,, \tag{3.1.2a}$$

$$x^2 \ = \ x^\mu\, \eta_{\mu\nu}\, x^\nu \ = \ x^1\, x^1 + \ x^2 x^2 \ + \ x^3\, x^3 \ - \ x^4\, x^4 \ \ \in \ R(n,+,\times)\,, \tag{3.1.2b}$$

where μ, ν = 1, 2, 3, 4, c_o is the *speed of light in vacuum*, η is the *Minkowski metric*; and the Riemannian spaces can be written

$$\Re = \Re(x,g,R), \quad x \ = \ (\,\mathbf{r}\,, x^4\,), \quad g = \ g(x) \ = \ g^\dagger\,, \quad \text{Det.}\, g \neq \ 0\,, \tag{3.1.3a}$$

$$x^2 \ = \ x^\mu\, g_{\mu\nu}(x)\, x^\nu \ \in \ \ R(n,+,\times)\,, \tag{3.1.3b}$$

where g(x) is the *Riemannian metric.*

By inspecting these structures, and as it already emerged from the study of isonumbers of the preceding chapter, it is evident that the isotopic generalization of numbers and related fields implies a corresponding, necessary generalization of all conventional spaces of current use in mathematics and physics.

At a deeper study, it emerges that, for evident mathematical consistency, *the isotopies of ordinary numbers imply compatible liftings of all mathematical structures used in quantum mechanics.* In fact, the isotopic generalization of conventional spaces implies the necessary, corresponding generalization of the

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INTERIOR DYNAMICAL PROBLEMS WITHIN PHYSICAL MEDIA: ISOEUCLIDEAN, ISOMINKOWSKIAN AND ISORIEMANNIAN SPACES

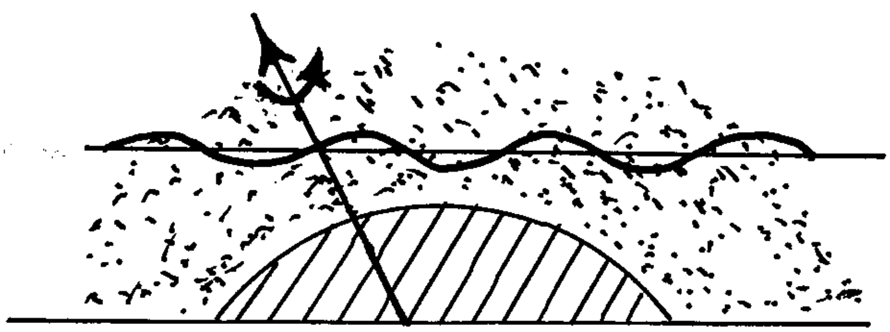

FIGURE 3.1.1.As well known, the Euclidean (3.1.1), Minkowskian (3.1.2) and Riemannian spaces (3.1.3) are the *Newtonian, relativistic and gravitational representation spaces, respectively, of the exterior dynamical problems.* As such, they provide corresponding *geometrizations of the homogeneity and isotropy of empty space (vacuum).* In this chapter we shall study the isotopies of the above spaces under the names of *isoeuclidean, isominkowskian and isoriemannian spaces* which were specifically conceived by this author [1] for the *description, this time, of the corresponding interior dynamical problems.* As such, they provide corresponding *geometrizations of the inhomogeneity and anisotropy of physical media.* In this figure we illustrate a central objective of isospaces, a quantitative representation of the *deviations* from motion of an electromagnetic wave in empty space caused by motion within a physical medium such as our atmosphere (which is manifestly inhomogeneous, because of the local variation of the density, and anisotropic, because of the intrinsic angular momentum of Earth). A

known deviation is the replacement of the constant speed of light in vacuum c_o with a locally varying speed $c = c_o/n$ where n is the local index of refraction. An objective of isospaces is to provide quantitative predictions suitable for experimental tests of additional deviations from motion in empty space expected from the inhomogeneity and anisotropy of the medium itself. Primary emphasis is put in achieving, first, a *purely classical description* of said inhomogeneity and anisotropy studied in this chapter and in Ch. I.5, with operator descriptions to be considered only thereafter in Vol. II.[16] Another important distinction is that between the isospaces themselves, studied in this chapter, and the isogeometries defined on them, which are studied in Ch. I.5.

transformations defined on them. In turn, the lifting of the transformations implies that of algebras, groups, geometries, etc., according to the sequence:

$$\begin{aligned} &\text{isonumbers} \rightarrow \text{isofields} \rightarrow \text{isospaces} \rightarrow \text{isotransformations} \\ &\rightarrow \text{isoalgebras} \rightarrow \text{isogroups} \rightarrow \text{isosymmetries} \rightarrow \\ &\rightarrow \text{isorepresentations} \rightarrow \text{isogeometries}, \text{etc.} \end{aligned} \tag{3.1.4}$$

In this chapter we shall study the isotopies of the conventional spaces proposed for the first time in ref. [1] of 1983, under the name of *isotopic spaces* or *isospaces* for short, as the foundations of the isotopic generalization of the Lorentz group O(3.1) and of Einstein's special relativity for interior dynamical problems. The isospaces were then applied in ref.s [2,3] for the construction of the isotopies of the rotational symmetry O(3), as well as for the formulation of a general theorem on symmetries under isotopies. Isospaces were then used in monographs [4,5] for comprehensive applications in classical mechanics.

The *isodual spaces* and *isodual isospaces* were identified for the first time by this author in ref.s [2,3], and then applied in classical mechanics in monographs [4,5]. The first operator applications of isodual isospaces were done in ref. [6] while the most recent advances can be found in ref. [7]. A mathematical presentations is available in memoirs [8,9].

A first experimental verifications of isospaces can be found in ref. [10] which computes a modification of the Minkowski metric in in the interior of pions and kaons via conventional gauge theories in the Higgs sector. Additional independent experimental verifications can be found in ref.s [11,12] on the

[16] This is done to void the predictable attitude of attempting the interpretation of interior conditions via conventional means, such as inelastic scatterings of photons on atoms which, as such, reduce the interior problem to conventional exterior conditions. This attitude is precluded in these volumes because it eliminates the central geometric characteristics to be described, the inhomogeneity and anisotropy of the medium (recall the "No reduction theorems" of Sect. 1.2). Needless to say, we shall indeed consider second quantization and related photons, but only *after* achieving a classical and direct representation of the inhomogeneity and anisotropy of the medium in which the dynamical evolution holds.

behaviour of the meanlives of unstable hadrons with speed. Numerous additional applications and experimental verifications will be studied in Vol. II.

The "direct universality" of isospaces was first proved by Aringazin in ref. [13]. Additional studies on isospace were conducted by Lopez [14] in gravitation. An independent mathematical review of isospaces can be found in monograph [15].

In this chapter we shall study isospaces at the purely classical level in Kadeisvili's classification (Sect.s I.1.5 and I.2.3). The study of pseudoisospaces will be left to the interested readers. The isogeometries built on isospaces will be studied in Ch. I.5 also at the classical level. Operator formulations of both isospaces and their isogeometries are studied in Vol. II.

3.2: ISOSPACES AND THEIR ISODUALS

Let $F(\alpha,+,\times)$ be a field (Def. 2.3.1) with elements α, β, ..., conventional sum $\alpha + \beta$, multiplication $\alpha\times\beta = \alpha\beta$ and related additive and multiplicative units 0, and I, respectively. A *linear space* $V(a,F)$ (see, e.g., ref.s [16–18] for mathematical studies) is a set of elements a, b, c,... over a field $F(\alpha,+,\times)$ such to verify the following laws for all a, b, c $\in$ V and $\alpha, \beta, \gamma \in F$

$$a + b = b + a; \quad a + (b + c) = (a + b) + c; \tag{3.2.1}$$

and

$$\alpha (\beta a) = (\alpha \beta) a; \quad \alpha (a + b) = \alpha a + \alpha b; \quad (a + b) \alpha = \alpha a + \beta a; \tag{3.2.2}$$

plus, for every $a \in V$, there exists an element $-a$ such that

$$a + (-a) = a - a = 0; \tag{3.2.2}$$

From the above structural lines we can introduce the following:

Definition 3.2.1 [1,3]: *Given a linear space* $V(a,F)$ *over a field* $F(\alpha,+,\times)$, *the Class I "isotopes"* $\hat{V}(a,\hat{F})$ *of* V *called "isolinear spaces", are the same set of elements* a, b, c,... $\in$ V *although defined over the isofield of Class I* $\hat{F}(\hat{\alpha},+,\hat{\times})$ (Def. 2.3.1) *with elements* $\hat{\alpha} = \alpha\hat{1}$, $\hat{\beta} = \beta\hat{1}$, *conventional sum* $\hat{\alpha} + \hat{\beta}$, *isomultiplication* $\hat{\alpha}\hat{\times}\hat{\beta} = \hat{\alpha}\hat{T}\hat{\beta}$, *additive unit* 0, and multiplicative unit $\hat{1} = \hat{T}^{-1}$, *such to preserve all original axioms of* V, *i.e.,*

$$\hat{\alpha} \hat{\times} (\hat{\beta} \hat{\times} a) = (\hat{\alpha} \hat{\times} \hat{\beta}) \hat{\times} a, \quad \hat{\alpha} \hat{\times} (a + b) = \hat{\alpha} \hat{\times} a + \hat{\alpha} \hat{\times} b, \tag{3.2.4a}$$

$$(\hat{\alpha} + \hat{\beta}) \hat{\times} a = \hat{\alpha} \hat{\times} a + \hat{\beta} \hat{\times} a, \; \hat{\alpha} \hat{\times} (a + b) = \hat{\alpha} \hat{\times} a + \hat{\alpha} \hat{\times} b, \tag{3.2.4b}$$

for all a, b $\in \hat{V}$ *and* $\hat{\alpha}, \hat{\beta} \in \hat{F}$. *The "isodual isolinear spaces"* $\hat{V}^d(a^d,\hat{F}^d)$ *are Class II images of* $\hat{V}(a,\hat{F})$ *under the isoduality*

$$\hat{1} \rightarrow \hat{1}^d = -\hat{1}, \quad a \rightarrow a^d = -a \ , \tag{3.2.5}$$

and, as such, are defined over an isodual isofield $\hat{F}^d(\hat{\alpha}^d,+,\hat{\times}^d)$ *of Class II.*

Note the lifting of the field, but the elements of the vector space remain unchanged. The interested reader can prove as an exercise a number of properties of isolinear spaces and their isoduals via a simple isotopy of the corresponding properties of linear spaces [16]. One which is particularly relevant for these volumes follows from the invariance of the elements a, b, c, ... under isotopy as well as under isoduality and can be expressed as follows.

Proposition 3.2.1 [4]: *The basis of a (finite-dimensional) linear space remains unchanged under isotopy up to possible renormalization factors.*

The above property essentially anticipates the fact that, when studying later on Lie-isotopic algebras and their isoduals, we shall expect no alteration of its basis because Lie algebras are, first of all, linear spaces. In turn, this implies that hadronic mechanics preserves the *conventional* total conservation laws because, as well known, the generator of Lie symmetries are conserved quantities.

Linear spaces V are also called *vector spaces* [16] in which case their elements a, b, c, are called *vectors.* The isotopes $\hat{V}$ are then called *isovector spaces* and $\hat{V}^d$ are called *isodual isovector spaces.* Their elements a, b, c are then called *isovectors* and *isodual isovectors* , respectively [4]. Note the existence of the simpler *isodual vector spaces* V^d with *isodual vectors.*

Finally, note that *the formulation of isospaces via Kadeisvili's Class III unifies: vector, isovector, isodual vector and isodual isovector spaces.*

A *metric space* [16] hereon denoted S(x,g,F) is a (universal) set of elements x, y, z,.. over the fields F = F(α,+,×) equipped with a nonsingular, and Hermitean map (function) g: S × S → F, such that:

$$g(x, y) \geqq 0, \tag{3.2.6a}$$

$$g(x, y) = g(y, x) \ \ \forall \ x, y \in S; \quad g(x, y) = 0 \text{ iff } x = y \ , \tag{3.2.6b}$$

$$g(x, y) \leqq g(x, z) + g(y, z) \ , \quad \forall \ x, y, z \in S. \tag{3.2.6c}$$

A *pseudo-metric space,* hereon also denoted by S(x,g,F), occurs when the first condition (3.2.6a) is relaxed. Finally, recall that only metric or pseudo-metric

spaces over the reals $F = R(n,+,\times)$ are used in contemporary physics to characterize our physical space-time. Spaces over the complex numbers, such as the complex Hermitean Euclidean spaces $E(z,\delta,C)$ are used for unitary symmetries, such as SU(2) or SU(3).

Suppose that the space $S(x,g,F)$ is n-dimensional, and introduce the contravariant components $x = (x^i)$, $y = (y^i)$, $i = 1, 2, ..., n$. Then, the familiar way of realizing the map $g(x, y)$ is that via a (Hermitean) *metric* g of the form

$$g(x,y) = x^i g_{ij} y^j, \qquad \mathrm{Det.}\, g \neq 0, \quad g = g^\dagger . \tag{3.2.7}$$

The axiom $g(x, y) > 0$ for metric spaces then implies the condition that g is positive-definite, $g > 0$.

A celebrated physical example of metric spaces is the Euclidean space (3.1.1). Pseudo-metric spaces of primary physical relevance are the Minkowski space (3.1.2), and the Riemannian spaces (3.1.3).

The simplest possible way of constructing an infinite family of isotopes of $S(x,g,F)$ is by introducing n-dimensional isounits of Class I

$$\hat{1} = (\hat{1}^j_i), \quad i, j = 1, 2,..., n. \tag{3.2.8}$$

with isotopic elements

$$\hat{T} = \hat{1}^{-1} = (\hat{T}_i^j), \tag{3.2.9}$$

Then, we can introduce the notion of the *isomap* $\hat{g}: \hat{S}\times \hat{S} \rightarrow \hat{F}$ with realization

$$\hat{g}(x, y) = (x^i \hat{g}_{ij} y^j)\hat{1} \in F \tag{3.2.10}$$

where the quantity

$$\hat{g} = \hat{T} g = (\hat{T}_i^k g_{kj}), \tag{3.2.11}$$

is the *isometric* [1].

The *basis* $e = (e_i)$, $i = 1, 2, ..., n$, of an n-dimensional space $S(x,g,F)$ can be defined via the rule

$$g(e_i, e_j) = g_{ij} . \tag{3.2.12}$$

Then, the *isobasis* is characterized by

$$\hat{g}(\hat{e}_i, \hat{e}_j) = \hat{g}_{ij}, \tag{3.2.13}$$

The above isotopic generalizations can be expressed as follows.

Definition 3.2.2 [1]: *The "isotopic liftings" of Class I of a given, n-dimensional,*

metric or pseudometric space S(x,g,F) *over the field* $F = F(\alpha,+,\times)$, *called "isospaces", are given by the infinitely possible "isospaces"* $\hat{S}(\hat{x},\hat{g},\hat{F})$ *characterized by: a) the same dimension* n *of the original space, b) the isotopies of the original metric* g *into one of the infinitely possible nonsingular, Hermitean "isometric"* $\hat{g} = \hat{T}g$ *with isotopic element* $\hat{T}$ *of Class I depending on the local variables* x, *their derivatives* $\dot{x}$, $\ddot{x}$, ... *with respect to an independent variable* t, the *local density* μ, *the local temperature* τ, *the local index of refraction* n, *as well as any needed additional quantity (such as wavefunctions* ψ *and their derivative for operator formulations)*

$$g \to \hat{g} = \hat{T}\, g, \tag{3.2.14a}$$

$$\hat{T} = \hat{T}(s, x, \dot{x}, \ddot{x}, \mu, \tau, n, ...), \quad \det \hat{T} \neq 0, \quad \hat{T}^{\dagger} = \hat{T} > 0, \tag{3.2.14b}$$

$$\det.\, \hat{g} \neq 0, \quad \hat{g} = \hat{g}^{\dagger}, \tag{3.2.14c}$$

$$\hat{x}^k \equiv x^k, \qquad \hat{x}_k = \hat{g}_{ki}\hat{x}^i = \hat{g}_{ki}x^i \neq x_k = g_{ki}x^i, \tag{3.2.14d}$$

and c) the lifting of the field $F(\alpha,+,\times)$ *into an isotope of Class I* $\hat{F}(\hat{\alpha},+,*)$ *whose isounit* $\hat{1}$ *is the inverse of the isotopic element* $\hat{T}$, *i.e.,*

$$\hat{1} = \hat{T}^{-1}, \tag{3.2.15}$$

with "isocomposition" on $\hat{F}$

$$(\hat{x}\, \hat{,}\, \hat{y}) = (x, \hat{T}\, y)\hat{1} = (\hat{T}\, x, y)\hat{1} = \hat{1}\, (x, \hat{T}\, y) = (x^i\, \hat{g}_{ij}\, y^j)\hat{1} \in \hat{F}. \tag{3.2.16}$$

The "isodual isospaces" of Class II $\hat{S}^d\, (\hat{x}^d, \hat{g}^d, \hat{F}^d)$ *are given by the image of* $\hat{S}(\hat{x},\hat{g},\hat{F})$ *under isoduality and are defined by the map*

$$\hat{g} \to \hat{g}^d = \hat{T}^d\, g, \quad \hat{T}^d = -\hat{T}, \tag{3.2.17a}$$

$$\hat{1} \to \hat{1}^d = (\hat{T}^d)^{-1} = -\hat{1}, \tag{3.2.17b}$$

$$\hat{x} \to \hat{x}^d = -\hat{x}, \tag{3.2.17c}$$

with "isodual isocomposition" in $\hat{F}^d$

$$(\hat{x}\, \hat{,}\, \hat{y})^d = (x, \hat{T}^d\, y)\hat{1}^d = (\hat{T}^d\, x, y)\hat{1}^d =$$

$$= \hat{1}^d\, (x, \hat{T}^d\, y) = (\hat{x}^{id}\, \hat{g}^d{}_{ij}\, \hat{y}^{jd})\hat{1}^d \in \hat{F}^d. \tag{3.2.18}$$

A few comments are now in order. The first and geometrically most dominant aspect is that, because of the unrestricted functional dependence of the isotopic element $\hat{T}$, the isometrics $\hat{g} = \hat{T}g$ are generally of *integral* type.

Thus, *the isotopic liftings* $S(x,g,F) \rightarrow \hat{S}(\hat{x},\hat{g},\hat{F})$ *imply a nonlocal-integral generalization of the original local-differential space.* In particular, isospaces require a suitable integral topology for their rigorous treatment which is vastly unexplored at this time at the pure mathematical level.

However, all integral terms are embedded, by construction, in the isounits $\hat{1}$. On the other hand, topologies are known to be insensitive to the functional dependence of their own units, provided that they are positive-definite. This implies the particular integro-differential topology of hadronic mechanics whereby conventional topologies hold everywhere except at the unit (see Fig. 1.4.1 for a conceptual basis and Chapter I.6 for a treatment).

Moreover, again from the arbitrariness of the functional dependence of the isotopic element $\hat{T}$, one can readily see that *the isotopies* $S(x,g,F) \rightarrow \hat{S}(\hat{x},\hat{g},\hat{F})$ *imply nonlinear, nonlocal and noncanonical generalizations of the original spaces*, where the nonlinearity is in all variables and their derivatives.

Finally note from an abstract viewpoint that the distinction in the use of different fields is meaningful in the conventional metric or pseudo-metric spaces. However, at the isotopic level such a distinction cease to exists because of the isotopic unification of all fields and isofields of Theorem I.2.9.1.

Isospaces can also be distinguished via Kadeisvili's classification depending on the characteristics of the unit (Sect. 1.5) into:

Isospaces properly speaking (Class I),
isodual isospaces (Class II)
Indefinite isospaces (Class III),
Singular isospaces (Class IV], and
General isospaces (Class V).

In this section we shall solely study isospaces of Classes I, II and III, with few comments on isospaces of Class IV.

An important property derived from Proposition I.3.2.1 is that *the basis of a metric or pseudo-metric space remains unchanged under isotopies (except for renormalization factors)*.

As indicated earlier, isospaces are bona-fide nonlinear, nonlocal and noncanonical generalizations of the original spaces. Despite the above differences, we have the following

Theorem 3.2.1 [1]: *Isospaces of Class I* $\hat{S}(\hat{x},\hat{g},\hat{F})$ *(isodual isospaces of Class II* $\hat{S}^d(\hat{x}^d,\hat{g}^d,\hat{F}^d)$*) are locally isomorphic to the original spaces* $S(x,g,\hat{F})$ *(isodual space* $S^d(x^d,g^d,F^d)$*).*

The above simple mathematical property has fundamental physical implications because, since a given space $S(x,g,F)$ and its isotope $\hat{S}(\hat{x},\hat{g},\hat{F})$ are locally isomorphic, so are expected to be the corresponding groups of isometries.

This implies that the isotopies of Class I of space-time symmetries such as the rotation, Lorentz, Poincaré and unitary symmetries will be locally isomorphic to the original symmetries. Nevertheless, the explicit form of the transformations will be generally nonlinear, nonlocal and noncanonical, thus achieving the desired structural generalization of conventional symmetry transformations to represent interior problems, while achieving a geometric unity with the axiomatic structure of the exterior problem.

Note the necessity for these isomorphisms of the *joint* liftings

$$g \rightarrow \hat{g} = \hat{T}\, g \quad \textit{and} \quad F \rightarrow \hat{F}, \quad \hat{1} = \hat{T}^{-1}. \tag{3.2.19}$$

In fact, a lifting of the type $S(x,g,F) \rightarrow S'(\hat{x},\hat{g},F)$, $\hat{g} = \hat{T}\, g$, without the joint lifting of the base field *is not* an isotopy and the spaces $S(x,g,F)$ and $S(x,\hat{g},F)$ are generally *non-isomorphic*.

The same mechanism of joint lifting of the metric and of the field characterizes the isodualities of the Euclidean spaces $E(x,\delta,R) \rightarrow E^d(x^d,\delta^d,R^d)$, the Minkowski space $M(x,\eta,R) \rightarrow M^d(x^d,\eta^d,R^d)$ and the Riemannian spaces $\Re(x,g,R) \rightarrow \Re^d(x^d,g^d,R^d)$, which are at the foundation of our characterization of antimatter [6].

From property (3.2.18) we have the following

Proposition 3.2.2 [4,5]: *Compositions* (x, y) *on a given space* $S(x,g,F)$ *and their isotopes* $(\hat{x},\hat{}\,\hat{y})$ *on isospaces* $\hat{S}(\hat{x},\hat{g},\hat{F})$ *are isoselfdual, i.e., invariant under isoduality*

$$(\hat{x},\hat{}\,\hat{y}) = (\hat{x}^i\, \hat{g}_{ij}\, \hat{y}^j)\, \hat{1} \;\equiv\; (\hat{x},\hat{}\,\hat{y})^d = (\hat{x}^{id}\, \hat{g}^d{}_{ij}\, \hat{y}^{jd})\, \hat{1}^d. \tag{3.2.20}$$

The above simple property has rather important physical implications studies later on, such as the novel *universal invariance of physical laws under isoduality* (established at the classical level in monograph [5] and studied at the particle level in these volumes), or the causality of motion backward in time referred to an isodual field (because evidently equivalent to motion forward in time referred to an ordinary field).

Scalar functions $f(\hat{x})$ on isospaces $\hat{S}(\hat{x},\hat{g},\hat{F})$ are ordinary functions in the new coordinates $\hat{x}$. An *scalar isofunction* $\hat{f}(\hat{x})$ on $\hat{S}(\hat{x},\hat{g},\hat{F})$ is a function with values on the isofield, i.e.,

$$\hat{f}(\hat{x}) = f(\hat{x})\, \hat{1} \in \hat{F}. \tag{3.2.21}$$

As it happens for isonumbers, conventional elements of a space can be preserved

(although their operations are lifted), or they can be themselves lifted. As we shall see, this implies nontrivial consequences in functional analysis, e.g., the existence of *two*, rather than one, isotopies of Dirac's delta function.

It should be indicated that in Definition 3.2.1 *the local coordinates* $\hat{x} \in \hat{S}(x,\hat{g},\hat{F})$ *are assumed to be ordinary scalars and not isoscalars* (that is, they do not have the isounit as factors). One can then build an isospace $\hat{S}(\hat{x},\hat{g},\hat{F})$ with *isocoordinates*

$$\hat{x}^k = x^k \hat{1}, \tag{3.2.22}$$

in which case all products have to be isotopic, thus resulting in the same composition law

$$\hat{x}^{\hat{2}} = \hat{x}^t \hat{\times} \hat{g} \hat{\times} \hat{x} = (x^t \hat{g} x)\hat{1}. \tag{3.2.23}$$

The interchange between the isotopic element and the isounit

$$\hat{T} \rightarrow \hat{1}, \tag{3.2.24}$$

is called *isoreciprocity map* [6].

We finally note the following important property:

Proposition 3.2.3 [1]: *The basic isounits of any given isospace coincide with that of the underlying isofield.*

This property does not hold for the conventional case in which the unit of the space is the n-dimensional *unit matrix* I = diag. (1, 1, ..., 1) while that of the base field is the *number* +1. Nevertheless, one should note that the isotopic methods of the preceding chapter readily permits the reformulation of the *conventional* metric or pseudo-metric spaces in which the unit of the space and of the field coincide and are given by I = diag. (1, 1, ..., 1).

By recalling that the basic unit of hadronic mechanics, Eq. (1.1.1) is outside conventional fields, and by recalling Proposition I.2.3.1 and I.2.3.2, the isospaces of primary relevance for hadronic mechanics are given by the structures $\hat{S}(x,\hat{g},\hat{F})$ of Definition 3.2.1 specialized to the cases of isoreal and isocomplex fields $\hat{F} = \hat{R}, \hat{C}$, plus their image under isoduality $\hat{S}(\hat{x}^d,\hat{g}^d,\hat{R}^d)$ and under isoreciprocity $\hat{S}(\hat{x},\hat{g},\hat{R})$.

A third class of generalized spaces are conveivable, those based on the hyperstructures, which are expected to characterize *hyperspaces* over the *hyperfields* indicated in Ch. I.2. These latter spaces are excessively general for the analysis of these volumes and will be ignored for simplicity.

3.3: ISOTOPIC UNIFICATION OF SPACES AND ISOSPACES

In Sect. I.2.9 we initiated our presentation of the unifying power of isotopic techniques, beginning with the unification of all conventional numbers into the single abstract notion of isoreal numbers $\mathfrak{R}$ of Class III.

We now illustrate this unifying power for spaces and isospaces. We shall then show in Sect. I.3.7 that this is not a sterile mathematical properties, because it permits the geometric unification of the special and general relativities which in turn, is at the foundation of their isotopies studied in Vol. II [5].

The capability of isospaces of unifying all conventional spaces was identified by the author in their original proposal [1]. By using subsequent advances, the property can be expressed as follows:

Theorem 3.3.1 [loc. cit.]: *All possible metric and pseudometric spaces in n-dimension* $S(x,g,F)$ *plus all their infinitely possible isotopic images* $\hat{S}(\hat{x},\hat{g},\hat{F})$, *the isodual spaces* $S^d(x^d,g^d,F^d)$ *and the isodual isospaces* $\hat{S}^d\ (\hat{x}^d,\hat{g}^d,\hat{F}^d)$ *can be unified into one, single notion, the abstract n-dimensional isoeuclidean space* $\hat{\mathfrak{E}}(\hat{x},\hat{\delta},\hat{\mathfrak{R}})$ *of Class III over the abstract isoreals* $\mathfrak{R}$.

In fact, the assumption of Class III implies the relaxation of the positive- or negative-definite character of the isounit. The property then follows from the fact that all infinitely possible metric g and isometrics $\hat{g}$, as well as all their possible isoduals g^d and $\hat{g}^d$ of the same dimension can be unified into the isotopies of the Euclidean metric $\delta = \text{diag.}\,(1, 1, ..., 1)$

$$\delta \quad \rightarrow \quad \hat{\delta} = \hat{T}\,\delta, \quad \hat{T} = g \quad \text{or } \hat{g} \text{ or } g^d \quad \text{or } \hat{g}^d \tag{3.3.1}$$

Thus, from a mathematical viewpoint, there is no need to study the isotopies of individual spaces, because those of the fundamental Euclidean space are sufficient, and inclusive of all others. This is the reason why ref. [1] studies the isotopies of the (3+1)-dimensional *Minkowski* space as a particular case of the isotopies of the 4-dimensional *Euclidean* space.

Note that the distinctions between spaces over the real or complex numbers are lost under Theorem 3.3.1 because all fields and isofields are particular case of the abstract isoreals $\mathfrak{R}$ (Theorem I.2.9.1).

3.4: ISOEUCLIDEAN SPACES AND THEIR ISODUALS

Isospaces are so fundamental for the study of hadronic mechanics to warrant a

brief individual study of the most important ones prior to the study of their geometries. We therefore begin with the following:

Definition 3.4.1 [1]: *The liftings of the conventional n-dimensional Euclidean spaces* E(r,δ,R) *over the reals* R(n,+,×), *Eq.s (3.1.1), into the "isoeuclidean spaces" of Class I are given by*

$$E(r,\delta,R) \rightarrow \hat{E}(\hat{r},\hat{\delta},\hat{R}), \tag{3.4.1a}$$

$$\delta = I_{n\times n} \rightarrow \hat{\delta} = \hat{T}(t, r, \dot{r}, \ddot{r}, \mu, \tau, n, \ldots)\,\delta, \tag{3.4.1b}$$

$$\det \delta = 1 \neq 0,\ \delta = \delta^{\dagger} \rightarrow \det.\hat{\delta} \neq 0,\quad \hat{\delta} = \hat{\delta}^{\dagger}, \tag{3.4.1c}$$

$$R(n,+,\times) \rightarrow \hat{R}(\hat{n},+,\hat{\times}),\ \hat{n} = n\hat{1},\quad \hat{1} = \hat{T}^{-1} = \hat{\delta}^{-1} \tag{3.4.1d}$$

$$r^2 = (r, r) = r^i\,\delta_{ij}\,r^j \rightarrow \hat{r}^{\hat{2}} = (\hat{r}\,\hat{,}\,\hat{r}) = (\hat{r}\,\hat{\delta}\,\hat{r})\hat{1} =$$

$$= (\hat{\delta}\hat{r}, \hat{r})\hat{1} = \hat{1}(\hat{r}, \hat{\delta}\hat{r}) = [\hat{r}^i\,\hat{\delta}_{ij}(r, \dot{r}, \ddot{r}, \ldots)\,\hat{r}^j]\,\hat{1} \in \hat{R}, \tag{3.4.1e}$$

$$\hat{r} = \{\hat{r}^k\} = \{r^k\} \equiv r,\ \hat{r}_k = \hat{\delta}_{ki}\,\hat{r}^i = \hat{\delta}_{ki}\,r^i \neq r_k, \tag{3.4.1f}$$

where the isofield $\hat{R}(\hat{n},+,\hat{\times})$ *is of Class I. The "isodual isoeuclidean spaces" of Class II are given by the isodual image of the preceding ones*

$$\hat{E}(\hat{r},\hat{\delta},\hat{R}) \rightarrow \hat{E}^d(\hat{r}^d,\hat{\delta}^d,\hat{R}^d), \tag{3.4.2a}$$

$$\hat{\delta} = \hat{T}\,\delta \rightarrow \hat{\delta}^d = \hat{T}^d\,\delta = -\hat{\delta},\ \hat{T}^d = -\hat{T}, \tag{3.4.2b}$$

$$\hat{R}(\hat{n},+,\hat{\times}) \rightarrow \hat{R}^d(\hat{n}^d,+,\hat{\times}^d),\ \hat{n}^d = \hat{n}\,\hat{1}^d,\ \hat{1}^d = -\hat{1} \tag{3.4.2c}$$

$$\hat{r}^{\hat{2}} = (\hat{r}\,\hat{,}\,\hat{r}) = (\hat{r}^i\,\hat{\delta}_{ij}\,\hat{r}^j)\,\hat{1} \rightarrow (\hat{r}^d)^{\hat{2}d} = (\hat{r}\,\hat{,}\,\hat{r})^d = (\hat{r}^{id}\,\hat{\delta}^d_{ij}\,\hat{r}^{jd})\,\hat{1}^d \equiv r^2 \tag{3.4.2d}$$

$$\hat{r} \rightarrow \hat{r}^d = -\hat{r}. \tag{3.4.2e}$$

The n-dimensional complex Hermitean Euclidean spaces E(z,δ,C) *with separation*

$$z^{\dagger}\,\delta\,z = \bar{z}^i\,\delta_{ij}\,z^j, \tag{3.4.3}$$

is lifted into the "complex isoeuclidean spaces" of Class I

$$\hat{E}(\hat{z},\hat{\delta},\hat{C}):\qquad (\hat{z}^{\dagger}\,\hat{\delta}\,\hat{z})\,\hat{1} = (\bar{z}^i\,\hat{\delta}_{ij}\,z^j)\,\hat{1}, \tag{3.4.4a}$$

$$\hat{\delta} = \hat{T}\,\delta,\ \hat{T} \equiv \hat{T}^{\dagger},\ \hat{1} = \hat{T}^{-1} > 0. \tag{3.4.4b}$$

where upper bar denotes complex conjugation. The "isodual complex Hermitean

isoeuclidean spaces" are instead given by

$$\hat{E}^d(\hat{z},\hat{\delta}^d,\hat{C}^d) : \qquad (\hat{z}^{d\dagger}\hat{\delta}^d\hat{z})\,\hat{1}^d = (\bar{z}^{id}\,\delta^d{}_{ij}\,z^{jd})\,\hat{1}^d, \tag{3.4.5a}$$

$$\hat{\delta}^d = \hat{T}^d\,\delta,\ \hat{T}^d = -\hat{T},\ \hat{1}^d = -\hat{1}. \tag{3.4.5b}$$

We now outline a few mathematical and physical aspects of isoeuclidean spaces for subsequent more detailed treatment. The applications of isoeuclidean spaces are of three primary types:

A) Geometric applications. Recall that the conventional Euclidean metric $\delta = \text{diag.}(1, 1, 1)$ is a geometrization of the perfect rigid sphere with unit radius. From their topological characteristics, isometrics of Class I can always be diagonalized. We can therefore always assume the realization of the isotopic element in the diagonal form

$$\hat{T} = \text{diag.}(b_1{}^2, b_2{}^2, b_3{}^2),\ b_k = b_k(t, r, \dot{r}, \ddot{r}, \ldots) > 0,\ k = 1, 2, 3, \tag{3.4.6}$$

where the b's are called *characteristics functions of the isospace.*

The first geometrical application of isospaces is therefore that of representing all infinitely possible deformations of the original perfect sphere $\delta = \text{diag.}(1, 1, 1)$ into the ellipsoids with semiaxes

$$\hat{1} = \text{diag.}(b_1{}^{-2}, b_2{}^{-2}, b_3{}^{-2}), \tag{3.4.7}$$

where the functional dependence expresses the physical origin of the deformations as due to local pressures, densities, temperatures, etc. We therefore have the following:

Geometric meaning: *The isounit permits a direct representation of the actual nonspherical shape of a given body as well as the representation of all its infinitely possible deformations.*

As we shall see, this capability exists at the pure classical level [5] and then simply persists under operator formulations prior to any second quantization or use of form factors.

The main geometric point addressed here is that *extended, generally nonspherical and deformable shapes are outside the representational capabilities of a Lagrangian or Hamiltonian.* This requires their classical representation with *any suitable quantity except the Lagrangian or the Hamiltonian.* The classical representation of extended, nonspherical and deformable shapes adopted in these volumes is that

permitted by the generalization of the basic unit of the carrier space. Even though not unique, the representation is simple, effective and "directly universal", that is, capable of representing all possible shapes (universality] directly in the coordinates of the observer (direct universality).

It should be noted that the above features render applicable the isoeuclidean spaces also for extended particles *in vacuum*, e.g., when they experience deformations due to external fields, such as a charged sphere in vacuum under the influence of an intense electric field. Even though there are no nonhamiltonian interactions, the generalization of the unit is still effective, as shown in details in ref. [5], because the physical event of deformation of shape is conceptually, geometrically and analytically outside the representational capabilities of the Hamiltonian.

B) Analytic applications. As ut is well known, nonrelativistic exterior dynamical problems are representable via conventional analytic equations, such as Lagrange equations, which are defined on the 3-dimensional Euclidean space $E(r,\delta,R)$ (plus an additional one dimensional space representing time, see below). In this case the trajectory in vacuum is solely characterized by *one single quantity,* the Lagrangian $L = K - V$, with kinetic K and potential energy V.

The main objective of the isotopies is the representation of interior dynamical problems with forces represented with the conventional potential V, plus contact, nonlinear-nonlocal-nonlagrangian[17] forces due to the medium. In this latter case, the system is represented by *two independent quantities,* the Lagrangian $L = K - V$ *and* the isounit $\hat{1}$. We therefore have the following

Analytic meaning: *The isounit permits a direct representation of contact, nonlinear-nonlocal-nonlagrangian forces for interior physical conditions.*

The Lagrangian L must now be properly written in isoeuclidean space $\hat{E}(\hat{r},\hat{\delta},\hat{R})$. This results in expression $\hat{L}(\hat{r},\hat{v}) = \hat{K}(\hat{v}) - \hat{V}(\hat{r})$,. $\hat{v} = d\hat{r}/dt$ which is defined in terms of the conventional contravariant coordinates $\hat{r}$ and velocities $\hat{v}$, although all their operations are now of isotopic character.

In particular, the *isokinetic energy* is given by

$$\hat{K} = \tfrac{1}{2} m \hat{v}^{\hat{2}} = \tfrac{1}{2} m \hat{v}^t \hat{\delta} \hat{v} = \tfrac{1}{2} m \hat{v}^i \hat{\delta}_{ij} \hat{v}^j , \qquad (3.4.8)$$

and the *isopotential energy* is given by the isotopic image of the function $V(\hat{r})$

[17] By "nonlagrangian" we mean hereon non-first-order Lagrangians, namely, equations of motion which violate the integrability conditions for their representation via first-order Lagrangians $L = L(t, r, \dot{r})$. Evidently, higher order Lagrangian may exist, e.g., $L = L(t, r, \dot{r}, \ddot{r})$. The point is that, there is no (conventional) Hamiltonian for a Lagrangian of order higher than the first. The term "noncanonical" is then used as a synonym of "non-first-order-Lagrangian".

(see Ch. 6), e.g., for the case of V(r) depending on the norm x

$$\hat{V}(\hat{r}) = V(\hat{r}): \quad \hat{r} = (\hat{r}^t \hat{\delta} \hat{r})^{\frac{1}{2}}. \tag{3.4.9}$$

The isotopies of the analytic equations in which the above isofunctions are defined with various examples are preliminarily studied in Ch. I.5 and then studied in detail in Vol. II.

Even though not claimed to be unique, the above isotopic representation is effective and "directly universal" in classical mechanics, that is, capable of representing all possible Newtonian and non-Newtonian nonlinear, integro-differential, nonselfadjoint systems [5], as we shall see in detail in Vol. II. Such classical effectiveness and "direct universality" will then imply corresponding properties in operator formulations, as studied in Vol.s Ii and III.

By adding the preceding geometric meaning of the isounit, one can see that *isospaces provide a direct geometrization of the inhomogeneity and anisotropy of physical media* (Fig. 3.1.1). In fact, the inhomogeneity can be represented in isospaces, e.g., via a dependence of the isometric $\hat{\delta}$ on the locally varying density μ. The anisotropy, e.g., due to the presence of an intrinsic angular momentum along the direction $\vec{n}$, is then representable via a factorization of such a preferred direction in the isometric much along the Finslerian geometry, or via the differentiations $b_1 \neq b_2 \neq b_3$.

Note that the representation is "direct" because occurring directly in the isometric itself, without any need of operator formulations or any use of artificial or indirect approaches.

Thus, the transition from exterior to interior conditions is done via a generalization of the basic unit $I \rightarrow \hat{1}$. A condition the reader should keep in mind to avoid undetected inconsistencies is that in most physical applications the isounits $\hat{1}$ are constructed in such a way to recover the conventional unit identically in the exterior problem.

This condition can be realized by assuming that the entire matter of the medium considered is enclosed in a minimal surface S° with local radius R° and density μ, in which case

$$\hat{1}_{r \geq R^\circ} \equiv I = \mathrm{diag.}(1, 1, 1), \text{ or } \qquad \mathrm{Lim}_{\mu \rightarrow 0} \hat{1} \equiv I . \tag{3.4.10}$$

Note that the 3-dimensional Euclidean "space" is one. On the contrary, there exist infinitely many 3-dimensional isoeuclidean "spaces". This is evidently due to the infinitely possible isometrics $\hat{\delta}$ representing the infinitely possible physical conditions of interior problems.

We finally remark that, when overall notions are needed, that is, the quantities are referred to the physical medium as a whole, the characteristic b-functions can be averaged into constants

$$b^{\circ}_k = < b_k(t, r, \dot{r}, ...) > , \quad k = 1, 2, 3. \qquad (3.4.11)$$

As we shall see, constant isotopic elements $\hat{T}$ and characteristic b°-quantities will have numerous applications. The point to keep in mind is that such constancy is in actuality an average over a rather complex functional dependence.

C) Algebraic applications: Recall that the unit I = diag. (1, 1, 1) of the Euclidean space is the fundamental unit of the related Lie's theory, e.g., the unit of the group of isometries of the Euclidean space, the orthogonal group O(3). The following property is then consequential

Algebraic meaning: *The isounit constitutes the basic generalized units of the Lie-isotopic theory(studied in the next chapter).*

As we shall see in the next chapter, the isotopies of Lie's theory for the achievement of nonlinear, nonlocal and noncanonical realizations of conventional space-time and unitary symmetries is based precisely on the isotopies $I \rightarrow \hat{1}$.

The following property is a consequence of Theorem 4.2.1.

Corollary 3.2.1A: *Isoeuclidean spaces* $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ *of Class I (isodual isoeuclidean spaces of Class II* $\hat{E}^d(\hat{r}^d,\hat{\delta}^d\hat{R}^d))$ *are locally isomorphic to the conventional Euclidean spaces of the same dimension* $E(r,\delta,R)$ *(isodual Euclidean spaces of the same dimension* $E^d(r^d,\delta^d R^d))$.

We shall say that, from a geometrical viewpoint, Euclidean spaces and their isotopes of Class I are equivalent, as ensured by the preservation of the original axioms, as well as the identity of the two spaces at the abstract level, and we shall write $E(r,\delta,R) \approx \hat{E}(\hat{r},\hat{\delta},\hat{R})$.

We close this section with the identification of the isoeuclidean spaces used in nonrelativistic hadronic mechanics, inclusive of a time component.

Definition 3.4.2 [5]: *The "nonrelativistic isotopic space-time" of hadronic mechanics of Class I is given by the Cartesian product of two isoeuclidean spaces, one representing time and the other representing space with corresponding isounits* $\hat{1}_t$ *and* $\hat{1}$*, isocomposition*

$$\hat{E}(t,\hat{R}_t) \times \hat{E}(\hat{r},\hat{\delta},\hat{R}) :. \quad \hat{t}^2 = (t\, T_t\, t)\hat{1}_t \in \hat{R}_t, \quad \hat{1}_t = \hat{T}_t^{-1}, \hat{t} = t \qquad (3.4.12)$$

$$\hat{r}^2 = (\hat{r}^t\, \hat{T}\, \delta\, \hat{r})\hat{1} \in \hat{R}, \quad \hat{1} = \hat{T}^{-1}, \qquad (3.4.11)$$

and diagonal realization

$$\hat{T}_t = b_4^2, \qquad b_4 = b_4(t, r, \dot{r}, \mu, \tau, n, \ldots) > 0, \tag{3.4.12a}$$

$$\hat{T} = \text{diag.}(b_1^2, b_2^2, b_3^2), \qquad b_k = b_k(t, r, \dot{r}, \ddot{r}, \ldots) > 0 \tag{3.4.12b}$$

The "isodual nonrelativistic isotopic space-time" of hadronic mechanics of Class II is then given by

$$\hat{E}^d(\hat{t}^d,\hat{R}^d{}_t) \times \hat{E}^d(\hat{r}^d,\delta^d,\hat{R}^d): \quad (\hat{t}^d)^{\hat{2}d} = (t\,\hat{T}^d{}_t\,t)\,\hat{1}^d{}_t \in \hat{R}^d{}_t, \tag{3.4.13a}$$

$$(\hat{r}^d)^{\hat{2}d} = (\hat{r}^{dt}\,\hat{T}^d\,\delta\,\hat{r}^d)\,\hat{1}^d \in \hat{R}, \tag{3.4.13b}$$

$$\hat{1}^d{}_t = (\hat{T}^d{}_t)^{-1} = -\hat{1}_t, \qquad \hat{1}^d = (\hat{T}^d)^{-1} = -I, \tag{3.4.13c}$$

$$\hat{T}^d{}_t = -b_4^2, \quad b_4{}^d = -b_4(t, r, \dot{r}, \mu, \tau, n, \ldots) > 0, \tag{3.4.13d}$$

$$\hat{T}^d = \text{diag.}(-b_1^2, -b_2^2, -b_3^2), \quad b_k{}^d = -b_k(t, r, \dot{r}, \ddot{r}, \ldots) > 0. \tag{3.4.13e}$$

As it is the case for all other quantities, the above definition implies the existence of four distinguishable nonrelativistic times in hadronic mechanics:

Time, as the usual element t of the field of real numbers $R(t,+,\times)$;
Isotime, the element $\hat{t} = t\,\hat{1}_t \in \hat{R}(\hat{t},+,\hat{\times}_t)$;
Isodual time, the element $t^d = t\,I^d = -t \in R^d(t^d,+,\times^d{}_t)$;
Isodual isotime, the element $\hat{t}^d = t\,\hat{1}^d{}_t = -\hat{t} \in \hat{R}^d(\hat{t}^d,+,\hat{\times}^d{}_t)$.

The following property is a consequence of the theory of isonumbers of the preceding chapter.

Proposition 3.4.1: *The direction of time (Eddington's "time arrow") changes sign in the transition from our space-time to its isodual.*

In fact, under isoduality, we have the map of our time $t \in R(t, R_t)$

$$t > 0 \quad \rightarrow \quad t^d = t\,I^d = -t < 0, \tag{3.4.14}$$

and the same result occurs under isotopy.

The fundamental aspect here is *the change of the unit of time in the transition from one to another of the about different times, and a similar situation occurs for space.* As we shall see in Vol.s Ii and II, this occurrence has rather fundamental physical implications.

The isotopies of time have been introduced here via the purely mathematical use of the methods studied until now. Nevertheless, as we shall see in the next section, the time isotopies emerge rather forcefully from the

nonrelativistic limit of relativistic isotopic theories.

3.5: ISOMINKOWSKI SPACES AND THEIR ISODUALS

We now study the central carrier spaces of the relativistic hadronic mechanics, which can be introduced according to the following:

Definition 3.5.1 [1]: *The isotopic liftings of Class I of the conventional (3+1)-dimensional Minkowski space* $M(x,\eta,R)$ *over the reals* $R(n,+,\times)$ *are given by the isotopes called "isominkowski spaces"*

$$M(x,\eta,R) \rightarrow \hat{M}(\hat{x},\hat{\eta},\hat{R}), \tag{3.5.1a}$$

$$\eta = \mathrm{diag.}(1,\ 1,\ 1,\ -1) \rightarrow \hat{\eta} = \hat{T}(x,\ \dot{x},\ \ddot{x},\ \tau,\ n,..)\ \eta, \tag{3.5.1b}$$

$$\det \eta = -1 \neq 0, \quad \eta = \eta^{\dagger} \rightarrow \det \hat{\eta} \neq 0, \quad \hat{\eta}^{\dagger} = \hat{\eta}, \tag{3.5.1c}$$

$$R(n,+,\times) \rightarrow \hat{R}(\hat{n},+,\hat{\times}), \quad \hat{n} = n\,\hat{1}, \quad \hat{1} = T^{-1}, \tag{3.5.1d}$$

$$x^2 = (x,\ x) = x^{\mu}\,\eta_{\mu\nu}\,x^{\nu} \rightarrow \hat{x}^{\hat{2}} = (\hat{x}\,\hat{,}\,\hat{x}) = (\hat{x},\ \hat{T}\,\hat{x})\,\hat{1} =$$

$$(\hat{T}\,\hat{x},\ \hat{y})\,\hat{1} = \hat{1}(\hat{x},\ \hat{T}\,\hat{y}) = [x^{\mu}\,\hat{\eta}_{\mu\nu}(x,\ \dot{x},\ \ddot{x},...)\,x^{\nu}\,]\,\hat{1}, \tag{3.5.1e}$$

$$\hat{x} = \{\,\hat{x}^{\mu}\,\} = \{\,x^{\mu}\,\} \equiv x, \quad \hat{x}_{\mu} = \hat{\eta}_{\mu\nu}\,\hat{x}^{\nu} \neq x_{\mu} = \eta_{\mu\nu}\,x^{\nu},\ \mu,\ \nu = 1,\ 2,\ 3,\ 4, \tag{3.5.1f}$$

with diagonal realization of the isounit and isoseparation

$$\hat{1} = \mathrm{diag.}\,(\,b_1^{-2},\ b_2^{-2},\ b_3^{-2},\ b_4^{-2}\,) > 0, \quad b_{\mu} = b_{\mu}(x,\ \dot{x},\ \ldots\,..) > 0, \tag{3.5.2a}$$

$$\hat{x}^2 = (\,x^1\,b_1^{\,2}\,x^1 + x^2\,b_2^{\,2}\,x^2 + x^3\,b_2^{\,2}\,x^3 - x^4\,b_4^{\,2}\,x^4\,)\,\hat{1} \in \hat{R}\,. \tag{3.5.2b}$$

invariant measure

$$d\hat{s}^{\hat{2}} = (\,-\,d\hat{x}^{\mu}\,\hat{\eta}_{\mu\nu}\,d\hat{x}^{\nu}\,)\,\hat{1}, \tag{3.5.3}$$

and characteristic constants

$$b^{\circ}_{\mu} = <b_{\mu}(x,\ \dot{x},\ \ldots.)>, \quad \mu = 1,\ 2,\ 3,\ 4 \tag{3.5.4}$$

derived via a given averaging procedure <...>. The "isodual isominkowski spaces" of Class II are given by

$$\hat{M}^d(\hat{x}^d,\hat{\eta}^d,\hat{R}^d): \; \hat{\eta}^d = \hat{T}^d(x, \dot{x}, \ddot{x}, \mu, \tau, n,..)\, \eta = -\hat{\eta}, \tag{3.5.5a}$$

$$\hat{T}^d = -\hat{T}, \qquad \hat{1}^d = (\hat{T}^d)^{-1} = -\hat{1}\,, \quad \hat{x}^d = -\hat{x}\,, \tag{3.5.5b}$$

$$(\hat{x}^d)^{\hat{2}\,d} = (\hat{x}\,\hat{,}\,\hat{x})^d = (\hat{x}^d, \hat{T}^d x^d)\hat{1}^d = (\hat{T}^d \hat{x}^d, \hat{y}^d)\hat{1}^d = \hat{1}^d (\hat{x}^d, \hat{T}^d \hat{y}^d) =$$

$$[\hat{x}^{\mu d}\, \hat{\eta}^d_{\mu\nu}(x, \dot{x}, \ddot{x},...)\, \hat{x}^{\nu d}\,]\hat{1}^d \equiv \hat{x}^2 = [x^\mu\, \hat{\eta}_{\mu\nu}(x, \dot{x}, \ddot{x},...)\, x^\nu\,]\hat{1} \tag{3.5.5c}$$

with diagonal realization of the isodual isounit and isodual isoseparation

$$\hat{1}^d = \text{diag.}\,(-b_1^{-2}, -b_2^{-2}, -b_3^{-2}, -b_4^{-2}) > 0, \quad b_\mu{}^d = -b_\mu(x. \ldots ..) > 0, \tag{3.5.6a}$$

$$(\hat{x}^d)^{2\,d} = (-x^1 b_1{}^2 x^1 - x^2 b_2{}^2 x^2 - x^3 b_2{}^2 x^3 + x^4 b_4{}^2 x^4)\,\hat{1}^d \in \hat{R}^d\,. \tag{3.5.6b}$$

invariant measure

$$d\hat{s}^{\hat{2}\,d} = (+\, dx^\mu\, \eta^d_{\mu\nu}\, dx^\nu\,)\,\hat{1}^d \equiv ds^{\hat{2}}, \tag{3.5.7}$$

and characteristic functions averaged into constants

$$b^\circ_\mu{}^d = -< b_\mu(x, \dot{x}, ...) >\,, \qquad \mu = 1, 2, 3, 4\,. \tag{3.5.8}$$

Again, we have four distinguishable types of spaces:

1) the conventional *Minkowski space* $M(x,\eta,R)$, used for the representation of *point-particles in vacuum*;

2) the *isominkowski spaces* $\hat{M}(x,\hat{\eta},\hat{R})$, used for the representation of *extended, nonspherical and deformable particles within physical media*;

3) the *isodual Minkowski space* $M^d(\eta^d,R^d)$, used for the representation of *point-antiparticles in vacuum*; and

4) and the *isodual isominkowski spaces* $\hat{M}^d(x,\hat{\eta}^d,\hat{R}^d)$, used for the representation of *extended, nonspherical and deformable antiparticles within physical media.*

The conventional Minkowski space is and will remain the fundamental space for the geometrization of the vacuum (Fig. 3.1.1). The primary function of isominkowskian spaces is to provide a relativistic geometrization, first, of classical physical media (see also Fig. 3.1.1) and, then of the interior of hadrons, upon suitable operator formulation. The primary geometric task is therefore the representation of the *departures* from the homogeneous and isotropic vacuum expected, classically, from physical media in general, and the deep superposition of the wavepackets of the particles, for the case of hadronic matter at large.

In the latter case we recall that all massive particles have an experimentally established wavepacket/wavelength of the order of 1 fm (10^{-13} cm). But all hadrons have a charge distribution with a radius also of the order of 1 fm. The region of space in the interior of hadrons are then expected to have a

nonlinear, nonlocal-integral and noncanonical-nonhamiltonian structure for which representation the isominkowskian spaces were built [1].

The isominkowskian characterization of the interior of hadrons has received numerous direct and indirect experimental verifications which will be studied in detail in Vol. III. We here limit ourselves to recall that phenomenological calculations conducted in ref. [10] via the conventional gauge theory in the Higgs sector identify the following modification of the Minkowski metric in the interior of pions and kaons

$$\hat{\eta} = \text{Diag.}\,[\,(1 - \alpha/3), (1 - \alpha/3), (1 - \alpha/3), -(1 + \alpha)\,], \tag{3.5.9}$$

with

$$\alpha = -\,3.79 \times 10^{-3} \text{ for pions and } \alpha = +6.1 \times 10^{-4} \text{ for kaons}, \tag{3.5.10}$$

It is evident that modified metric (3.5.9) is a particular case of the general class (3.5.5.2b) for $b°_1 = b°_2 = b°_3 = b°$

$$\hat{\eta} = \text{diag.}\,(\,b°_1{}^2, b°_1{}^2, b°_3{}^2, -\ b°_4{}^2), \tag{3.5.11a}$$

$$b°^2 \cong 1 + 1.2 \times 10^{-3}, \quad b_4{}^2 \cong 1 - 3.79 \times 10^{-3} \quad \text{for pions}, \tag{3.5.11b}$$

$$b°^2 \cong 1 - 2 \times 10^{-4}, \quad b_4{}^2 \cong 1 + 6.1 \times 10^{-4} \quad \text{for kaons}, \tag{3.5.11c}$$

Similarly, the phenomenological studies of ref.s [11] conducted also for the kaons yield the numerical values

$$b°_1{}^2 = b°_2{}^2 = b°_3{}^2 = b°^2 \simeq 0.909080 \pm 0.0004, \ b°_4{}^2 \simeq 1.002 \pm 0.007, \tag{3.5.12}$$

which are remarkably close to value (3.5.11c) for kaons.

Note the change of value (as well as of sign of the α-parameter) in the transition from pions to kaons thus confirming the expectation that the characteristic b°-quantities are different for different physical conditions. In fact, the charge radius of hadrons is approximately the same for all particles, thus implying *different densities for different hadrons*, which result in *different interior conditions for different particles.* We can therefore conclude by saying that *different hadrons necessarily have different values of the characteristic* $b_\mu°$*-constants evidently because they have different densities.*

This point has been here illustrated to prevent the customary tendency of looking for universal constants, which is inapplicable under isotopies because quantities that are constants in quantum mechanics, such as Planck's constant $\hbar$, the speed of light c_0, etc., are replaced with *locally varying* values.

By no mean, one should therefore search for the "universal values" of the

characteristic $b_\mu{}^\circ$-constants because such constants provide an average of the physical characteristics of the medium considered and, as such, vary from medium to medium.

The primary physical application of the isodual Minkowski space $M^d(x^d,\eta^d,R^d)$ is the geometrization of the vacuum for antiparticles, that for negative-energy solutions of conventional relativistic field equations (such as Dirac's equation). In fact, it is easy to see that the isodual energy is negative-definite

$$E^d = p^d{}_4 = -E. \tag{3.5.13}$$

The aspect we have to show later on in Volume II is that negative-energy solutions behave in a fully physical way when interpreted via isodual spaces.

Recall that, by putting $x^2 = R^2 = \text{const.}$, the non-relativistic limit of the Minkowski space is the familiar structure

$$\text{Lim}_{\,R/c_o = 0}\, M(x,\eta,R) = E(t,R_t) \times E(r,\delta,R). \tag{3.5.14}$$

Along the same lines, by assuming $\hat{x}^{\hat{2}} = R^2 = \text{cost.}$, it has been shown in ref. [5], Ch. VI, that

$$\text{Lim}_{\,R/c_o = 0}\, \hat{M}(\hat{x},\hat{\eta},\hat{R}) = \hat{E}(\hat{t},\hat{R}_t) \times \hat{E}(\hat{r},\hat{\delta},\hat{R}), \tag{3.5.15}$$

thus recovering the isoeuclidean space of nonrelativistic hadronic mechanics of Definition I.3.4.2.

Note that, jointly with the "decoupling" of space and time, we have a corresponding "decoupling" of the space and time components of the isotopic element $\hat{T}$ and isounit $\hat{1}$. Then, the isorelativistic quantity $b_4{}^{-2}$ becomes the nonrelativistic isounit of time.

An important application of isominkowski spaces is provided by the realization

$$b_\mu = 1 / n_\mu \quad \mu = 1, 2, 3, 4. \tag{3.5.16}$$

under which the isoseparation becomes

$$\hat{x}^{2} = \left(x^1 \frac{1}{n_1{}^2} x^1 + x^2 \frac{1}{n_2{}^2} x^2 + x^3 \frac{1}{n_3{}^2} x^3 - t \frac{c_o{}^2}{n_4{}^2} t \right) \hat{1}. \tag{3.5.17}$$

As one can see, the quantity n_4 represents the *index of refraction of light* propagating within physical media. We reach in this way the following important:

Proposition 3.5.1 [1]: *The isominkowski space permits a direct representation*

(that is, a representation directly with the basic isometric and related isointerval) of the locally varying speed of light within physical media,

$$c = c_o / n_4 < c_o = c(x, \mu, \tau, ...) = c_o / n_4(x, \mu, \tau, ...) . \qquad (3.5.18)$$

The transition from the conventional metric in vacuum to the isotopic metric within physical media can then be derived from the local variation of the speed of light.

In fact, the replacement of c_o^2 with $c^2 = c_o^2/n_4^2$ in the fourth component of the separation implies the introduction of the corresponding space components n_k^{-2}, as requested from the customary space-time symmetrization. Independently from that, the emergence of the space components n_k^2 would follow anyhow from the application of the Lorentz transformations to an interval with the local speed $c^2 = c_o^2/n_4^2$. The understanding is that a more appropriate symmetry will be the Lorentz-isotopic symmetry because the conventional symmetry only holds for constant speed of light in vacuum c_o while the isotopic symmetry has been built for locally varying speeds $c = c_o/n_4$ [1].

When the medium is no longer transparent, the quantity n_4 geometrizes the density of the medium itself, in a way mathematically similar to (although physically different than) the geometric meaning of the fourth component g_{44} of the Riemannian metric.

We can say in this way that *the isominkowski spaces extend to all possible physical media, whether transparent or not, the notion of index of refraction for transparent media.*

The average quantity $b°_4 = 1/n°_4$ provides "global" value, such as the *average value of the index of refraction throughout our atmosphere.*

To begin the illustration of the uses of the isominkowski spaces, consider first the case of the homogeneous and isotropic water. Then simple considerations lead to the identities $b_\mu = 1/n°_4$, $\mu = 1, 2, 3, 4$, with corresponding lifting of the separation

$$x^2 \big|_{\text{empty space}} \rightarrow \hat{x}^2 \big|_{\text{water}} = \frac{1}{n°_4{}^2} x^2 . \qquad (3.5.19)$$

This establishes that *the transition from empty space to homogeneous and isotropic media such as water is directly representable via the simplest possible isotopy called "scalar isotopy",* that with the common factor $b_4^2 = 1/n_4^2$. Such a representation is not merely formal, because it permits the resolution of a number of inconsistencies of the special relativity when applied to physical media.

We are here referring to the basic assumption of the special relativity that

the speed of light is the maximal causal speed which, when applied to water, is in contradiction with the experimental evidence according to which electrons in water can travel faster than the local speed of light (this is the blue Cherenkov light we see in the pool of nuclear reactors). If the basic postulate of the maximal causal speed is relaxed, one differentiates the maximal causal speed from the speed of light and assumes for maximal causal speed *in water* the speed of light *in vacuum* c_o, then the principe of causality is salvaged, but one encounters other inconsistencies, such as the violation of the relativistic sum of speeds because the sum of two light speeds in water does not yield the speed of light in water,

$$V_{Tot} = (c + c) / (1 + c^2 / c_o^2) \neq c_o \text{ and } \neq c . \qquad (3.5.20)$$

All the above inconsistencies are resolved by the isominkowskian geometry as we shall see in details in Vol. II.

If the medium is inhomogeneous and anisotropic, such as our atmosphere, we have even greater inconsistencies because, as it is well known, the homogeneity and isotropy of the vacuum are the geometric pillars of the special relativity. These inconsistencies too are resolved by our isominkowskian media.

In the final analysis one should remember that the Minkowski space and the special relativity were specifically conceived for the propagation of electromagnetic waves and particles *in vacuum* where the speed of light is a universal constant. The selection of the appropriate generalization is evidently open to scientific debates. However, the insistence in their necessary applicability within physical media, in which the speed of light is a locally varying quantity, is outside the boundaries of science.

The following property is a corollary of Theorem I.3.2.1:

Corollary 3.2.1A: *Isominkowskian spaces of Class I* $\hat{M}(\hat{x},\hat{\eta},\hat{R})$ *(isodual isominkowskian spaces of Class II* $\hat{M}^d(\hat{x}^d,\hat{\eta}^d,\hat{R}^d)$*) are locally isomorphic to the conventional Minkowski space* $M(x,\eta,R)$ *(isodual isominkowski space* $M^d(x^d,\eta^d,R^d)$*).*

Despite the profound differences in functional dependence, conventional and isotopic Minkowski spaces are "geometrically equivalent". In fact, all the original geometric axioms of the space are preserved under isotopies as studied in detail in Ch. I.5.

This implies that certain operations are equivalently done in both spaces. As an example, one can introduce the *contravariant isometric tensor* $\hat{\eta}^{-1}$ in $\hat{M}(\hat{x},\hat{\eta},R(\hat{n},+,\hat{x}))$ with elements

$$\hat{\eta}^{\mu\nu}(x, \dot{x}, ...) = (| \hat{\eta}_{\alpha\beta}(x, \dot{x}, ...)|^{-1})^{\mu\nu} . \qquad (3.5.21)$$

Then, the transition from covariant to contravariant indices, and viceversa, is done as in the conventional case

$$\hat{x}_\mu = \hat{\eta}_{\mu\nu}\,\hat{x}^\nu, \qquad \hat{x}^\mu = \hat{\eta}^{\mu\nu}\,\hat{x}_\nu. \tag{3.5.22}$$

This implies that, by ignoring the multiplicative isounit, the isoseparation can formally be written in a way identical to the conventional one,

$$x_\mu\, x^\mu{}_{|M} = x^\mu\, \eta_{\mu\nu}\, x^\nu \;\rightarrow\; \hat{x}_\mu\, \hat{x}^\mu{}_{|\hat{M}} = x^\mu\, \hat{\eta}_{\mu\nu}(x, \dot{x}, ...)\, x^\nu. \tag{3.5.23}$$

In this sense, most of the relativistic isotopic formulations are "hidden" in the conventional ones. To identify them, one must identify the basic unit and related multiplication.

Despite this "isoequivalence", the physical differences between the isotopic and conventional formulations are considerable and experimentally measurable, classically and operationally. In fact, isorelativistic theories can directly represent:

A) the actual, generally nonspherical shape of the considered hadrons, say, an oblate spheroidal ellipsoid, via the space components of the isometric $b_k{}^2$;

B) all infinitely possible deformation of the above original shape due to sufficiently intense external fields or collisions, which are easily representable via a suitable functional dependence of the b_k-quantities on the external forces;

C) the density of physical media via the fourth component of the isometric b_4;

D) the nonlinear, nonlocal and nonhamiltonian dynamics of the interior particle problems (that is, particles moving inside other particles);

E) the inhomogeneity and anisotropy of matter;

all this in such a way to admit the conventional Minkowskian formulations at the limit $\hat{1} \rightarrow I$.

The following Corollary of Proposition I.3.3.1 is also important:

Corollary: 3.3.1A: *The conventional Minkowski space* $M(x,\eta,R)$ *in (3+1) space-time dimensions is an isotope* $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ *of the 4-dimensional Euclidean space* $E(x,\delta,R)$ *of Class III characterized by the isotopy of the metric*

$$\delta = \mathrm{Diag.}\,(1, 1, 1, 1) \rightarrow \hat{\delta} = \hat{T}\delta = \eta = \mathrm{diag.}\,(1, 1, 1, -1), \tag{3.5.24}$$

under the redefinition of the fields

$$R(n,+,\times) \rightarrow \hat{R}(\hat{n},+,\hat{\times}), \quad \hat{1} = \hat{T}^{-1} = \eta^{-1} = \eta. \tag{3.5.25}$$

In fact, the isominkowskian spaces were first derived [1] via the *"isotopies of isotopies"*

$$E_4(x,\delta,R) \rightarrow \hat{E}_{3+1}(\hat{x},\hat{\delta},\hat{R}) \approx M(x,\eta,\hat{R}) \rightarrow \hat{M}(\hat{x},\hat{\eta},\hat{R}). \tag{3.5.26}$$

The reader should remember that the isotopy of the field is a feature needed for mathematical consistency, but it does not affect the practical numbers of the theory. In fact, as pointed out in the preceding chapter, the product of an isonumber $\hat{n}$ by a quantity Q in isominkowski space coincides with the conventional product

$$\hat{n} \hat{\times} Q \equiv n\,Q. \tag{3.5.27}$$

Also, as we shall see in the next chapter, the symmetries of $\hat{E}_{3+1}(\hat{x},\hat{\delta},\hat{R})$ and those of $M(x,\eta,\hat{R})$ coincide because characterized by the same metric $\hat{\delta} = \eta$.

This essentially means that, at the isotopic level of Class III (but not so in conventional theories), there is no essential geometrical distinction between the 4-dimensional Euclidean space $E(x,\delta,R)$ and the (3+1)-dimensional Minkowski space $M(x,\eta,R)$. These notions are important pre-requisite for their isotopic liftings [1].

We finally close this section with the following important property proved by Aringazin [15]

Theorem 3.5.1 [loc. cit.]: *Isominkowski spaces of Class I are "directly universal" for all infinitely possible deformations of the Minkowski metric preserving the original signature* (+, +, +, −), *i.e., they are capable of representing all possible modifications of the Minkowski space of the class considered, directly in the frame of the experimenter. A similar occurrence holds for the remaining classes with different signatures.*

The above property follows from the fact that any signature preserving deformation of the Minkowski metric $\eta \rightarrow \hat{\eta}$ can always be expressed in the isotopic form $\hat{\eta} = \hat{T}\eta$, owing to the unrestricted functional dependence of the 4×4 isotopic matrix $\hat{T}$.

As we shall review in details in Vol. II, Aringazin [loc. cit.] illustrated the above property by showing that all generalizations of the Einsteinian expression for the behaviour of the meanlife with speed existing in particle physics are particular cases of the single, unified, geometric expression characterized by the isominkowski space. The difference between one or the other of the existing expressions is merely due to the assumption of different expansion with different parameters and different truncations.

The above property should be kept in mind because other approaches to the interior problem are indeed possible, and their study is indeed encouraged. However, other approaches generally imply the loss of the original Minkowskian axioms, while the isominkowskian spaces are conceived to preserve such axioms. This occurrence is illustrated by the so-called *deformed Minkowski spaces* $\hat{M}(\hat{x},\hat{\eta},R)$ in which the metric is deformed $\hat{\eta} = \hat{T}\eta$, but the basic unit I and related field $R(n,+,\times)$ are left unchanged. The differences between η and $\hat{\eta}$ then imply the lack of preservation of the basic axioms in the lifting $M(x,\eta,R) \rightarrow \hat{M}(\hat{x},\hat{\eta},R)$.

Under isotopies we have the deformation of the metric $\eta \rightarrow \hat{\eta} = \hat{T}\eta$ while jointly lifting the basic unit of the inverse amount $I \rightarrow \hat{1} = \hat{T}^{-1}$ and reconstructing the field with respect to the new unit. All original geometric axioms are then preserved under the corresponding liftings $M(x,\eta,R) \rightarrow \hat{M}(\hat{x},\hat{\eta},\hat{R})$, as we shall see in Ch. I.5.

The reason for our preference of the "isominkowskian spaces" over the "deformed Minkowski spaces" is that the former will allow us in Vol. II to preserve Einstein's axioms at the abstract level and merely realize them in a more general way, while for the latter spaces the Einsteinian axioms are lost, thus creating the problems of first identifying new axioms and then proving them experimentally.

We close this section by illustrating the rather profound modifications of the Minkowski space which are possible under *nondiagonal isounit of Class I.* Consider for instance the following lifting of the Minkowskian unit

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \hat{1} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \qquad (3.5.28)$$

It is easy to see that Det $\hat{1} = 1$ and, thus, $\hat{1}$ is of Kadeisvili Class I with isotopic element

$$\hat{T} = -\hat{1} = \hat{1}^{d}, \quad \hat{T}^2 = \hat{1}^2 = -I . \qquad (3.5.29)$$

The corresponding lifting of the Minkowskian metric $\eta \rightarrow \hat{\eta} = \hat{T}\eta$ is therefore isotopic.

However, the latter isotopy implies the following rather profound structural change of the line element

$$x^{\mu}\eta_{\mu\nu}x^{\nu} = x^1x^1 + x^2x^2 + x^3x^3 - x^4x^4 \rightarrow$$

$$\rightarrow x^{\mu}\hat{\eta}x^{\nu} = x^1x^3 - x^2x^4 - x^3x^1 - x^4x^2 = 2x^2x^4, \qquad (3.5.30)$$

as the reader can easily verify.

We learn in this way that under a *regular* (that is, invertible) isotopy, the line element can be *degenerate*, in the sense of being contracted from four– to two–dimensions in which the values of the other two coordinates remain completely arbitrary. This particular lifting will be called *degenerate isotopy of Class I.*

As we shall see in Vol. III, the above particular type of isotopy appears to have an important role for the understanding of a fundamental physical process: the synthesis of the neutron in the core of stars from protons and electrons *only.*

This completes our preliminary presentation of isominkowski spaces. We will have ample opportunities of additional studies during the course of our analysis.

3.6: ISORIEMANNIAN SPACES AND THEIR ISODUALS

The additional spaces of particular relevance for isotopic studies can be presented via the following:

Definition 3.6.1 [1,5]: *The liftings of a given n–dimensional Riemannian or pseudoriemannian space* $\Re(x,g,R)$ *over the reals* $R(n,+,\times)$ *into the infinitely possible isotopes* $\hat{\Re}(x,\hat{g},\hat{R})$ *of Class I called "isoriemannian spaces" are given by*

$$\Re(x,g,R) \;\rightarrow\; \hat{\Re}(\hat{x},\hat{g},\hat{R}), \tag{3.6.1a}$$

$$g = g(x) \;\rightarrow\; \hat{g} \;=\; \hat{T}(x, \dot{x}, \ddot{x}, \mu, \tau, n, \ldots)\, g(x), \tag{3.6.1b}$$

$$\text{Det. } g \neq 0, \quad g = g^{\dagger} \;\rightarrow\; \text{Det. } \hat{g} \neq 0, \quad \hat{g} = \hat{g}^{\dagger}, \tag{3.6.1c}$$

$$R(n,+,\times) \;\rightarrow\; \hat{R}(\hat{n},+,\hat{\times}), \;\; \hat{n} = n\,\hat{1}, \quad \hat{1} \;=\; \hat{T}^{-1}, \;\; x^k \;\rightarrow\; \hat{x}^k \;=\; x^k\,, \tag{3.6.1d}$$

$$(x,\, y) \;= [\, x^i\, g_{ij}(x)\, x^j\,]\, I \;\rightarrow\; (\hat{x}\,\hat{,}\,\hat{x}) \;=\; (\hat{x}, \hat{T}\,\hat{x})\,\hat{1} \;=\; (\hat{T}\,\hat{x}, \hat{x})\,\hat{1} \;=$$

$$= \hat{1}\,(\hat{x}, \hat{T}\,\hat{x}) \;= [\, x^i\, \hat{g}_{ij}(x, \dot{x}, \ddot{x}, \ldots)\, x^j\,]\,\hat{1} \in \hat{R}\,. \tag{3.6.1e}$$

with "invariant isoseparation"

$$d\hat{s}^{\hat{2}} \;= [\, -\, dx^{\mu}\, \hat{g}_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots)\, dx^{\nu}\,]\,\hat{1}\,. \tag{3.6.2}$$

The "isodual isoriemannian spaces" of Class II are given by

$$\hat{\Re}^d(\hat{x}^d,\hat{g}^d,\hat{R}^d)\!: \; \hat{g}^d \;=\; \hat{T}^d(x, \dot{x}, \ddot{x}, \mu, \tau, n, \ldots)\, g(x) = -\,\hat{g} \tag{3.6.3a}$$

$$\hat{R}^d(\hat{n}^d,+,\hat{\times}^d), \quad \hat{1}^d \;=\; (\,\hat{T}^d\,)^{-1} \;=\; -\,\hat{1} \tag{3.6.3b}$$

$$(\hat{x}\,\hat{;}\,\hat{x})^d = (\hat{x}^d, \hat{T}^d \hat{x}^d)\hat{1}^d = (\hat{T}^d \hat{x}^d, \hat{x}^d)\hat{1}^d \in \hat{R}^d . \tag{3.6.3c}$$

with "isodual invariant isoseparation"

$$d\hat{s}^{\hat{2}d} = (+ dx^\mu \hat{g}^d_{\mu\nu}(x, \dot{x}, \ddot{x}, ...) dx^\nu]\hat{1}^d . \tag{3.6.4}$$

As now familiar, the above definition characterizes four important spaces:

Riemannian spaces $\Re(x,g,R)$, which will be used for the representation of *the exterior gravitational problem of matter*;

Isoriemannian spaces $\hat{\Re}(x,\hat{g},\hat{R})$, which will be used for the representation of *the interior gravitational problem of matter*;

Isodual Riemannian spaces $\Re^d(x,g^d,R^d)$, which will be used for the representation of *the exterior gravitational problem of antimatter*, and

Isodual isoriemannian spaces $\hat{\Re}^d(x,\hat{g}^d,\hat{R}^d)$ which will be used for the representation of *the interior gravitational problem of antimatter.*

The conventional Riemannian spaces are (and will remain) the basic spaces for the gravitational geometrization of the vacuum. The primary physical application of isoriemannian spaces for which they were built in the first place [4,5], is a more adequate representations of interior gravitational problems with nonlinear, nonlocal and nonlagrangian effects, such as the study of the interior of a neutron star and, more specifically, of gravitational collapse.

In fact, the latter systems are composed by a large number of extended particles/wavepackets/charge distributions, not only in condition of total mutual penetration, but also of compression in large numbers into a small region of space. Under these conditions, the emergence of interior nonlinear, nonlocal and nonlagrangian interactions is beyond credible doubts, and so is the lack of exact applicability of the conventional Riemannian spaces and related geometries in favor of structurally more general spaces and geometries.

The clear understanding, stressed in Ch. I.1, is that the *approximate validity* of Riemannian spaces for the above interior conditions remains unquestionable.

From the isodual spaces and isospaces the reader can begin to see the "hidden character" of the *isodual Universe* made up of antimatter, in the sense that it does not appears at a first study of the Riemannian spaces and related geometry. This is due to the fact that *Riemannian and isodual Riemannian spaces share the same separation by construction,*

$$(\hat{x}^d)^{\hat{2}d} = [(-\hat{x})^t (-\hat{g})(-\hat{x})] (-\hat{1}) \equiv (\hat{x}^t \hat{g} \hat{x}) \hat{1} = \hat{x}^{\hat{2}} . \tag{3.6.5}$$

As such, the isodual space cannot be identified with Riemannian techniques,

although it is a mathematically and physically distinct space.

As one can see, the functional dependence of the elements of the $n \times n$ isotopic matrix $\hat{T}$ remains unrestricted under isotopies. Thus, *isoriemannian spaces are bona-fide nonlinear (in the velocities), nonlocal-integral and nonpotential-nonlagrangian generalizations of the conventional spaces.*

Despite these physical differences, the two spaces are geometrically equivalent, as expressible via the following particular case of Theorem I.3.2.1:

Corollary 3.2.1B: *A given (3+1)-dimensional Riemannian space* $\Re(x,g,R)$ *(isodual space* $\Re^d(x^d,g^d,R^d)$*) and all its infinitely possible isotopes of Class I* $\hat{\Re}(\hat{x},\hat{g},\hat{R})$ *(isotopes of Class II* $\hat{\Re}^d(\hat{x}^d,\hat{g}^d,\hat{R}^d)$*) are locally isomorphic.*

We should again recall that this is possible because of joint liftings

$$g \to \hat{T} g , \qquad I \to \hat{1} = \hat{T}^{-1} . \tag{3.6.6}$$

which ensures that all deviations from the Riemannian spaces (velocity-dependent, etc.) are embedded in the isounit. In particular, the above mechanism permits the use of the integro-differential topology indicated earlier, with considerable simplifications over the conventional integral topology.

As an example, a conventional integral generalization of the Riemannian metric $g \to \hat{g}$ without the joint lifting of the unit would require a full integral geometry, without any local isomorphism, in general, between the old and new spaces.

Corollary 3.2.1C implies that some of the operations in isoriemannian spaces can be conducted in a way geometrically equivalent to the conventional ones, as it is the case for Minkowski and isominkowski spaces. Nevertheless, as we shall see in Ch. I.5, the isoriemannian geometry is structurally different than the conventional Riemannian geometry, evidently because of the explicit dependence in the velocities and accelerations.

Note that we have the Euclidean "space" and Minkowski "space" because their metric is unique, while we have Riemannian "spaces" because we have an infinite number of different (but geometrically equivalent) metrics g. By the same token, we now have an infinite number of isoriemannian spaces for *each* given Riemannian space. This multiple variety is necessary to represent physical reality. In fact, for each given total gravitational mass M, and, thus, for each given exterior metric g, there exist infinitely different interior conditions depending on size, density, temperature, etc. Thus, each given exterior total gravitational mass M admit an infinite number of interior isometrics $\hat{g}$ for the representation of all its possible physical realizations.

This point is important to understand that, under no condition, one should expect isotopic techniques to predict the numerical values of the isotopic element

$\hat{T}$ on mere geometric grounds because this would be exactly the same as requiring Einstein's gravitation to predict the numerical value of the mass from pure geometry without physical input.

On the contrary, a beauty and effectiveness of Einstein's gravitation is that it applies for all infinitely possible masses M whose explicit value in a given case must be obtained from experimental measures. By the same token, the physical effectiveness of isotopic theories is that they apply for all infinitely possible interior conditions whose characteristics must be identified via experiments.

In the final analysis, one should remember that no theory, whether conventional or isotopic, can predict the numerical value of its own unit.

It is best to provide some explicit example of isoriemannian metrics which can later on be of guidance in further studies.

Recall that the Riemannian spaces are locally Minkowskian. This property is evidently preserved under isotopies, according to which *isoriemannian spaces are locally isominkowskian,* as evident from the preservation of the signature (+, +, +, −),

As shown in ref. [7], the above property essentially implies that the isotopic element $\hat{T}$ in gravitation is considerably similar to that in isominkowski space. As indicated earlier, the isounit of Class I can be diagonalized into the form

$$\hat{T} = \text{diag.}(n_1^2, n_2^2, n_3^2, n_4^2), \ n_\mu = n_\mu(x, \dot{x}, \ldots) > 0, \tag{3.6.7}$$

An isoriemannian line element with space isotropy, $n_1 = n_2$ n3 = ns, but space-time anisotropy, $n_s \neq n_4$, is then given by the following *isoschwartzschild line element* [7]

$$ds^2 = [\ -(1-2\ M\ /r)^{-1} dr^2 - r^2\ d\theta^2 - r^2\ \sin^2\ \theta d\phi^2\ ]\ /\ n_s^2 + (1-2M\ /r)\ dt^2\ c_o^2/n_4^2, \tag{3.6.8}$$

As one can see, the first capability of the isoriemannian spaces is therefore the direct geometrization of the locally varying speed of light within physical media, $c = c_o/n_4$. By recalling that this feature is not possible in conventional Riemannian spaces, its physical significance appears in its proper light.

As an example, gravitational horizons are computed via the local use of the light cones, thus assuming the local speed of light c_o. But the region of space in the exterior of gravitational horizons is not empty, and it is filled up instead of hyperdense and very large chromospheres in which the speed of light is not c_o, but rather c. The possibility of a more accurate study of gravitational horizon via the use of the isoriemannian spaces and their local isominkowskian spaces is then evident.

Moreover, the characteristic n-quantities represent the deformation of the conventional metric expected from nonlinear, nonlocal and nonlagrangian internal effects. Also, the characteristic n-quantities can be effectively averaged for all "global" treatments, such as the speed of light throughout our entire

chromosphere considered above

$$\hat{1}^\circ = <\hat{1}> = \text{diag.}\,(n^\circ{}_1{}^2, n^\circ{}_2{}^2, n^\circ{}_3{}^2, n^\circ{}_4{}^2), \quad n^\circ{}_\mu = \text{const} > 0, \qquad (3.6.8)$$

by setting the foundation for quantitative predictions of interior effects which are verifiable with contemporary experiments (see Vol. III).

Note that the degenerate isotopies of Class I introduced in the preceding sections, when applied to the Riemannian spaces, appear to be useful for the representation of gravitational singularities.

3.7: ISOTOPIC UNIFICATION OF MINKOWSKI AND RIEMANNIAN SPACES

As indicated in Ch. I.1, isotopic techniques also have significant applications for *conventional* theories in vacuum. The best way to illustrate this possibility is by showing the new geometrical and physical insights permitted by the isotopies in gravitation. In turn, this can set the foundations for novel advances studied in Vol. II, such as an unambiguous operator form of conventional gravitation, a novel approach to singularities, and others.

Let us begin our study with the following evident property:

Corollary 3.3.1B: *The conventional, (3+1)-dimensional Riemannian spaces* $R(x,g,R)$ *are locally isomorphic to the isotope of Class III* $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ *of the 4-dimensional Euclidean space* $E(x,\delta,\mathfrak{R})$ *characterized by the lifting of the Euclidean metric* $\delta = \text{diag.}\,(1, 1, 1, 1)$ *into the Riemannian metric* g

$$\delta = I_{4\times 4} \rightarrow \hat{T}\delta = g, \qquad (3.7.1)$$

and by the corresponding lifting of the field

$$R(n,+,\times) \rightarrow \hat{R}(\hat{n},+,\hat{\times}), \; \hat{n} = n\hat{1}, \quad \hat{1} = \hat{T}^{-1} = g^{-1}. \qquad (3.7.2)$$

By recalling Corollary 3.3.1A, we lose any distinction at the abstract isotopic level between Euclidean,Minkowskian and Riemannian spaces of the same dimension. The following additional property also holds

Corollary 3.2.1C: *The conventional (3+1)-dimensional Riemannian spaces* $\mathfrak{R}(x,g,R)$ *is locally isomorphic to the Class I isotopes* $\hat{M}(\hat{x},\hat{\eta},\hat{R})$ *of the Minkowski space* $M(x,\eta,R)$ *characterized by the lifting of the metric*

$$\eta = \mathrm{diag.}\,(1, 1, 1, -1) \;\rightarrow\; \hat{T}_{gr}(x)\,\eta = g(x), \tag{3.7.3}$$

and of the field

$$R(n,+,\times) \rightarrow \hat{R}(\hat{n},\hat{+},\hat{\times}), \quad \hat{n} = n\,\hat{1}_{gr}, \approx \hat{R}\hat{1}, \quad \hat{1}_{gr} = [\,\hat{T}_{gr}(x)\,]^{-1}. \tag{3.7,4}$$

In fact, from their locally Minkowskian character, all possible Riemannian spaces must verify the isotopic decomposition of the metric

$$g(x) = T_{gr}(x)\,\eta\;, \tag{3.7.5}$$

where the 4×4 isotopic matrix $\hat{T}_{gr}(x)$ is positive definite. The above isominkowskian reformulation of Riemannian spaces then follows.

A simple example is provided precisely by the Schwarzschild metric in spherical polar coordinates

$$ds^2 = (1-2\,M\,/\,r\,)^{-1}\,dr^2 - r^2\,d\theta^2 - r^2\,\sin^2\theta\,d\phi^2 + (\,1 - 2\,M\,/\,r\,)\;dt^2\,, \tag{3.7.6}$$

which exhibits a manifest isotopic structure with respect to the Minkowski space with characteristic b–functions

$$T = \mathrm{diag.}\,\{\,(1 - 2\,M\,/\,r\,)^{-1}(1, 1, 1),\;\;(\,1 - 2\,M\,/\,r\,)\,\} \tag{3.7.7}$$

The above properties imply that *the transition from relativistic to gravitational formulations is an isotopy* [5]. This concept is at the foundations of the advances in gravitation permitted by the isotopic techniques and studied in Vol.s Ii and III, such as the identification of the universal symmetry for all possible Riemannian metrics, an unambiguous operator formulation of gravitation without Hamiltonian, the study of the gravitational field of antimatter beginning at the classical level, the study on the *origin* of the gravitational field, and others.

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4: LIE–SANTILLI ISOTHEORY AND ITS ISODUAL

4.1: STATEMENT OF THE PROBLEM

Lie's theory (see ref.s [31–33] for recent accounts and ref. [4] for historical notes) is the true structural foundation of quantum mechanics in view of the celebrated product

$$[A, B] = A \times B - B \times A, \tag{4.1.1}$$

where A×B = AB is the conventional associative product. In fact, most quantum mechanical laws, such as the unitary time evolution or Heisenberg's equation, can be simply "read–off" Lie's theory via a mere interpretation of its generators as operators on a Hilbert space.

The isotopic generalization of Lie's theory under the name of *Lie–isotopic theory* was submitted by the author in memoir [1] of 1978 with basic product

$$[A \hat{,} B] = A \hat{\times} B - B \hat{\times} A = A \times \hat{T} \times B - B \times \hat{T} \times A = \tag{4.1.2}$$

$$= A\hat{T}(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, \ldots)\, B - B\hat{T}(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, \ldots)\, A,$$

because it implies a step–by–step generalization of quantum mechanics with new dynamical equations, new interactions represented by the isotopic operator $\hat{T}$, new notions of space–time and internal symmetries, etc.. The existence of the new mechanics was confirmed in memoir [2] of the same year, and proposed for study under the name of *hadronic mechanics*.[18]

The isotopic content of memoir [1] was then developed in monographs [3,4] and in papers [5–9]. Additional structural advances in the Lie–isotopic theory were made in memoir [16–18], in the mathematical papers [19–20] and in monographs

[18] It should be noted that the Lie–isotopic theory was submitted as a particular case of the yet more general Lie–admissible theory subsequently studied in monographs [11,12] and in papers [13–15].

[21,22].

THE STRUCTURE OF LIE'S THEORY

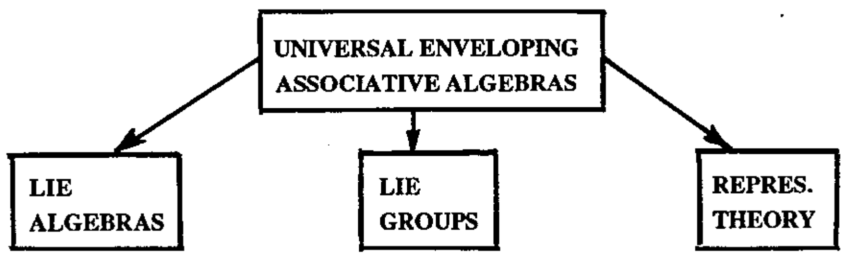


FIGURE 4.1.1: *Lie's theory* is today an articulated body of inter-related methods in algebras, geometries, functional analysis and other fields virtually encompassing all branches of mathematics [31-33]. Its most fundamental structure is the *universal enveloping associative algebra* $\xi(L)$ of a Lie algebra L [31] with conventional associative product AB among vector-fields A, B on a cotangent bundle or operators on a Hilbert space. In fact, the knowledge of ξ permits the construction of: the Lie algebra L as the attached antisymmetric algebra $\xi \approx [\xi(L)]^-$; the corresponding connected Lie group G via exponentiations in $\xi(L)$; the representation theory; etc. In memoir [1] this author submitted the elements of the *Lie-isotopic theory* conceived as a step-by-step isotopic generalization of the above formulation of Lie theory, beginning with the isotopies of universal enveloping algebras, and then passing to the isotopies of Lie's algebras and groups, the isotopies of the representation theory, etc. The dominant motivation of the proposal is of purely *physical* character and consists in: a) achieving methods for the construction of *nonlinear-nonlocal-noncanonical symmetries for interior dynamical problems;* b) in such a way to preserve the abstract axioms of the contemporary *linear-local-canonical symmetries of exterior dynamical problems*; c) so as to achieve a unity of mathematical and physical thought admitting of both, exterior and interior problems as different realizations.

Numerous *physical* contributions on the Lie-isotopic theory by various authors have appeared in the literature since 1978. An independent review of contributions up to 1990 of primarily physical character is available in the monograph by Aringazin, Jannussis, Lopez, Nishioka and Veljanoski [30]. Applications of the Lie-isotopic theory have also been presented at various physics meetings (see, e.g., contributions [40-51] of meetings in 1993). An update and further development of these physical applications to include subsequent contributions is presented in Vol.s II and III.

By comparison, pure mathematical studies on the Lie-isotopic theory (as referred to in Fig. 4.1.1) have been conspicuously absent until recently[19]. In fact, to the author's best knowledge, the first contribution in a mathematical Journal mentioning the words "Lie-isotopic algebras" is the review by (the physicists) Aringazin *et al.* [23] of 1990, some twelve years following their original proposal [1] in a physics journal. The only additional studies on Lie-isotopic theory appeared in the mathematical literature prior to the summer of 1993 are memoirs [19,20].

This situation is now changing rapidly. In fact, comprehensive *mathematical* studies in the Lie-isotopic theory are today available by the mathematicians Sourlas and Tsagas in monograph [24] and papers [25]. Other comprehensive studies, this time with emphasis on nonassociative algebras, are presented by the mathematicians Lôhmus, Paal and Sorgsepp in monograph [39]. Studies in the structure and isorepresentation of Lie-isotopic algebras have been conducted by Kadeisvili in papers [26,27] and monograph [29], by the mathematician Klimyk and this author [52]., and by others A mathematical study on isonumbers and isofields was recently published by Kamiya [53]. Additional articles by pure mathematicians are in print.

In the above quoted mathematical and physical literature the Lie-isotopic theory is called the *Lie-Santilli isotheory* and intended to including the isotopies of enveloping algebras, Lie algebras, Lie groups and representation theory. The more general *Lie-admissible theory* is called *Lie-Santilli genotheory* is intended to including the genotopies of enveloping algebras, Lie algebras, Lie groups, and representation theory. A third, still more general formulation of hyperstructural character [57] is conceivable but will not be treated at this time.

The primary differences among these three layers of generalizations are the following. The *isotheory* is based on the isotopic product [1] $[A,\hat{}B] = A\hat{T}B - B\hat{T}A$ where $\hat{T}$ is a Hermitean matrix or operator, $\hat{T} = T^{\dagger}$ [1]. The *genotheory* is based on the product $(A, B) = A\hat{T}B - B\hat{T}^{\dagger}A = A\hat{R}B - B\hat{S}A$, where $\hat{T}$ is now a nonhermitean matrix or operator, $\hat{T} = \hat{R} \neq \hat{T}^{\dagger} = \hat{S}$ [2]. The *hypertheory* is based on a product of the type $A \odot B = A\hat{R}B - B\hat{S}A$ where $\hat{R}$ and $\hat{S}$ are *sets*.

This chapter is solely devoted to the isotheory while the genotheory are treated in Ch. I.7. Nevertheless, the reader should be aware that most of the properties of the isotheory studied in this chapter, specifically, for Hermitean isotopic elements $\hat{T}$, admits generalized formulations when $\hat{T}$ is nonhermitean.

The difficulties in a first inspection (and appraisal) of the Lie-Santilli isotheory are, again, of mathematical nature. They are due to the understandable expectation that the current formulation of Lie's theory (see, e.g., ref.s [31-33] and literature quoted therein) encompasses all possible realizations, thus including the isotopic formulation.

[19] This is not the case for mathematical studies on Lie-admissible algebras which, as we shall see in Ch. I.7, have been quite numerous.

As it is well known (see, the contemporary Lie theory is constructed with respect to a conventional unit, e.g., the N-dimensional unit matrix $I = \text{diag.}(1, 1, ..., 1)$. The central idea of the Lie-Santilli isotheory [1,2] is the reconstruction of Lie's theory with respect to the most general possible isounit $\hat{1}$ with a nonlinear, nonlocal and noncanonical dependence in all possible local variables and quantities. The lifting of the unit $I \rightarrow \hat{1}$ therefore implies a corresponding compatible lifting of *all* branches of the conventional Lie theory (Sect. I.4.1).

From the very outset one can therefore see the reachness of the Lie-Santilli isotheory as compared to the conventional Lie theory with Kadeisvili's classification [28]

Lie Santilli isotheory (Class I);
Isodual Lie-Santilli isotheory (Class II);
Indefinite Lie-Santilli isotheory (Class III);
Singular Lie-Santilli isotheory (Class IV);
General Lie-Santilli isotheory (Class V).

which applies to each of the branches of the generalized theory, thus resulting in isoenveloping algebras of Classes I-V, Lie-isotopic algebras of Classes I-V, Lie-isotopic groups of Classes I-V, isorepresentations of Classes I-V, etc.[20], each of which can be of isocharacteristic zero or p (Sect.I. 2.3).

Moreover, the isotopies imply the possibility of introducing fundamentally novel notions, such as *"Lie's theory on a singular unit"*, or formulating the *"Lie-isotopic theory of discrete groups over continuously varying units"*, or, viceversa, studying the *"Lie-isotopic theory of continuous groups over discrete units"*, etc.

The above classes of isotopies admits corresponding classes of genotopic and hyperstructural type, thus illustrating the truly remarkable richness of Lie's theory which emerges from the lifting of the unit.

It is important to understand beginning with these introductory words that the Lie and Lie-isotopic theories are structurally inequivalent for the following reasons:

1) The map interconnecting Lie product (4.1.1) and its Lie-isotopic generalization (4.1.2) is *nonunitary*,

$$U U^{\dagger} = \hat{1} \neq I \tag{4.1.3a}$$

$$U [A, B] U^{\dagger} = U (A B - B A) U^{\dagger} = A' \hat{T} B' - B' \hat{T} A' = [A' \hat{,} B'], \tag{4.1.3b}$$

$$A' = U A U^{\dagger} , \quad B' = U B U^{\dagger}, \tag{4.1.3c}$$

[20] The reader should keep in mind that, as originally presented in memoir [1], all these formulations are still particular cases of the more general Lie-admissible theory (Ch. 7).

with isotopic element $\hat{T}$ given precisely by the Hermitean inverse of $\hat{1}$ as needed for a correct isotopic formulation,

$$\hat{T} = (U\, U^{\dagger})^{-1} = \hat{1}^{-1} = \hat{T}^{\dagger} ; \tag{4.1.4}$$

2) Lie's theory is linear-local-canonical in its structure, while the Lie-isotopic theory has a nonlinear-nonlocal-noncanonical structure (when projected in the original carrier space, see Sect. 4.2) as a necessary condition to be directly applicable to interior dynamical problems. This implies a generalization under isotopies of the basic symmetries of contemporary physics, such as rotations, Lorentz transformations, etc., into the most general possible nonlinear-nonlocal-noncanonical forms;

3) The isotopies alter conventional weights and, in general, the spectra of eigenvalues of the conventional Lie theory. Let X be a Hermitean generator of a Lie algebra with spectrum of eigenvalues S° with respect to a basis $| \, b >$. Then, under isotopies the *same* generator X admits a *different* spectrum S, according to the lifting

$$X \,|\, b > \; = S^{\circ} \,|\, b > \;\; \rightarrow \;\; X \hat{\times} \,|\, \hat{b} > \; = X\, \hat{T} \,|\, \hat{b} > \; = S \,|\, \hat{b} > \; , S \neq S^{\circ} ; \tag{4.1.5}$$

4) The isotopies map Cartan's tensor and other structural elements of Lie's theory into suitable integro-differential forms;

5)The topology of the current formulation of Lie's theory is notoriously local-differential, while that of the covering Lie isotopic theory is integro-differential (Fig. I.1.1.4); and other reasons.

This chapter has been specifically written for physicists to outline only those aspects of the Lie-isotopic theory that are essential for the physical applications of Volumes II and III. Unless otherwise indicated, the presentation is specifically intended for the *Lie-isotopic theory of Class I* (that with isounits $\hat{1}$ which are sufficiently smooth, bounded, nowhere degenerate, Hermitean and positive-definite, see Sect.s I.1.4 and I.2.3). An outline of the *Lie-isotopic theory of Class II* (with negative-definite isounits) is also presented because it is important for our subsequent study of antimatter and antiparticles. We shall also study a few aspects of the *Lie-isotopic theory of Class III* because it unifies those of Classes I and II. The *Lie-isotopic theories of Classes IV* (singular isounits) *and V* (unrestricted isounits) are vastly unknown at this writing and will be discussed only briefly.

During the course of our analysis we shall assume that: all Lie algebras are finite dimensional; all Lie algebras basis and corresponding parameters are ordered; and all fields have characteristic zero (Def. I.2.3.1). Mathematically inclined readers are suggested to consult the above quoted mathematical literature, e.g., ref.s [24,29,39].

A clear understanding is that the Lie-isotopic theory is still at its first infancy, particularly when compared to the current status of Lie's theory with vast mathematical and physical contributions by a large number of mathematicians and physicists for over one century.

A technical knowledge of the conventional Lie theory is an evident pre-requisite for the understanding of this chapter. A prior reading of Appendix I.4.A on basic notions of algebras and their isotopies is recommendable to the noninitiated reader.

We should also mention that the formulation of the Lie-isotopic groups presented in this chapter is in reality the *Lie-isotopic transformation groups*. The formulation of the *topological Lie-isotopic groups* is considerably more involved on technical grounds and it is still lacking at this writing. Also lacking are the *discrete Lie-isotopic groups*. Additional intriguing and significant open problems will be identified during the course of our analysis.

We should finally mention that, on strict pedagogical grounds, the presentation of the isotopies of Lie's theory of this chapter should have been postponed to follow those of differential calculus, functional analysis, geometries and mechanics. We have preferred instead the presentation of the foundations of the Lie-isotopic theory *prior* to these other isotopies, not only to follow their original lines of derivation, but also to illustrate the character of the Lie-isotopic theory of being of truly fundamental guidance in the identification of the unique and unambiguous isotopies of all other formulations.

4.2: ISOLINEARITY, ISOLOCALITY AND ISOCANONICITY

The primary limitations of quantum mechanics are that the theory is linear, local and canonical (Ch. I.1). The primary objectives of hadronic mechanics are the achievement a covering theory which is structurally nonlinear, nonlocal and noncanonical ≅Sect. I.1.1) while admitting conventional formulations as a particular cases.

The primary reason for selecting the isotopies as the basic mathematical tools for the construction of hadronic mechanics is that they permit the reconstruction of linearity, locality and canonicity on isospaces over isofields, in which case the latterproperties are called *isolinearity, isolocality and isocanonicity.*

In turn, the preservation at the abstract level of the original linearity, locality and canonicity will prove to be crucial for the achievement of physical consistency under nonlinear, nonlocal and nonhamiltonian interactions.

These main structural lines see their maximal emphasis and realization in the Lie-Santilli isotheory which, when formulated in the original, conventional

spaces and fields is nonlinear, nonlocal and noncanonical, while it is isolinear, isolocal and isocanonical when formulated on its proper isospaces over isofields.

Let $S(x,R)$ be a conventional, real vector space with local coordinates x over the reals $R(n,+,\times)$, and let

$$x' = A(w)\, x\,, \quad w \in F\,, \quad x'^{\,t} = x' \; A^t(w) \tag{4.2.1}$$

be a conventional *right, and left, linear, local and canonical transformation* on $S(x,R)$, where t denotes transpose.

The isotopic lifting $S(x,R) \rightarrow \hat{S}(\hat{x},\hat{R})$ studied in the preceding chapter requires a corresponding necessary isotopy of the transformation theory. In fact, it is instructive for the interested reader to verify that the application of transformations (4.2.1) to the isospace $\hat{S}(\hat{x},\hat{R})$ implies the loss of linearity, transitivity and other basic properties.

For these and other reasons, the author submitted in the original proposals [1,2] the isotopy of the transformation theory, called *isotransformation theory* [24,29,30], which is characterized by *isotransforms*

$$\hat{x}\,' = \hat{A}(\hat{w}) \hat{\times} \hat{x} = \hat{A}(\hat{w})\, \hat{T}\, \hat{x}, \quad \hat{x}'^{t} = x^t \hat{\times} \hat{A}^{\dagger}(\hat{w}) = \hat{x}^t\, \hat{T}\, \hat{A}^{\dagger}(\hat{w})\,, \tag{4.2.2a}$$

$$\hat{T} = \hat{T}^{\dagger} = \text{fixed}, \quad \hat{x} \in \hat{S}(\hat{x},\hat{R}), \quad \hat{w} \in \hat{R}(\hat{n},+,\hat{\times}), \quad \hat{1} = \hat{T}^{-1}. \tag{4.2.2b}$$

where the isotopic element $\hat{T}$ is here assumed to be of Kadeisvili's Class I.

The most dominant aspect in the transition from transforms (4.2.1) to isotransforms (4.2.2) is that, while the former are linear, local and canonical, the latter are *nonlinear* in the coordinates as well as other quantities and their derivatives of arbitrary order, *nonlocal-integral* in all needed quantities, and *noncanonical when projected in the original spaces* $S(x,\hat{F})$. In fact, from the unrestricted nature of the isotopic element $\hat{T}$, the projection of isotransform (4.2.2) in $S(x, R)$ reads (for $\hat{x} = \{\hat{x}^k\} = \{x^k\}$)

$$x' = \hat{A}(\hat{w})\, T(t, x, \dot{x}, \ddot{x}, \psi, \psi^{\dagger}, \partial\psi, \partial\psi^{\dagger}, \mu, \tau, n, ...)\, x\,. \tag{4.2.3}$$

In turn, the above features are crucial for the desired achievement of nonlinear, nonlocal and noncanonical formulations of space-time symmetries and related mechanics.

We learn in this way a feature which is general for all possible isotopic methods. While conventional theories, including Lie's theory, is unique in the sense that it admits only the conventional formulation over conventional spaces and fields, *all isotopic theories admit two different formulations, the first on their proper isospaces over isofields and the second given by their projection on conventional spaces over conventional fields.*

This is precisely the case of the isotransforms for which the first formulation is given by expression (4.2.2) on $\hat{S}(\hat{x},\hat{F})$ while the second is given by projection (4.2.3) on S(x,F).

But transforms (4.2.1) and their covering (4.2.2) coincide at the abstract level where we have no distinction between the modular action "Ax" and its isotopic form "$\hat{A}\hat{\times}\hat{x}$". We therefore have the following

Proposition 4.2.1 [22]: *Isotransforms (4.2.2) are "isolinear" when formulated on isospaces* $\hat{S}(\hat{x},\hat{F})$ *because they verify the conditions of linearity in isospaces,*

$$\hat{A}\hat{\times}(\hat{a}\hat{\times}\hat{x} + \hat{b}\hat{\times}\hat{y}) = \hat{a}\hat{\times}(\hat{A}\hat{\times}\hat{x}) + \hat{b}\hat{\times}(\hat{A}\hat{\times}\hat{y}), \tag{4.2.4a}$$

$$\forall\, \hat{x}, \hat{y} \in \hat{S}(\hat{x},\hat{F}), \quad \hat{a}, \hat{b} \in \hat{F}(\hat{a},+,\hat{\times}), \tag{4.2.4b}$$

while coinciding with linear transforms at the abstract level.

More directly, we can say that a Lie algebra is linear because it can be interpreted as a linear vector space over a conventional field. By the same token, we can that that a *Lie-isotopic algebra is isolinear because it can be interpreted as an isolinear vector space over an isofield* (Sect. I.2.4).

Note that conventional transforms (4.2.1) are characterized by the *right modular associative action* Ax of A on $x \in S(x,R)$. Isotransforms (4.2.2) are then characterized by the *right isomodular associative action* action $\hat{A}\hat{\times}\hat{x}$ of $\hat{A}$ on $\hat{x} \in \hat{S}(\hat{x},\hat{R})$. In fact, the preservation of the associativity is established by the properties

$$\hat{A}\hat{\times}\hat{B}\hat{\times}\hat{C}\hat{\times}\hat{x} = \hat{A}\hat{\times}(\hat{B}\hat{\times}\hat{C}\hat{\times}\hat{x}) = (\hat{A}\hat{\times}\hat{B}\hat{\times}\hat{C})\hat{\times}\hat{x}, \text{ etc.} \tag{4.2.5}$$

while the preservation of the modular character under isotopies also holds and it is discussed in Sect. I.4.7.

The situation for locality and canonicity follows the same lines, even tough their technical treatment requires additional advances studied in subsequent chapters.

It is known that Lie's theory is local because it possesses a local-differential topology [31-33]. By the same token, we can say that *the Lie-isotopic theory is isolocal because it possesses the Tsagas-Sourlas isolocal topology* [25] reviewed in Ch. I.6. At this point it is sufficient for the physical objective of this presentation to recall the remarks in Sect. I.1.4 and Fig. I.1.4.1 that Lie's theory describes a local trajectory intended as *the trajectory* x(t) *of a particle in point-like approximation under only action-at-a-distance interactions.* The Lie-isotopic theory also describes the local trajectory x(t), this time, intended as that of the center-of-mass of an extended, nonspherical and deformable particle, plus integral corrections due to its motion within a physical medium. In this sense, the theory is isolocal, that is, *everywhere local except at the unit.*

Still a similar situation occurs for the notions of canonicity and isocanonicity. One of the primary physical characteristics of Lie's theory is that it is canonical in the sense, e.g., that the time evolution is characterized by a one–parameter Lie transformation group which is characterized by the sole knowledge of the Hamiltonian. This notion is embedded in the primitive principle of conventional mechanics, the *action principle* on the cotangent bundle (phase space)

$$\delta A = \int_{t_1}^{t_2} (p_k \times dr^k - H \times dt) = 0 . \qquad (4.2.6)$$

where × is the conventional product in the base field R(n,+,×).

As we shall study in details in Vol. II, isotopic theories are derivable instead by the *isoaction principle*

$$\delta \hat{A} = \int_{t_1}^{t_2} (p_k \hat{\times} dr^k - H \hat{\times}_t dt) =$$
$$= \int_{t_1}^{t_2} [p_i \times \hat{1}^i{}_j(t, r, p, \dot{p} ...) \times dr^j - H \times \hat{1}_t(t, r, p, \dot{p} ...) \times dt) = 0 , \qquad (4.2.7)$$

where, as we shall see, $\hat{1}$ is the *space isounit* and $\hat{1}_t$ is the *time isounit*. Isotopic theories are then called *isocanonical* because *the isoaction principle coincides at the abstract level with the conventional action principle*, as one can see by comparing the first formulation of principle (4.2.7) with principle (4.2.6).

The power of the isotopies is expressed by the fact that, despite the above abstract identify, the isoaction principle is vastly broader than the conventional principle, because it is directly universal for all possible nonlinear, integro–differential, nonhamiltonian systems, as studied in Vol. II.

Moreover, while principle (4.2.6) is strictly of *first order*, isoprinciple (4.27) is of *arbitrary order when written on isospaces over isofields*. As we shall see in Vol. II, these manifestly fundamental properties have rather profound physical implications.

Note again the *dual formulation of isotopic theories*, the first on isospaces over isofields (the first formulation of isoprinciple (4.2.7)) and the second given by its projection on the original space over conventional fields (the second formulation of isoprinciple (4.2.7)).

The notion of isocanonicity is better focused for the Lie–isotopic theory in the study of the time evolution of physical systems. In classical Hamiltonian mechanics the time evolution is characterized by the one–parameter Lie group of canonical transforms on phase space) with local coordinates $a = \{a^\mu\} = \{r^k, p_k\}$, $\mu = 1, 2, ..., 6$, $k = 1, 2, 3$

$$a' = \{ e^{X \times t} \} \times a , \qquad (4.2.8)$$

where X is a Hamiltonian vector field (see next chapter). Similarly, the time evolution in quantum mechanics is equally given by the one-dimensional Lie group, this time, of unitary transform on a Hilbert space with states $|\psi>$ (see Ch. I.6)

$$|\psi'> = \{e^{i H \times t}\} \times |\psi>, \tag{4.2.9}$$

with interconnecting map called *naive or symplectic quantization* (see Vol. II).

The dominant characteristic in both cases (4.2.8) and (4.2.9) is that the time evolution is solely characterized by the knowledge of *one* quantity, the Hamiltonian H, under the generally tacit assumption that the basic unit is the trivial quantity I = diag. (1, 1, ..., 1).

In the transition to isotopic theories the situation is different because the characterization of physical systems now requires the knowledge of *two* quantities, the Hamiltonian H for the characterization of all action-at-a-distance interactions and the isounit $\hat{1}$ for the characterization of the contact nonpotential interactions and the geometry of interior media (Ch. I.3).

We shall learn in the next section that conventional time evolutions (4.2.8) and (4.2.9) are lifted under isotopies to the classical form

$$\hat{a}' = \{\hat{e}^{\hat{X} \hat{\times} \hat{t}}\} \hat{\times} \hat{a} = \{e^{X \times T \times t}\} \times \hat{a}, \tag{4.2.10}$$

with corresponding operator form

$$|\hat{\psi}'> = \{\hat{e}^{i \hat{H} \hat{\times} \hat{t}}\} \hat{\times} |\psi> = \{e^{i H \times T \times t}\} \times |\hat{\psi}>. \tag{4.2.11}$$

with unique and unambiguous interconnecting map studied in Vol. II, where $\hat{e}$ denotes the new exponentiation under the generalization of the unit studied in the next section.

The fundamental characteristics which is important for these introductory comments is that *isotopic theories are isocanonical because the isotopic, classical and operator time evolutions coincide at the abstract level with their corresponding conventional counterparts,* as one can see by comparing the first formulation of laws (4.2.10) and (4.2.11) with the corresponding ones (4.2.8) and (4.2.9). As we shall see in this chapter, the above property extends to all Lie-Santilli isogroups of Class I.

Again, the power of the isotopies is expressed by the act that, despite the above abstract identity, the isotopic time evolution can represent physical systems vastly more general than the conventional ones, as transparently exhibited by the second formulation of laws (4.2.10) and (4.2.11).

The following property is important for the understanding of isotopic

theories:

Proposition 4.2.2 [1,2]: *All possible nonlinear, nonlocal and noncanonical transforms on a vector space* S(x,R)

$$x' = B(w, x, ...)\, x , \quad x \in S(x,R), \quad w \in R(n.+,\times), \tag{4.2.12}$$

can always be rewritten in an identical isolinear, isolocal and isocanonical form, that is, there always exists at least one isotopy of the base field, $R(n,+,\times) \rightarrow \hat{R}(\hat{n},+,*)$, *and a corresponding isotopy of the space* $S(x,R) \rightarrow \hat{S}(\hat{x},\hat{R})$ *under which*

$$x' = B(w, x, ...)\, x \equiv \hat{A}(\hat{w}) \hat{\times} x , \quad \hat{T} = \hat{A}^{-1} B . \tag{4.2.13}$$

The above property is at the foundation of the "direct universality' of the Lie-isotopic theories" , that is, its applicability to all possible nonlinear, nonlocal and noncanonical systems (universality) in the frame of the experimenter (direct universality). In order to apply a Lie-Santilli isosymmetry to a nonlinear, nonlocal and noncanonical system, one has merely to identity one of its possible isolinear, isolocal and isocanonical *identical reformulation in the same system of (contravariant) local coordinates.* The applicability of the methods studied in this chapter then follows.

Thus, the first role of isotopic techniques is that of generalizing conventional linear, local and canonical theories into lesser trivial nonlinear, nonlocal and noncanonical forms. The subsequent role is that of *turning conventionally nonlinear, nonlocal and noncanonical theories into "identical" isolinear, isolocal and isocanonical forms,* with evident simplification of their treatment.

We now study the image of the above notions under isoduality.

Definition 4.2.2 [8,9]: *The "isodual isotransforms" of Class II are given by the image of isotransforms (4.2.2) under isoduality, i.e., are defined on the isodual isospace* $\hat{S}^d(\hat{x}^d, \hat{R}^d)$,

$$\hat{x}^{d\prime} = \hat{A}^d (\hat{w}^d) \hat{\times}^d \hat{x}^d = - \hat{A}^d (\hat{w}^d) \hat{\times} \hat{x}^d , \quad x \in \hat{S}^d(x,\hat{R}^d), \ \hat{w}^d \in \hat{R}^d(\hat{n},+,*^d) , \tag{4.2.14a}$$

$$\hat{x}^{d\,t\,\prime} = \hat{x}^d \hat{\times}^d \hat{A}^{d\,t} (\hat{w}^d) = - \hat{x}^d \hat{\times} \hat{A}^{d\,\dagger} (\hat{w}^d) \tag{4.2.14b}$$

where $\hat{A}^t$ *and* $\hat{A}^{d\dagger}$ *will be identified later on in this chapter.*

Isodual isotransforms characterize the isodual Lie isotopic theory which, in turn, characterizes the isodual symmetries for our treatment of antiparticles.

4.3: ISOENVELOPES AND THEIR ISODUALS

In this section we study the isotopies of universal enveloping associative algebras called *universal enveloping isoassociative algebras* (or *isoenvelopes* for short) for the case of Class III over an isofield of characteristic zero. The use of Class III permits a unified formulation of the isotopies of Classes I and II with consequential unification of the envelopes of simple, compact and noncompact Lie algebras of the same dimension in Cartan's classification into one single isotope.

Isoenvelopes were first identified by this author in the original proposal of the Lie-isotopic theory [1] and then studied in details in monograph [4]. An independent study can be found in monograph [24].

All quantities belonging to the conventional Lie theory will be indicated hereon with conventional symbols for generators, such as A, B, X, etc., when belonging to conventional linear spaces $S(x,F)$ and for parameters, such as w, θ, v, etc., when belonging to conventional fields $F = F(a,+,\times)$.

All quantities belonging to the Lie-isotopic theory will be denoted with the symbols for generators $\hat{A}$, $\hat{B}$, $\hat{X}$, etc., when belonging to isolinear spaces, and for parameters $\hat{w}$, $\hat{\theta}$, $\hat{v}$, etc., when belonging to isofields $\hat{F}(\hat{a},+,\hat{\times})$. In particular, the symbols $\hat{A}$, $\hat{B}$, $\hat{X}$, etc., mean the original generators A, B, X, etc., now recompute in $\hat{S}(\hat{x},\hat{F}(\hat{a},+,\hat{\times}))$, while for the parameters we have $\hat{w} = w\times\hat{1}$, $\hat{\theta} = \theta\times\hat{1}$, $\hat{v} = v\times\hat{1}$, etc. Similarly, we shall write T to denote the isotopic element computed on $S(x,F)$ and $\hat{T}$ to denote its image computed on $\hat{S}(\hat{x},\hat{F})$. For the case of the isounit we shall only use the symbol $\hat{1}$ to prevent confusion with the conventional unit I.

To begin, let $\xi = \xi(L)$ be a universal enveloping associative algebra of an N-dimensional Lie algebra L (see, e.g., ref. [31] and Fig. 4.3.1) with generic elements A, B, C,... , trivial associative product $A\times B = AB$ (say, of matrices) and unit matrix in N-dimension $I = \mathrm{diag.}\ (1, 1, ..., 1)$.

Let the (ordered) basis of L be given by $\{X_k\}$, $k = 1, 2, ..., N$, over a field $F(a,+,\times)$. An (ordered) *standard monomial* of dimension n is the product of n-generators $X_i\times X_j\times ... \times X_k$ with the ordering $i \leq j \leq ... \leq k$. The infinite-dimensional basis of $\xi(L)$ is then expressible in terms of monomials and given by the *Poincaré-Birkhoff-Witt theorem* [31]

$$I\,,\quad X_k \qquad X_i \times X_j \quad (i \leq j), \qquad X_i \times X_j \times X_k \quad (i \leq j \leq k),\ ...... \qquad (4.3.1)$$

The *universal enveloping isoassociative algebra*, or *isoenvelope*) $\hat{\xi}(L)$ of the *Lie* algebra L [1] (see Fig. 4.3.1 for their definition) coincide with ξ as vector spaces (because the basis of a vector space is unchanged under isotopies). The

basis of $\hat{\xi}(L)$ is therefore constructed with the same generators X_k only computed on the new isospace $\hat{S}(\hat{x},\hat{F}(\hat{a},+,\hat{\times}))$ and denoted $\hat{X}_k$ and now equipped with the isoproduct $\hat{A}\hat{\times}\hat{B}$ so as to admit $\hat{1} = \hat{T}^{-1}$ as the correct (right and left) unit

$$\hat{\xi} \;:\; \hat{A}\,\hat{\times}\,\hat{B} = \hat{A}\times\hat{T}\times\hat{B} = \hat{A}\hat{T}\hat{B}, \quad \hat{T} \quad \text{fixed}, \qquad (4.3.2a)$$

$$\hat{1}\,\hat{\times}\,\hat{A} = \hat{A}\,\hat{\times}\,\hat{1} \equiv \hat{A} \qquad \forall \hat{A} \in \hat{\xi}, \quad \hat{1} = \hat{T}^{-1}. \qquad (4.3.2b)$$

The (ordered) standard monomials of dimension n of $\xi(L)$ are then mapped into the (ordered) *standard isomonomials* of the same dimension $\hat{X}_i\hat{\times}\hat{X}_j\hat{\times} \dots \hat{\times}\hat{X}_k$, $i \leq j \leq \dots \leq k$ of $\hat{\xi}(L)$.

A fundamental property from which most of the Lie-isotopic theory and hadronic mechanics can be derived is the following

Theorem 4.3.1- Isotopic generalization of the Poincare´-Birkhoff-Witt Theorem [1]: *The cosets of* $\hat{1}$ *and the standard isomonomials form an infinite-dimensional basis of the universal enveloping isoassociative algebra* $\hat{\xi}(L)$ *of a Lie algebra* L *of Class III*

$$\hat{1}, \; \hat{X}_k \qquad \hat{X}_i\,\hat{\times}\,\hat{X}_j \;\; (i \leq j), \qquad \hat{X}_i\,\hat{\times}\,\hat{X}_j\,\hat{\times}\,\hat{X}_k \;\; (i \leq j \leq k), \;\dots\dots \qquad (4.3.3)$$

A detailed proof can be found in ref. [4], pp. 154-163, or ref. [24], pp. 74-93, and it is not repeated here for brevity (although its knowledge is assumed for more advanced treatments).

Algebraically, the above theorem essentially expresses the property that non singular isotopies of the basic product, i.e.,

$$A \times B: \;\; (A{\times}B){\times}C = A{\times}(B{\times}C) \;\rightarrow\; A\,\hat{\times}\,B: \;\; (A\hat{\times}B)\hat{\times}C = A\hat{\times}(B\hat{\times}C), \qquad (4.3.4)$$

imply the existence of consistent isotopies of the basis (4.3.2). Note the abstract unity of the conventional and isoenvelopes. In fact, at the level of realization-free formulation the "hat" can be ignored and bases (4.3.1) and (4.3.3) coincide. Nevertheless, the isoenvelope $\hat{\xi}(L)$ is structurally broader then the conventional envelope $\xi(L)$, e.g., because it unifies compact and noncompact structures as shown below, and this begins to illustrate the nontriviality of the Lie-Santilli isotheory.

Theorem 4.3.1 and isobasis (4..3.3) have fundamental mathematical and physical implications. Recall that the conventional exponentiation is defined precisely via a power series expansions in ξ

$$e^{iwX} = e_{\xi}^{iwX} = I + (i\; w{\times}X)\;/1! + (i\; w{\times}X) \times (i\; w{\times}X)\;/\;2! + \dots\dots, \qquad w \in F(a,+,\times)\;. \qquad (4.3.5)$$

The above exponentiation is then inapplicable under isotopies because the

quantity I is no longer the basic unit of the theory, the conventional product $\times$ has no mathematical or physical meaning, etc.

In turn, this implies that all quantum mechanical quantities depending on the conventional exponentiation, such as time evolution, unitary groups, Dirac's delta distributions, Fourier transforms, Gaussian, etc. have no mathematical or physical meaning under isotopies and must be suitably lifted.

Isobasis (4.3.3) then permits the following

Corollary 4.3.1.A [1,4]: *The "isoexponentiation" of an element* $X \in \hat{\xi}$ *via isobasis (4.3.3) over an isofield* $\hat{F}(\hat{a},+,\hat{\times})$ *is given by*

$$\hat{e}^{i\,\hat{w}\,\hat{\times}\,\hat{X}} \equiv \hat{e}^{i\,w\,X} = e_{\xi}^{i\,\hat{w}\,\hat{\times}\,\hat{X}} \equiv e_{\xi}^{i\,w\,X} =$$

$$= \hat{1} + (i\,\hat{w}\,\hat{\times}\,\hat{X})\,/\,1! + (i\,\hat{w}\,\hat{\times}\,\hat{X})\;\hat{\times}\;(i\,\hat{w}\,\hat{\times}\,\hat{X})\,/\,2! + \ldots\ldots =$$

$$= \hat{1} \times \{ e^{iw\times T\times X} \} = \{ e^{iX\times T\times w} \} \times \hat{1}, \qquad \hat{w} \in \hat{F}(\hat{a},+,\hat{\times}). \qquad (4.3.6a)$$

$$\hat{W}\,\hat{\times}\,\hat{X} = (w \times \hat{1}) \times \hat{T} \times \hat{X} \equiv w\,X, \quad \hat{X}_{\hat{S}(\hat{x},\hat{F}(\hat{a},+,\hat{\times}))} \equiv X_{S(x,F(a,+,\times))}\,. \qquad (4.3.6b)$$

The first, and most fundamental application of the isoexponentiation is the characterization of classical and operator time evolutions of isotopic theories. In fact, isoexponentiation (4.3.6) yields precisely the desired realization of the generalized exponentiation of time evolutions (4.2.10) and (4.2.11), according to the rules in full compatibility with the isotransformation theory on isospaces over isofields

$$\hat{a}' = \{ \hat{e}^{\hat{X}\hat{\times}\hat{t}} \}\,\hat{\times}\,\hat{a} = \{ e^{X\times T\times t} \} \times \hat{a}, \qquad (4.2.7a)$$

$$|\hat{\psi}'\rangle = \{ \hat{e}^{\,i\,\hat{H}\,\hat{\times}\,\hat{t}} \}\,\hat{\times}\,|\psi\rangle = \{ e^{\,i\,H\times T\times t} \} \times |\hat{\psi}\rangle. \qquad (4.2.7b)$$

The significant of the Lie-isotopic theory as a basic guidance for other generalizations is now evident. In fact, our task in the following analysis of this and of Vol. II is that of searching for algebras, geometries and mechanics compatible with fundamental time evolutions (4.3.7).

The isoexponentiation will also have fundamental relevance in numerous other aspects of hadronic mechanics, such as a structural generalization of Dirac's delta distribution, Fourier transform, Gaussians, etc.

The nontriviality of the isotopies of Lie's theory is clearly expressed by *the appearance of the nonlinear, nonlocal and noncanonical isotopic element* $T(t, x, \dot{x}, \ddot{x}, \ldots)$ *directly in the exponent of isoexponentiations* (4.3.6). This is sufficient to see that the Lie-isotopic space-time and internal symmetries will be nonlinear,

nonlocal and noncanonical, as desired for strong interactions.

One should keep in mind the *uniqueness of isoexponentiation* (4.3.6). It originates from a crucial requirement of the Poincaré-Birkhoff-Witt theorem, the existence of a well defined left and right unit [31] which, in turn, implies the uniqueness of the isobasis (4.3.3). This property can then be compared with the *lack of uniqueness* of the exponentiations in other theories. As an example we shall study in Vol. II, the so-called q-deformations, which do not possess a unique exponentiation because they do not possess a unit [42].

By recalling the results of the preceding analysis on isodual isofields and isodual isospaces (particularly Proposition I.3.2.1), we can see that the *isodual isoenvelopes* $\hat{\xi}^d(\hat{L}^d)$ [8,9] are characterized by: the *isodual basis* and the *isodual parameters*

$$\hat{X}^d{}_k = -\hat{X}_k\,, \quad \hat{w}^d = w\,\hat{1}^d = -\hat{w}. \tag{4.3.8}$$

Corollary 4.3.1B: *The "isodual isoexponentiation" is the isodual image of isoexponentiation (4.3.6) on the isodual isofield* $\hat{F}^d(\hat{w}^d,+,\hat{\times}^d)$

$$\hat{e}^{d\, i^d\,\hat{w}^d\,\hat{\times}^d\,\hat{X}^d} = e_{\hat{\xi}^d}{}^{i^d\,\hat{w}^d\,\hat{\times}^d\,\hat{X}^d} = -\{e^{\,i\,X\,T\,w}\}\hat{1} \tag{4.3.9}$$

Note that the preservation of the sign in the exponent is only apparent, i.e., when projected in an isofield, because, when properly written in the isodual isofield, one can use the expression

$$e_{\hat{\xi}^d}{}^{-i^d\,\hat{w}^d\,\hat{\times}^d\,\hat{X}^d} \equiv -\{e^{-i\,X\,T\,w^d}\}\hat{1} \tag{4.3.10}$$

Isodual isoexponentiations play an important role for the construction of the isodual isosymmetries for antiparticles.

It is easy to see that Theorem 4.3.1 holds for envelopes of Class III, as originally formulated [1], thus unifying isoenvelopes $\hat{\xi}$ and their isoduals $\hat{\xi}^d$. In fact, Theorem 4.3.1 was conceived to unify with one single Lie algebra basis X_k, but arbitrary isotopies in the envelope $\hat{\xi}(L)$, nonisomorphic compact and noncompact algebras of the same dimension N.

To clarify this aspect, recall [31] that a conventional envelope $\xi(L)$ represents only *one* algebra (up to local isomorphism),

$$L \approx [\,\xi(L)\,]^-. \tag{4.3. 11}$$

The study of a nonisomorphic Lie algebra L′ then requires the use of a *different basis* X'_k, resulting in a *different envelope* $\xi'(L')$. Thus, in the conventional Lie

theory *nonisomorphic Lie algebras of the same dimension are represented via different bases and different envelopes.*

This scenario is altered under isotopy because the isoenvelopes are now characterized by *two* quantities, the basis $\hat{X}_k$ *and* the isounit $\hat{1}$. We therefore have the novel possibility of using the same basis and changing instead the isounit. In fact, one isoenvelope $\hat{\xi}(L)$ of Class III with a fixed N-dimensional basis X_k and an arbitrary N-dimensional isounit $\hat{1}$ represents a family of generally nonisomorphic Lie algebras $\hat{L}$ as the attached antisymmetric algebras

$$\hat{L} \approx [\hat{\xi}(L)]^- . \tag{4.3.12}$$

In particular, it was proved in the original proposal [1] that, the isoalgebra $\hat{L}$ constructed via rule (4.5.12) *is not* in general isomorphic to the original algebra L, $\hat{L} \not\approx L$, unless the isotopic element is positive-definite.

Theorem 4.3.1 therefore offers the possibility of unifying of all simple Lie algebra in Cartan's classification of the same dimension. This implies in particular the reduction of compact and noncompact structures of the same dimension to only one isotopic structure, and, for each given structure, the reduction of linear and nonlinear, local and nonlocal, canonical and noncanonical realizations to one primitive algebraic notion, the isoenvelope $\hat{\xi}(L)$ (see Fig. 4.3.1 below for more details).

The above unification was illustrated in the original proposal [1] with an example that is still valid today. Consider the conventional Lie algebra **so(3)** of the rotational group SO(3) on the Euclidean space E(r,δ,R) with unit I = diag. (1, 1, 1). The adjoint representation of **so(3)** is given by the familiar expressions

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.3.13}$$

The universal enveloping associative algebra ξ(**so(3)**) is then characterized by the unique infinite-dimensional basis from the conventional Poincaré–Birkhoff–Witt theorem [31]

$$I, \quad J_k, \quad J_i\, J_j \;(i \leq j), \quad J_i\, J_j\, J_k \quad (j_i \leq j \leq k), \ldots \tag{4.3.14}$$

and characterizes only one algebra as the attached antisymmetric algebra

$$[\, \xi\, (\mathbf{so(3)}\,)\,]^- \approx \mathbf{so(3)}\,. \tag{4.3.15}$$

The isotopies $\hat{\xi}$(**so(3)**) of the envelope ξ(so(3)) of Class III are characterized by the the lifting of the basic carrier space E(r,δ,R) into the isoeuclidean space $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ with isometric, isotopic element and isounit

$$\hat{\delta} = \hat{T}\delta\,, \quad \hat{T} = \mathrm{diag.}\,(g_{11}, g_{22}, g_{33})\,, \quad \hat{1} = \mathrm{diag.}\,(g_{11}{}^{-1}, g_{22}{}^{-1}, g_{33}{}^{-1})\,, \qquad (4.3.16)$$

where the characteristic quantities g_{kk} are real-valued, non-null but arbitrary functions of the local coordinates $g_{kk}(t, r, \dot{r}, \ddot{r}, ...)$ which, as such, can be either positive or negative. From Theorem 4.3.1, the isoenvelope $\hat{\xi}(\mathbf{so(3)})$ is then characterized by the original generators (4.3.13), although expressed now in terms of the isoassociative product $J_i \hat{\times} J_j = J_i T J_j$ and isounit $\hat{1}$ with unique infinite-dimensional basis from Theorem 4.3.1

$$\hat{1}, \quad J_k, \quad J_i\,T\,J_j \;(i \leq j), \quad J_i\,T\,J_j\,T\,J_k \quad (j_i \leq j \leq k), ... \qquad (4.3.17)$$

It is now easy to see that the algebra characterized by the attached antisymmetric part of $\hat{\xi}(so(3))$ *is not unique*, evidently because it depends on the explicit values of the characteristic quantities g_{kk}. It was shown in ref.s [1,9] that the isoenvelope $\hat{\xi}(so(3))$ unifies: all possible compact and noncompact three-dimensional Lie algebra of Cartan classification, the algebras **so(3)** and so**(2.1)**; all their infinitely possible isotopes $\hat{so}(3)$ and $\hat{so}(2.1)$; the compact and noncompact isodual algebras $so^d(3)$ and $so^d(3)$; as well as all their infinitely possible isodual isotopes $\hat{so}^d(3)$ and $\hat{so}^d(2.1)$, according to the classification

$$[\hat{\xi}(so(3))]^- : \quad \begin{array}{l} so(3) \text{ for } T = \mathrm{diag.}\,(1, 1, 1); \\ so(2.1) \text{ for } T = \mathrm{diag.}\,(1, -1, 1); \\ \hat{so}(3) \text{ for sign. } T = (+, +, +); \\ \hat{so}(2.1) \text{ for sign. } T = (+, -, +) \\ so^d(3) \text{ for } T = (-1, -1, -1); \\ so^d(2.1) \text{ for } T = \mathrm{diag.}\,(-1, +1, -1); \\ \hat{so}^d(3) \text{ for sign. } T = (-, -, -); \\ \hat{so}^d(2.1) \text{ for sign. } T = (-, +, -). \end{array} \qquad (4.3.18)$$

The only improvement we can now provide over the above original formulation is in notation but not in results. In fact, the basis J_k could today be rewritten as the *isobasis* $\hat{J}_k$ which are essentially given by *isomatrices*, i.e., by matrices (4.13) in which all elements are replaced by isonumbers (that is, the number 1 by the isounit $\hat{1}$). But in this case all products are isotopic, thus implying the elimination of the isotopic character of the matrix representation and the preservation of the preceding results.

The unification of all simple Lie algebras of dimension 6 in Cartan's classification was also identified by this author in ref. [7] and it will be studied later on. The unification of all simple algebras of the same dimension in Cartan's classification into one single Lie-Santilli isoalgebra is currently under study by the mathematicians Gr. Tsagas and D. Sourlas.

The explicit form of the Lie-isotopic algebras will be studied in the next section; an illustration of the isoexponentiation will be provided in Sect. 4.5; and a first example of physical applications will be given in Sect. I.4.7. The isotopes and isoduals of so(3) and so(4) will then be studied in detail in Vol. II their applications in Vol. III.

Whenever needed for clarity, isoenvelopes will be denoted with the symbol $\hat{\xi}_{\hat{T}}$ identifying the selected isotopic element $\hat{T}$.

As concluding remarks, note that the lifting $\xi \rightarrow \hat{\xi}$ is *necessary* under the isotopy of the unit because, in general, $\hat{1}A \neq A\hat{1} \neq A$.

UNIVERSAL ISOASSOCIATIVE ENVELOPING ALGEBRAS

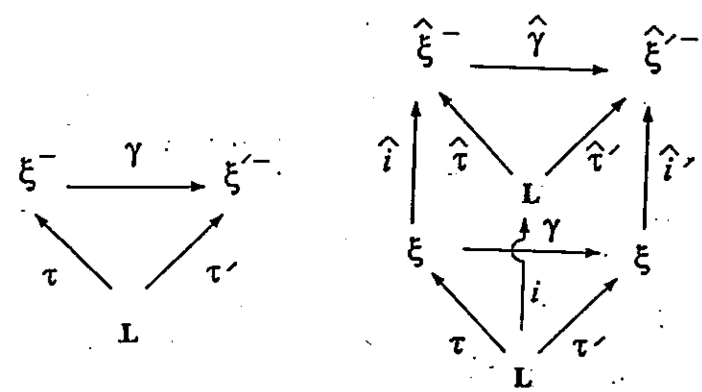
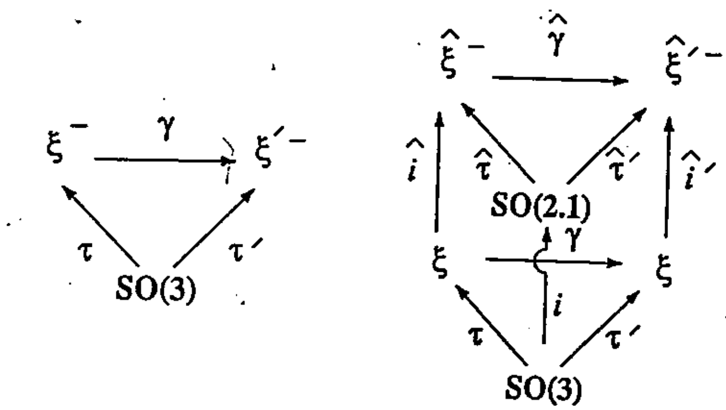


FIGURE 4.3.1: The *universal enveloping associative algebra* $\xi(L)$ of a Lie algebra L [31] is the set (ξ, τ) where ξ is an associative algebra and τ is a homomorphism of L into the antisymmetric algebra ξ^- attached to ξ such that: if ξ' is another associative algebra and τ' is another homomorphism of L into ξ'^- a unique isomorphism γ between ξ and ξ' exists in such a way that the diagram (a) above is commutative. The above definition evidently expresses the *uniqueness* of the Lie algebra L (up to local isomorphisms) characterized by its universal envelopes $\xi(L)$.

With reference to diagram (b) above, the *universal enveloping isoassociative algebra* $\hat{\xi}(L)$ of a Lie algebra L was introduced [1] as the set $\{(\xi, \tau), i, \hat{\xi}, \hat{\tau}\}$ where: (ξ, τ) is a conventional envelope of L; i is an isotopic mapping $L \rightarrow i\ L = \hat{L} \neq L$; $\hat{\xi}$ is an associative algebra generally nonisomorphic to ξ; $\hat{\tau}$ is a homomorphism of $\hat{L}$ into $\hat{\xi}^-$; such that: if $\hat{\xi}'$ is another associative algebra and $\hat{\tau}'$ another homomorphism of $\hat{L}$ into $\hat{\xi}'^-$, there exists a unique isomorphism $\hat{\gamma}$ of $\hat{\xi}$ into $\hat{\xi}'$ with $\hat{\tau}' = \hat{\gamma}\hat{\tau}$, and two unique isotopies $\hat{i}\xi = \hat{\xi}$ and $\hat{i}'\xi' = \hat{\xi}'$.

A primary objective of the isotopic definition is the achievement of *the lack of*

uniqueness of the Lie algebra characterized by the isoenvelope or, equivalently, the characterization of a family of generally nonisomorphic Lie algebras via the use of only one basis. The illustration of the above notions for the case of the rotational algebra **so(3)** studied in the text is straighforward and can be expressed via the diagrams (c) and (d) below



(c) (d)

where the isotopy is given by $I = \mathrm{diag.}\,(1, 1, 1) \Rightarrow \hat{1} = \mathrm{diag.}\,(1, -1, 1)$. The above definition then provides all infinitely possible isotopes and isodual isotopes.

The above notion of isoenvelope represents the essential mathematical structure of hadronic mechanics, namely, the preservation of the conventional basis, i.e., the set of observables, and the generalization of the operations on them via an infinite number of isotopies so as to admit a new class of interactions structurally beyond the possibilities of quantum mechanics.

The isoenvelopes are denoted $\hat{\xi}(L)$ and *not* $\hat{\xi}(\hat{L})$ to stress the preservation of the original basis of L under isotopies (Proposition I.3.2.1), as well as to emphasize the existence of an infinite family of isoenvelopes for *each* original Lie algebra L.

The isotopy $\xi \rightarrow \hat{\xi}$ is not a conventional map because the local coordinates x, the infinitesimal generators X_k and the parameters w_k are not changed except for their redefinition in isospace over isofields. In particular, the map from $\xi(L)$ to $\hat{\xi}(l)$ is *nonunitary* as indicated in Sect. I.4.1, and this illustrates their inequivalence. We therefore have the following

Proposition 4.3.1 [2]: *A conventional envelope ξ and its isotopic image $\hat{\xi}$ are not unitarily equivalent.*

The above algebraic property illustrates the fact that *hadronic mechanics*

is not unitarily equivalent to quantum mechanics, which is an evident necessary condition for novelty. Despite the above lack of unitary equivalence, a given Lie algebra L and its isotope $\hat{L}$ of Class I are indeed isomorphic as we shall see in the next section.

As now familiar, our main realization of the Lie-isotopic product is expression (I.4.1.2) where the the product $[\hat{A},\hat{,}\hat{B}]$ is *nonassociative* while the product of the underlying envelope $\hat{A}\hat{T}\hat{B}$ is *isoassociative.* The preservation of the *associative* character of the envelope under isotopies deserves an elaboration because important for the construction of any generalization of quantum mechanics, whether of isotopic type or not.

In principle, the lack of associativity of an algebra is not per se a compelling reason reason for its exclusion, because there are nonassociative algebras U such that the attached antisymmetric algebra U^- is Lie. In fact, as we shall see in Ch. I.7, this is precisely the definition of Lie-admissible algebras with product $A \odot B = ARB - BSA$ which is *nonassociative* yet the product $A \odot B - B \odot A = ATB - BTA$, $T = R+S$, is Lie-isotopic. The reason for the exclusion is that the notion of universal enveloping algebra has been essentially developed for *associative* algebras [31-33]. Nonassociative enveloping algebras are known only for very restricted algebras of the so-called *flexible Lie-admissible* type (see in this respect ref. [12]). In fact, the ordering of monomials is generally lost under nonassociative products, resulting in the general impossibility to formulate the Poincaré-Birkhoff-Witt theorem.

The physical reasons for excluding nonassociative envelopes are however deeper then the above. They are related to the fact that associative envelopes of the type herein considered admit a consistent unit which is at the foundation of physical applications such as the measurement theory. On the contrary, nonassociative envelopes generally do not admit a unit[21], thus prohibiting the very formulation of the measurement theory.

Moreover, in Volume II we shall review *"Obuko's no-go theorem"* which prohibits the use of a nonassociative envelope for a consistent generalization of quantum mechanics, e.g., because of the loss of equivalence between the Heisenberg-type and the Schrödinger-type representations.

We reach in this way the following:

Fundamental condition on Lie-Santilli isotheory 4.4.1 [2]: *All studies on the Lie-Santilli isotheory and hadronic mechanics will be restricted throughout our analysis to formulations based on an isoassociative character of the enveloping algebra with a well defined left and right isounit.*[22]

[21] Recall that the fundamental unit I of the conventional Lie's theory is the unit of the *associative envelope* and not of the Lie algebra. In fact, the product [A, B], per se, admits no consistent unit because it would require an element E such that $[E, A] = [A, E] \equiv A$, $\forall A \in L$. Exactly the same situation occurs under isotopies.

[22] As we shall see in Ch. I.7, this fundamental condition will persist also for the more

4.4: LIE-SANTILLI ISOALGEBRAS AND THEIR ISODUALS

We are now equipped to introduce the fundamental notion of hadronic mechanics according to the following

Definition 4.4.1 [1]: *A (finite-dimensional) isospace* $\hat{L}$ *over an isofield* $\hat{F}(\hat{a},+,\hat{\times})$ *of isoreal numbers* $\hat{R}(\hat{n},+,\hat{\times})$, *isocomplex numbers* $\hat{C}(\hat{c},+,\hat{\times})$ *or isoquaternions* $\hat{Q}(\hat{q},+,\hat{\times})$ *with isotopic element* $\hat{T}$ *and isounit* $\hat{1} = \hat{T}^{-1}$ *is called a "Lie-Santilli isoalgebra" over* $\hat{F}$ *when there is a composition* $[\hat{A}, \hat{,} \hat{B}]$ *in* $\hat{L}$, *called "isocommutator", which verifies the following "isolinear and isodifferential rules" for all* $\hat{a}, \hat{b} \in \hat{F}$ *and* $\hat{A}, \hat{B}, \hat{C} \in \hat{L}$

$$[\hat{a} \hat{\times} \hat{A} + \hat{b} \hat{\times} \hat{B} \hat{,} \hat{C}] = \hat{a} \hat{\times} [\hat{A} \hat{,} \hat{C}] + \hat{b} \hat{\times} [\hat{B} \hat{,} \hat{C}], \tag{4.4.1a}$$

$$[\hat{A} \hat{\times} \hat{B} \hat{,} \hat{C}] = \hat{A} \hat{\times} [\hat{B} \hat{,} \hat{C}] + [\hat{A} \hat{,} \hat{C}] \hat{\times} \hat{B}. \tag{4.4.1b}$$

and the "Lie-Santilli isoaxioms",

$$[\hat{A} \hat{,} \hat{B}] = - [\hat{B} \hat{,} \hat{A}], \tag{4.4.2a}$$

$$[\hat{A} \hat{,} [\hat{B} \hat{,} \hat{C}]] + [\hat{B} \hat{,} [\hat{C} \hat{,} \hat{A}]] + [\hat{C} \hat{,} [\hat{A} \hat{,} \hat{B}]] = 0. \tag{4.4.2b}$$

Note that the use of isoreals, isocomplexes and isoquaternions preserves the associative character of the underlying envelope. The use instead of isooctonions $\hat{O}(\hat{o},+,\hat{\times})$ (Sect. I.2.8) would imply the loss of such an associative character and, for this reason, isooctonions have been excluded as possible isofields in Definition 2.3.1 in a way fully parallel to conventional lines in number theory. Nevertheless, one should keep in mind that the *Lõhmus-Paal-Sorgsepp octonionization process* [39] resolves the above problematic aspects.

In the original proposal [1] this author proved the existence of consistent *isotopic generalization of the celebrated Lie's First, Second and Third Theorems.* For brevity, we refer the interested reader to ref. [4], pp. 163–184 or to the ref. [24], Ch. II. We here quote the *Isotopic second and third Theorems* because useful in applications for the speedy construction of one realizations of Lie-isotopic algebras (see later on for more complex realizations).

Theorem 4.4.1 - Lie-Santilli Second Theorem [1]: *Let* $X = \{X_k\}$, $k = 1, 2, ..., N$, *be the (ordered set of) generators in adjoint representations of a Lie algebra* L *with commutation rules*

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general Lie-admissible formulations for which the underlying envelope must remain still isoassociative, and the units must still exist, although they are differentiated for the right and left multiplications.

$$\mathbf{L}: \ [X_i, X_j] = X_i \times X_j - X_j \times X_j = C_{ij}^{\ k} X_k , \tag{4.4.3)}$$

where $C_{ij}^{\ k}$ *are the "structure constants". Then, one realization of the Lie-isotopic images* $\hat{\mathbf{L}}$ *of* $\mathbf{L}$ *is characterized by the same generators now computed in isospaces over isofields* $\hat{X}$ *with isocommutation rules*

$$\hat{\mathbf{L}}: \ [\hat{X}_i \hat{,} \hat{X}_j] = \hat{X}_i \hat{\times} \hat{X}_j - \hat{X}_j \hat{\times} \hat{X}_i = \hat{X}_i \times \hat{T} \times \hat{X}_j - \hat{X}_j \times \hat{T} \times \hat{X}_i =$$

$$= X_i\, T(x, \dot{x}, ..)\, X_j - X_j\, T(x, \dot{x}, ...)\, X_i = \hat{C}_{ij}^{\ k}(t, x, \dot{x}, \ ...) \hat{\times} \hat{X}_k =$$

$$= \ \hat{C}_{ij}^{\ k}(x, \dot{x}, \ ...)\, \hat{X}_k , \tag{4.4.4}$$

where the $\hat{C}_{ij}^{\ k}$ *are the "structure functions" in the isofield.*

Theorem 4.4.2 - Lie-Santilli Third Theorem [loc. cit.]: *The structure functions* $\hat{C}_{ij}^{\ k}$ *of a Lie-isotopic algebra* $\hat{\mathbf{L}}$ *verify the conditions*

$$\hat{C}_{ij}^{\ k} = - \hat{C}_{ji}^{\ k} , \tag{4.4.5}$$

and the property (when commuting with the generators [23]*)*

$$\hat{C}_{ij}^{\ p} * \hat{C}_{pk}^{\ q} + \hat{C}_{jk}^{\ p} * \hat{C}_{pi}^{\ q} + \hat{C}_{ki}^{\ p} * \hat{C}_{pj}^{\ q} = 0 . \tag{4.4.6}$$

The Lie-isotopic theorems have fundamental mathematical and physical relevance for all isotopic theories (for which reason they were given first attention in the original proposal [1]). Mathematically they identify the type of algebra and physically they identify the brackets of the basic time evolution of the theory.

In fact, the classical time evolution of a quantity Q(t) in the isotopic theories is given by the left and right, bimodular formulation of one-sided isotransforms (4.2.10) and 4.3.7),

$$Q(t) = \{ \hat{e}^{\,X t} \} \hat{\times} Q(0) \hat{\times} \{ \hat{e}^{\,t X} \} = \{ e^{\,X T t} \} Q(0) \{ \hat{e}^{\,-t T X} \} \tag{4.4.7}$$

which, for infinitesimal valued of times, $t \rightarrow dt$, yields the *fundamental brackets of the classical time evolution* in terms of a isohamiltonian vector field X [1,4]

$$dQ / dt = [Q(dt) - Q(0)] / dt = [Q \hat{,} H] = Q T H - H T Q , \tag{4.4.8}$$

[23] If not, more general properties are easily derivable from Jacobi's law.

which results to be of the Lie-isotopic type precisely according to the Lie-isotopic Theorems.

At the operator level we have a much similar situation. In fact, the time evolution of an operator Q(t) in the Heisenberg-type formulations is given again by the right and left, bimodular extension of the one-sided isotransforms (4.2.11) and (4.3.7),

$$Q(t) = \{ \hat{e}^{\,i X t} \} \hat{\times} Q(0) \hat{\times} \{ \hat{e}^{\,i t X} \} = \{ e^{\,i X T t} \} Q(0) \{ \hat{e}^{\,-i t T X} \} \quad (4.4.9)$$

which, for infinitesimal valued of times, $t \rightarrow dt$, yields the *fundamental brackets of the operator time evolution* (2,4]

$$dQ / dt = [\, Q(dt) - Q(0)] / dt = [\, Q \,\hat{,}\, H \,] = Q\,T\,H - H\,T\,Q, \quad (4.4.10)$$

which result to be again in full compliance with the Lie-isotopic Theorems.

Most of the efforts conducted in the last decades on the Lie-isotopic formulations, as reported in these volumes, have been devoted to the identification of algebraic, geometric and analytic formulations which are compatible with fundamental time evolution laws (4.4.8) and (4.4.10).

We learn in this way that the structure "constants" of Lie's theory acquire a dependence on local variables similar to that of the isotopic element $\hat{T}$, thus becoming structure "functions'.

It is important to illustrate the above theorems with an example. Consider the generators of the **su(2)** Lie algebra in their adjoint representation, which are given by the celebrated *Pauli's matrices* and related commutation rules

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.4.11a)$$

$$[\, \sigma_n \,, \sigma_m \,] = \sigma_n \times \sigma_m - \sigma_m \times \sigma_n = 2\, i\, \epsilon_{nmk}\, \sigma_k \,, \quad (4.4.11b)$$

Theorem 4.4.1 states that the *same generators* σ_k , when written in isospaces over isofields, can characterize *one realization* of the Lie-isotopic **$\hat{su}$(2)** algebra via the lifting of the structure constants into suitable functions.

This property is readily verified by introducing a Class III isotopic element assumed diagonal for simplicity, and then identifying the structure functions under which the algebra is closed. By ignoring for notational simplicity the rewriting of the basis in isospace, we have the following illustration of the Lie-isotopic Second Theorem

$$[\, \sigma_n \,\hat{,}\, \sigma_m \,] = \sigma_n \times T \times \sigma_m - \sigma_m \times T \times \sigma_n = 2\, i\, \hat{\epsilon}_{nmk} \times T \times \sigma_k \,, \quad (4.4.12a)$$

$$T = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}, \quad g_{kk} \neq 0, \quad \Delta = \det T = g_{11}\, g_{22}, \tag{4.4.12b}$$

$$\hat{1} = \begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & g_{22}^{-1} \end{pmatrix} \quad = \quad \Delta^{-1} \begin{pmatrix} g_{22} & 0 \\ 0 & g_{11} \end{pmatrix}, \tag{4.4.12c}$$

$$\hat{\epsilon}_{ijk} = \epsilon_{ijk} \begin{pmatrix} g_{22}/g_{11} & 0 \\ 0 & g_{11}/g_{22} \end{pmatrix}. \tag{4.4.12d}$$

Note that the original structure "constants" $C_{ij}^{\ k}$ are elements of a field $F(a,+,\times)$ and, as such, are ordinary numbers. On the contrary, the structure "functions" $\hat{C}_{ij}^{\ k}$ are now elements of the isofield $\hat{F}(\hat{a},+,\hat{\times})$ and, as such, are isonumbers and, thus, matrices. As such, they should be called more properly *structure isofunctions*, where the prefix "iso" stands precisely to represent their matrix character.

Note finally that Theorem 4.4.1 provide only *one method* for the speedy construction of an isotope $\hat{L}$ of a given Lie algebra L. In general, the above methods is not applicable because Lie and Lie-isotopic algebras are connected by a nonunitary transform (Sect. I.4.1), thus implying *different generators*. In fact, another way of constructing Class I isotopes $\hat{L}$ of a given Lie algebra L is by generalizing the generators X_k and keeping instead the old structure constants. This alternative approach will be used in a number of applications because it evidently ensures the local isomorphism $\hat{L} \approx L$ *ab initio*, while lifting conventional symmetries into the desired nonlinear-nonlocal-noncanonical form.

Theorems 4.4.1 and 4.4.2 were however conceived for specific physical needs. Recall that the generators of a Lie algebra represent physical quantities, such as linear momentum, angular momentum, energy, etc. As such, these quantities cannot be changed under isotopies, thus explaining the preservation of the original basis. An additional motivation is that, among all possible realizations, the method of Theorem 4.4.1 results to be most effective in the computation of the symmetries of nonlinear-nonlocal-noncanonical systems, as we shall see in Sect. I.4.6.

It is easy to prove the following:

Theorem 4.4.3 [2]: *The isotopies* $L \to \hat{L}$ *of an N-dimensional Lie algebra* L *preserve the original dimensionality.*

In fact, the basis e_k, $k = 1, 2, ..., N$ of a vector space and, thus, of a Lie algebra L is not changed under isotopy, except for renormalization factors denoted $\hat{e}_k$. Let then the commutation rules of L be given by

$$[\, e_i \, , e_j \,] \;=\; C_{ij}{}^k \, e_k. \tag{4.4.13}$$

The isocommutation rules of the isotopes $\hat{L}$ are

$$[\hat{e}_i \,\hat{,}\, \hat{e}_j \,] \;=\; \hat{e}_i \,\hat{\uparrow}\, \hat{e}_j \;-\; \hat{e}_j \,\hat{\uparrow}\, \hat{e}_i \;=\; \hat{C}_{ij}{}^k(t, x, \dot{x}, \;\ldots) \,\hat{e}_k \,. \tag{4.4.14}$$

One can see again in this way the necessity of lifting the structure "constants" into structure "functions", as correctly predicted by the Lie-isotopic Second Theorem. A number of examples will be provided during the course of our analysis.

We now review a few basic notions of Lie-isotopic algebras $\hat{L}$ which can be derived via an easy isotopy of the corresponding conventional notions (as available, e.g., in ref.s [31–33]). Lie-isotopic algebras $\hat{L}$ are said to be:

a) *isoreal (isocomplex)* when $\hat{F} = \hat{R}$ ($\hat{F} = \hat{C}$);

2) *isoabelian* when $[\hat{A} \hat{,} \hat{B}] \equiv 0$, $\forall$ A, B $\in \hat{L}$;

3) A subset $\hat{L}_0$ of $\hat{L}$ is said to be an *isosubalgebra* of $\hat{L}$ when

$$[\, \hat{L}_0 \hat{,} L_0 \,] \;\subset\; \hat{L}_0 \,; \tag{4.4.15}$$

4) An *isoideal* occurs when

$$[\, \hat{L} \hat{,} \hat{L}_0 \,] \;\subset\; \hat{L}_0; \tag{4.4.16}$$

5) The *isocenter* of a Lie-isotopic algebra is the maximal isoideal L_0 which verifies the property

$$[\, \hat{L} \hat{,} \hat{L}_0 \,] \;=\; 0. \tag{4.4.17}$$

Definition 4.4.2 [27]**:** *The "general isolinear and isocomplex Lie-isotopic algebras", here denoted with* $G\hat{L}(n,\hat{C})$, *are the vector isospaces of all n×n complex matrices over* $\hat{C}(\hat{c},+,\hat{*})$, *and are evidently closed under isocommutators. The "isocenter" of* $G\hat{L}(n,\hat{C})$ *is then given by* $\hat{c}\hat{*}\hat{1}$, $\forall$ $\hat{c} \in \hat{C}$. *The subset of all complex n×n matrices with null trace is also closed under isocommutators, it is called the "special, isolinear, isocomplex, Lie-isotopic algebra", and denoted with* $S\hat{L}(n,\hat{C})$. *The subset of all antisymmetric n×n real matrices* X, $X^t = -X$, *is also closed under isocommutators, is called the "isoorthogonal algebra", and is denoted with* $\hat{o}(n)$.

By proceeding along similar lines, one can classify all classical, non-exceptional, Lie-Santilli isoalgebras into the isotopes of the conventional forms, denoted with $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$ and $\hat{D}_n$ according to the general rules [27]

$$\begin{aligned} \textbf{Class } \hat{A}_{n-1} &= \hat{SL}(n, \hat{C}); \\ \textbf{Class } \hat{B}_n &= \hat{O}(2n+1, \hat{C}); \\ \textbf{Class } \hat{C}_n &= \hat{SP}(n, \hat{C}); \text{ and} \\ \textbf{Class } \hat{D}_n &= \hat{O}(2n, \hat{C}). \end{aligned}$$

plus the *isoexceptional algebras* here ignored for brevity.

Each one of the above algebras then needs its own classification (evidently absent in the conventional case), depending on whether $\hat{1}$ is positive-definite (Class I), negative definite (Class II), indefinite (Class III), singular (IV) and general (Class V), as well as whether of isocharacteristic zero or p, thus illustrating the richness of the isotopic theory indicated above.

The notions of *homomorphism, automorphism and isomorphism* of two Lie-isotopic algebras $\hat{L}$ and $\hat{L}'$ are the conventional ones. Similarly, all properties of Lie algebras based on the addition, such as the *direct* and *semidirect sums* , carry over to the isotopic context unchanged (because of the preservation of the conventional additive unit 0).

By following Kadeisvili [27] we now introduce an *isoderivation* $\hat{D}$ of a Lie-isotopic algebra $\hat{L}$ as an isolinear map of $\hat{L}$ into itself satisfying the property

$$\hat{D}([\hat{A} \hat{,} \hat{B}]) = [\hat{D}(\hat{A}) \hat{,} \hat{B}] + [\hat{A} \hat{,} \hat{D}(\hat{B})] \quad \forall \hat{A}, \hat{B} \in \hat{L} . \tag{4.4.18}$$

If two maps $\hat{D}_1$ and $\hat{D}_2$ are isoderivations, then $\hat{a}\hat{\times}\hat{D}_1 + \hat{b}\hat{\times}\hat{D}_2$ is also an isoderivation, and the isocommutators of $\hat{D}_1$ and $\hat{D}_2$ is also an isoderivation. Thus, the set of all isoderivations forms a Lie-isotopic algebra as in the conventional case.

The isolinear map $\hat{ad}(\hat{L})$ of $\hat{L}$ into itself defined by

$$\text{Isoad } \hat{A}(\hat{B}) = [\hat{A} \hat{,} \hat{B}], \quad \forall \hat{A}, \hat{B} \in \hat{L}, \tag{4.4.19}$$

is called the *isoadjoint map* . It is an isoderivation, as one can prove via the Jacobi identity (4.4.2b). The set of all $\hat{ad}(A)$ is therefore an isolinear Lie-isotopic algebra, called *isoadjoint algebra* and denoted $\hat{L}_a$. It also results to be an isoideal of the algebra of all isoderivations as in the conventional case.

Consider the algebras

$$\hat{L}^{(o)} = \hat{L}, \quad \hat{L}^{(1)} = [\hat{L}^{(o)} \hat{,} \hat{L}^{(o)}], \quad \hat{L}^{(2)} = [\hat{L}^{(1)} \hat{,} \hat{L}^{(1)}], \quad \text{etc.}, \tag{4.4.20}$$

which are also isoideals of $\hat{L}$. $\hat{L}$ is then called *isosolvable* if, for some positive integer n, $\hat{L}^{(n)} \equiv 0$.

Consider also the sequence

$$L_{(o)} = L, \quad \hat{L}_{(1)} = [\hat{L}_{(o)} \hat{,} \hat{L}], \quad \hat{L}_{(2)} = [\hat{L}_{(1)} \hat{,} \hat{L}], \text{ etc.} \tag{4.4. 21}$$

Then $\hat{L}$ is said to be *isonilpotent* if, for some positive integer n, $\hat{L}_{(n)} \equiv 0$. One can

then see that, as in the conventional case, an isonilpotent algebra is also isosolvable, but the converse is not necessarily true.

Let the *isotrace* of a matrix be given by the element of the isofield

$$\text{Isotr } A \;=\; (\text{Tr } A)\hat{1} \;\in \hat{F}, \tag{4.4.22}$$

where TrA is the conventional trace. Then

$$\text{Isotr}\,(A \hat{\times} B) = (\text{Isotr } A) \hat{\times} (\text{Isotr } B), \qquad \text{Isotr}\,(B \hat{\times} A \hat{\times} B^{\hat{-1}}) = \text{Isotr } A\,. \tag{4.4.23}$$

Thus, Isotr A preserves the axioms of Tr A, by therefore being a correct isotopy. Then, the isoscalar product

$$(\hat{A}\,\hat{,}\,\hat{B}) \;=\; \text{Isotr}\,[\,(\text{Isoad } \hat{A}) \hat{\times} (\text{Isoad } \hat{B})\,] \tag{4.4.24}$$

is called the *isokilling form* as first studied by Kadeisvili [27]. It is easy to see that $(\hat{A}\,\hat{,}\,\hat{B})$ is symmetric, bilinear, and verifies the property

$$(\text{Isoad } \hat{X}\,(\hat{Y})\,\hat{,}\,\hat{Z}) \;+\; (\hat{Y}\,\hat{,}\,\text{Isoad } \hat{X}\,(\hat{Z}) \;=\; 0, \tag{4.4.25}$$

thus being a correct, axiom-preserving isotopy of the conventional Killing form.

Let e_k, $k = 1, 2, \ldots, N$, be the basis of L with one-to-one invertible map $e_k \to \hat{e}_k$ into the basis $\hat{e}_k$ of $\hat{L}$ Generic elements in $\hat{L}$ can then be written in terms of local coordinates x, y, z,

$$\hat{A} = x^i\,\hat{e}_i\,,\ \hat{B} = y^j\,\hat{e}_j\,,\ C = z^k\,\hat{e}_k \;=\; [\,\hat{A}\,\hat{,}\,\hat{B}\,] = x^i\,y^j\,[\,\hat{e}_i\,\hat{,}\,\hat{e}_j\,] \;=$$

$$=\; x^i\,x^j\,\tilde{C}_{ij}{}^k\,\hat{e}_k\,. \tag{4.4.26}$$

Thus,

$$[\,\text{Isoad } \hat{A}\,(\hat{B})\,]^k \;=\; [\,\hat{A}\,\hat{,}\,\hat{B}\,]^k \;=\; \tilde{C}_{ij}{}^k\,x^i\,x^j\,. \tag{4.4.27}$$

By following again Kadeisvili [27], we now introduce the *isocartan tensor* $\tilde{g}_{ij}$ of a Lie-isotopic algebra $\hat{L}$ via the definition $(\hat{A}\,\hat{,}\,\hat{B}) = \tilde{g}_{ij}\,x^i\,y^j$ yielding

$$\tilde{g}_{ij}(t, x, \dot{x}, \ddot{x}, \ldots) \;=\; \tilde{C}_{ip}{}^k\,\tilde{C}_{jk}{}^p\,. \tag{4.4.28}$$

Note that the isocartan tensor has the general dependence of the isometric tensor of the preceding chapter, thus confirming the inner consistency among the various branches of the isotopic theory. In particular, the isocartan tensor is generally *nonlinear, nonlocal and noncanonical* in all variables $x, \dot{x}, \ddot{x}, \ldots$.

The isocartan tensor also clarifies another important point of the preceding

analysis, that the isotopies naturally lead to an arbitrary dependence in the velocities and accelerations, exactly as needed for realistic models of interior dynamical problems, and that their restriction to the nonlinear dependence on the coordinates x only, as needed for the exterior gravitational problem, would be manifestly un-necessary.

The isotopies of the structure theory of Lie algebras then follow, including the notion of *simplicity, semisimplicity,* etc. (see the monograph [24]) Here we limit ourselves to recall the following

Definition 4.4.3 [27]: *A Lie-isotopic algebra $\hat{L}$ is called "compact" ("noncompact") when the isocartan form is positive- (negative-) definite.*

Numerous additional, more refined definitions of compactness and noncompactness are possible via the isotopies of the corresponding conventional definitions [31-33]. The above definition is however sufficient for our needs.

We now study a few implications of the isotopic lifting of Lie's theory.

Theorem 4.4.4 [1]: *The isotopes of Class III $\hat{L}$ of a compact (noncompact) Lie algebra L are not necessarily compact (noncompact).*

The identification of the remaining properties which are not preserved under liftings of Class III is an instructive task for the reader interested in becoming an expert in isotopic theories. For instance, if the original structure is irreducible, its isotopic image is not necessarily so even for Class I, trivially, because the isounit itself can be reducible, thus yielding a reducible isotopic structure.

Definition 4.4.4 [8]: *Let $\hat{L}$ be a Lie-isotopic algebra with generators $\hat{X}_k$ and isounit $\hat{1} = \hat{T}^{-1} > 0$. The "isodual Lie-isotopic algebras" $\hat{L}^d$ is the isoalgebra with isodual generators $\hat{X}_k{}^d = -\hat{X}_k$ conventional structure functions over the isodual isofield $\hat{F}^d(\hat{a}^d,+,\hat{\times}^d)$ with "isodual isocommutators"*

$$[\hat{X}_i \hat{,} \hat{X}_j]^d = -[\hat{X}^d{}_i \hat{,} \hat{X}^d{}_j] = -[\hat{X}_i \hat{,} \hat{X}_j] = \tilde{C}_{ij}{}^k \hat{X}^d{}_k = -\tilde{C}_{ij}{}^k \hat{X}_k. \qquad (4.4.29)$$

When the original algebra is a Lie algebra L the "isodual Lie algebra" is given by the structure L^d over the isodual field $F^d(a^d,+,\times^d)$ with "isodual commutators"

$$[\hat{X}_i , \hat{X}_j]^d = \hat{X}_i \times^d \hat{X}_j - \hat{X}_j \times^d \hat{X}_i = -[\hat{X}_i , \hat{X}_j] = -C_{ij}{}^k \hat{X}_k . \qquad (4.4.30)$$

$\hat{L}$ and $\hat{L}^d$ are then anti-isomorphic. Note that the isoalgebras of Class III contain all algebras $\hat{L}$ and all their isoduals $\hat{L}^d$. The above remarks therefore show that the Lie-isotopic theory can be naturally formulated for Class III, as implicitly done in ref. [1].

Note the necessity of the isotopies for the very construction of the isodual of conventional Lie algebras. In fact, they require the nontrivial lift of the unit I $\rightarrow I^d = (-I)$, with consequential necessary generalization of the Lie product AB − BA into the isotopic form ATB − BTA.

The following property is mathematically trivial, yet carries important physical applications.

Theorem 4.4.5 [1,5]: *All infinitely possible, isotopes $\hat{L}$ of Class I of a (finite-dimensional) Lie algebra L are locally isomorphic to L, and all infinitely possible isodual isotopes $\hat{L}^d$ of Class II are locally isomorphic to L^d*

The simplest possible proof is via the redefinition of the basis $\hat{X}_k \rightarrow \hat{X}'_k = X_k\hat{1}$, under which isotopic algebras $\hat{L}$ acquire the same structure constants of L,

$$[\hat{X}_i \hat{,} \hat{X}_j] \rightarrow [\hat{X}'_i \hat{,} \hat{X}'_j] = [\hat{X}_i, \hat{X}_j]\hat{1} = C_{ij}{}^k \hat{X}'_k . \tag{4.4.31}$$

We should however indicate that, even though the above reduction is possible, in general we have $\hat{C}_{ij}{}^k \neq C_{ij}{}^k \hat{1}$, as it is the case of example (4.4.8), thus rendering inapplicable the realization $\hat{X}' = X\,\hat{1}$. Also the realization $\hat{X}'_k = X_k\hat{1}$ does not yield the desired nonlinear-nonlocal-nonhamiltonian isosymmetries as we shall see in Sect. 4.6.

Despite the local isomorphism $L \approx \hat{L}$, the lifting $L \rightarrow \hat{L}$ is not mathematically trivial because these two algebras are not unitarily equivalent. The physical relevance of the isotopies originates precisely from their local isomorphism, because it permits the construction of nonlinear, nonlocal and noncanonical isotopes of the rotational $S\hat{O}(3)$, Galilean $\hat{G}(3.1)$, Lorentz $\hat{O}(3.1)$, Poincaré $\hat{P}(3.1)$, $S\hat{U}(3)$ and other space-time and internal symmetries which are locally isomorphic to the original algebras.

Theorem 4.4.5 therefore represents the property which has permitted the achievement of methods for the nonlinear-nonlocal-noncanonical interior problems by preserving the analytic, algebraic and geometric axioms of the conventional, linear-local-canonical methods of the exterior problems [4].

For additional technical studies of the Lie-isotopic algebras we refer the reader to the forthcoming book [24].

We now illustrate the results of this sections with the isotopies and isodualities of the rotational algebra **so(3)** with generators in their adjoint form (4.3.13). For this purpose, the isounit and isotopic element of Class III, Eq. (4.3.16), can be realized in the form

$$\hat{1} = \mathrm{diag.}\,(\pm b_1^{-2}, \pm b_2^{-2}, \pm b_3^{-2}), \quad b_k(t, r, \dot{r}, \ddot{r}, \ldots) \neq 0, \tag{4.4.32a}$$

$$\delta = \hat{T} = \mathrm{diag.}\,(\pm b_1^2, \pm b_2^2, \pm b_3^2), \tag{4.4.32b}$$

The Isotopic Second Theorem 4.4.1 then yields

$$[\mathfrak{J}_i \hat{,} \mathfrak{J}_j] = \mathfrak{J}_i \hat{T} \mathfrak{J}_j - \mathfrak{J}_j \hat{T} \mathfrak{J}_i = \hat{C}_{ij}{}^k(t, r, \dot{r}, \ddot{r}, \ldots) \hat{T} \mathfrak{J}_k \tag{4.4.33}$$

where the $\mathfrak{J}$'s are the conventional adjoint generators (4.3.13) only rewritten in isospace and the $\hat{C}$'s are the structure functions.

It is easy to see that all possible isoalgebras (4.4.8) are those of classification (4.3.18), and are given by [1,9]:

1) **so(3)** for $\hat{T} \equiv I = \text{diag.}\ (1, 1, 1)$ with commutation rules

$$[J_1, J_2] = J_3, \ [J_2, J_3] = J_1, \ [J_3, J_1] = J_2; \tag{4.4.34}$$

2) **so(2.1)** for $\hat{T} = \text{diag.}\ (1, -1, 1)$ with rules

$$[\mathfrak{J}_1 \hat{,} \mathfrak{J}_2] = \mathfrak{J}_3, \ [\mathfrak{J}_2 \hat{,} \mathfrak{J}_3] = -\mathfrak{J}_1, \ [\mathfrak{J}_3 \hat{,} \mathfrak{J}_1] = \mathfrak{J}_2; \tag{4.4.35}$$

3) An infinite family of isotopes $\hat{so}(3)$ isomorphic to so(3) for $\hat{T} = \text{diag.}\ (b_1^2, b_2^2, b_3^2)$ with rules

$$[\mathfrak{J}_1 \hat{,} \mathfrak{J}_2] = b_3^2 \mathfrak{J}_3, \ [\mathfrak{J}_2 \hat{,} \mathfrak{J}_3] = b_1^2 \mathfrak{J}_1, \ [\mathfrak{J}_3 \hat{,} \mathfrak{J}_1] = b_2^2 \mathfrak{J}_2; \tag{4.4.36}$$

4) An infinite family of isotopes $\hat{so}(2.1)$ isomorphic to so(2.1) for $\hat{T} = \text{diag.}\ (b_1^2, -b_2^2, b_3^2)$ and rules

$$[\mathfrak{J}_1 \hat{,} \mathfrak{J}_2] = b_3^2 \mathfrak{J}_3, \ [\mathfrak{J}_2 \hat{,} \mathfrak{J}_3] = -b_1^2 \mathfrak{J}_1, \ [\mathfrak{J}_3 \hat{,} \mathfrak{J}_1] = b_2^2 \mathfrak{J}_2; \tag{4.4.37}$$

5) The isodual $so^d(3)$ of **so(3)** for $\hat{T} = \text{diag.}\ (-1, -1, -1)$ and rules

$$[\mathfrak{J}_1 \hat{,} \mathfrak{J}_2] = -\mathfrak{J}_3, \ [\mathfrak{J}_2 \hat{,} \mathfrak{J}_3] = -\mathfrak{J}_1, \ [\mathfrak{J}_3 \hat{,} \mathfrak{J}_1] = -\mathfrak{J}_2; \tag{4.4.38}$$

6) The isodual $so^d(2.1)$ of **so(2.1)** for $\hat{T} = \text{diag.}\ (-1, 1, -1)$ and rules

$$[\mathfrak{J}_1 \hat{,} \mathfrak{J}_2] = -\mathfrak{J}_3, \ [\mathfrak{J}_2 \hat{,} \mathfrak{J}_3] = \mathfrak{J}_1, \ [\mathfrak{J}_3 \hat{,} \mathfrak{J}_1] = -\mathfrak{J}_2; \tag{4.4.39}$$

7) The infinite family of isotopes $\hat{so}^d(3) \approx so^d(3)$ for $\hat{T} = \text{diag.}\ (-b_1^2, -b_2^2, -b_3^2)$ and rules

$$[\mathfrak{J}_1 \hat{,} \mathfrak{J}_2] = -b_3^2 \mathfrak{J}_3, \ [\mathfrak{J}_2 \hat{,} \mathfrak{J}_3] = -b_1^2 \mathfrak{J}_1, \ [\mathfrak{J}_3 \hat{,} \mathfrak{J}_1] = -b_2^2 \mathfrak{J}_2; \tag{4.4.40}$$

8) The infinite family of isotopes $\hat{so}^d(2.1) \approx SO^d(2.1)$ for $\hat{T} = \text{diag.}\ (-b_1^2, b_2^2, -b_3^2)$ and rules

$$[\mathfrak{J}_1 \hat{,} \mathfrak{J}_2] = -b_3{}^2 \mathfrak{J}_3, \; [\mathfrak{J}_2 \hat{,} \mathfrak{J}_3] = b_1{}^2 \mathfrak{J}_1, \; [\mathfrak{J}_3 \hat{,} \mathfrak{J}_1] = -b_2{}^2 \mathfrak{J}_2; \tag{4.4.41}$$

The reader can readily verify the above indicated local isomorphisms via the redefinition of the basis

$$\mathfrak{J}'_1 = b_1{}^{-1} b_3{}^{-1} \mathfrak{J}_1, \quad \mathfrak{J}'_2 = b_1{}^{-1} b_3{}^{-1} \mathfrak{J}_2, \quad \mathfrak{J}'_3 = b_1{}^{-1} b_2{}^{-1} \mathfrak{J}_3, \tag{4.4.42}$$

in which case the b-terms in the r.h.s. of the commutation rules disappear and one recovers conventional structure constants of so(3) and so(2.1) under isotopies (see Ch. II-6 for details).

It is also significant that exactly the same classification exists for the isotopies of **so(3)** in classical mechanics, in which case the isoproduct is given by an isotopy of the conventional Poisson brackets (see ref. [22] for details). This latter occurrence is important to understand that the conventional quantization of the classical rotational symmetry carries over in its entirety to the isotopic and isodual coverings (Vol. II).

It is instructive for the interested reader to verify with the above examples various other notions introduced in this section, such as the isocartan's tensor, the isokilling form, etc. We shall have plenty of opportunities to study additional examples of Lie-isotopic algebras in Vol.s II and III.

As final comments, we discourage the reader from applying conventional notions of Lie's theory to the covering Lie-isotopic theory without their specific isotopic reformulation. This is due to the lack of general preservation of structural properties of the original Lie algebras, such as compactness, irreducibility, etc.

The reader should also be aware of the physical importance of preserving under isotopies the original generators X_k (i.e., the original basis). In fact, the generators represent physical quantities, such as total energy, linear momentum, angular momentum, etc. which, as such, cannot be changed by isotopies or other techniques. Similarly, the parameters represent physics measurable quantities such as angles of rotation, velocities, etc. This also illustrates the preservation under isotopies of the conventional parameters $w \in F$ merely .lifted into the form $\hat{w} = w\hat{1} \in \hat{F}$.

In Vol. II we shall identify a classical and an operator realization of the Lie-Santilli isoalgebras with a simple, yet unique and unambiguous interconnecting map. Within such a setting, *the conventional total conservation laws of the classical and operator theories can be simply read-off the generators of the Lie-isotopic symmetries.*

4.5: LIE-SANTILLI ISOGROUPS AND THEIR ISODUALS

As indicated earlier, the isotopies of a topological space are still lacking at this writing and so are the isotopies of topological Lie groups. Only the isotopies of Lie's transformation groups are available essentially according to the original proposals [1,2,4].

Definition 4.5.1 [1]: *A "right Lie-Santilli (transformation) isogroup" $\hat{G}$, or "isogroup" for short, on an isospace $\hat{S}(\hat{x},\hat{F})$ over an isofield $\hat{F}(\hat{a},+,\hat{\times})$ (of isoreal numbers $\hat{R}$ or isocomplex numbers $\hat{C}$ or isoquaternions $\hat{Q}$) is a group which maps each element $\hat{x} \in \hat{S}(\hat{x},\hat{F})$ into a new element $\hat{x}' \in \hat{S}(\hat{x},\hat{F})$ via the isotransformations*

$$\hat{x}' = \hat{U} \hat{\times} \hat{x} = \hat{U}\, \hat{T}\, \hat{x}, \qquad \hat{T} \textit{ fixed}, \tag{4.5.1}$$

such that:
1) The map $(\hat{U}, \hat{x}) \rightarrow \hat{U} \hat{\times} \hat{x}$ of $\hat{G} \hat{\times} \hat{S}(\hat{x},\hat{F})$ onto $\hat{S}(\hat{x},\hat{F})$ is differentiable;
2) $\hat{1} \hat{\times} \hat{U} = \hat{U} \hat{\times} \hat{1} = \hat{U}$, $\forall \hat{U} \in \hat{G}$; and
3) $\hat{U}_1 \hat{\times} (\hat{U}_2 \hat{\times} \hat{x}) = (\hat{U}_1 \hat{\times} \hat{U}_2) \hat{\times} \hat{x}$, $\forall \hat{x} \in \hat{S}(\hat{x},\hat{F})$ *and* $\hat{U}_1, \hat{U}_2 \in \hat{G}$.
A "left Lie-Santilli (transformation) isogroup" is defined accordingly.

Right or left Lie-isotopic groups are characterized by the following *isogroup laws* first introduced in ref. [1]

$$\hat{U}(\hat{0}) = \hat{1}, \tag{4.5.2a}$$

$$\hat{U}(\hat{w}) \hat{\times} \hat{U}(\hat{w}') = \hat{U}(\hat{w}') \hat{\times} \hat{U}(\hat{w}) = \hat{U}(\hat{w} + \hat{w}'), \tag{4.5.2b}$$

$$\hat{U}(\hat{w}) \hat{\times} \hat{U}(-\hat{w}) = \hat{1}, \qquad \hat{w} \in \hat{F}, \tag{4.5.2c}$$

The most important meaning of the isogroups is that of identifying the group structure of the classical and operator time evolution of isotopic theories. In fact, it is easy to verify that the isotransforms from (4.2.10), (4.2.11), (4.37), (4.4.8) and (.4.4.10),

$$\hat{x}' = \hat{U}(\hat{t}) \hat{\times} \hat{x}, \tag{4.5.3a}$$

$$\hat{U} = \hat{e}^{\hat{X} \hat{\times} \hat{t}} \quad \text{or} \quad = \hat{e}^{i \hat{H} \hat{\times} \hat{t}}, \quad \hat{t} = t \times \hat{1} \tag{4.5.3b}$$

do indeed constitute Lie-Santilli isogroups as per Definition (4.5.1).

Note the insufficiency of the conventional Lie groups for the characterization of structures (4.5.3) on numerous independent grounds, such as:

Lie (transformation) groups have a linear, local and canonical structure while structures (3.5.3) are nonlinear, nonlocal and noncanonical; Lie groups are fundamentally dependent on the form I = diag. (1, 1, ..., 1) of the basic unit, while structures (4.5.3) have arbitrary integro-differential quantities $\hat{1}$ for basic unit; etc.

Most opf the studies conducted on isotopies until now have been focused on the achievement of a formulation of functional analysis, geometries and mechanics compatible with the isotopic structure of groups (4.5.3). In the following we identify only those rudimentary properties of the isogroups which are necessary for the physical studies of Vol. II.

The notions of *connected or simply connected transformation groups* (see, e.g., ref.s [31-33]) carry over to the Lie-isotopic groups in their entirety. We consider hereon the connected Lie-isotopic transformation groups (see Sect. 4.6 for the discrete parts).

Evidently, Eq.s (4.5.3) hold for some open neighborhood N of the isoorigin of $\hat{L}$ and which, in this way, characterizes some open neighborhood of the isounit of $\hat{G}$ (see in this respect [27,28]).

Still another important property permitting the isocomposition of Lie-isotopic groups is given by the following

Theorem 4.5.1 - Isotopic Baker-Campbell-Hausdorff theorem [1,4]: *The conventional group composition laws admit a consistent isotopic lifting, resulting in the following "isotopic composition law"*

$$\hat{U}_1 \hat{\times} \hat{U}_2 = \{ e_{\xi}^{\hat{X}_1} \} \hat{\times} \{ e_{\xi}^{\hat{X}_2} \} = \hat{U}_3 = e_{\xi}^{\hat{X}_3}, \tag{4.5.4a}$$

$$\hat{X}_3 = \hat{X}_1 + \hat{X}_2 + [\hat{X}_1 \hat{,} \hat{X}_2] / 2 + [(\hat{X}_1 - \hat{X}_2) \hat{,} [\hat{X}_1 \hat{,} \hat{X}_2]] / 12 + ... \tag{4.5.4b}$$

By following Kadeisvili [27], we now study the connection between Lie-Santilli isogroups and isoalgebras. Let $\hat{L}$ be a (finite-dimensional) Lie-isotopic algebra with (ordered) basis $\hat{X}_k$, k = 1, 2, ..., N. For a sufficiently small neighborhood N of the isoorigin of $\hat{L}$, a generic element of $\hat{G}$ can be written

$$\hat{U}(\hat{w}) = \hat{\prod}_{k=1,2,...,N} e_{\xi}^{i\hat{X}_k \hat{\times} \hat{w}_k}. \tag{4.5.6}$$

which characterizes some open neighborhood M of the isounit $\hat{1}$ of $\hat{G}$.

The map

$$\Phi_{\hat{U}_1}(\hat{U}_2) = \hat{U}_1 \hat{\times} \hat{U}_2 \hat{\times} \hat{U}_1^{-\hat{1}}, \tag{4.5.7}$$

for a fixed $\hat{U}_1 \in \hat{G}$, characterizes an *inner isoautomorphism* of $\hat{G}$ onto itself. The

corresponding isoautomorphism of the algebra $\hat{L}$ can be readily computed by considering expression (4.5.7) in the neighborhood of the isounit $\hat{1}$, in which case we have

$$\hat{U}'_2 = \hat{U}_1 \hat{\times} \hat{U}_2 \hat{\times} \hat{U}_1^{-1} \cong \hat{U}_2 + \hat{w}_1 \hat{\times} \hat{w}_2 \hat{\times} [\hat{X}_2 \hat{,} \hat{X}_1] + O^{(2)} . \qquad (4.5.8)$$

By recalling the differentiability property of $\hat{G}$, we also have the following isotopy of the conventional expression in one dimension[24]

$$(1/i) \frac{d}{dw} \hat{U}_{|\hat{w}=0} = (1/i) \frac{d}{dw} e_{\xi}^{\;iwX}{}_{|w=0} = \hat{X} \hat{\times} e_{\xi}^{\;iwX}{}_{|w=0} = \hat{X} , \quad (4.5.9)$$

Thus, to every inner isoautomorphism of $\hat{G}$ there corresponds an inner isoautomorphism of $\hat{L}$ which can be expressed in the form [27]

$$(\hat{L})_i^{\,j} = \hat{C}_{ki}^{\;\;j} w^k . \qquad (4.5.10)$$

The Lie-isotopic group $\hat{G}_a$ of all inner isoautomorphism of $\hat{G}$ is called the *isoadjoint group.* It is possible to prove that the Lie-isotopic algebra of $\hat{G}_a$ is the isoadjoint algebra $\hat{L}_a$ of $\hat{L}$.

We mentioned before that the direct sum of Lie-isotopic algebras is the conventional operation because the addition is not lifted in our studies. The corresponding operation for groups is the semidirect product which, as such, demands care in its formulation.

Let $\hat{G}$ be a Lie-isotopic group and $\hat{G}_a$ the group of all its inner isoautomorphisms. Let $\hat{G}^o{}_a$ be a subgroup of $\hat{G}_a$, and let $\hat{\Lambda}(\hat{g})$ be the image of $\hat{g} \in \hat{G}$ under $\hat{G}^o{}_a$. The *semidirect isoproduct* $\hat{G} \hat{\times} \hat{G}^o{}_a$ of $\hat{G}$ and $\hat{G}^o{}_a$ is the Lie-isotopic group of all ordered pairs $(\hat{g}, \hat{\Lambda})$ with group isomultiplication

$$(\hat{g}, \hat{\Lambda}) \hat{\times} (\hat{g}', \hat{\Lambda}') = (\hat{g} \hat{\times} \hat{\Lambda}(\hat{g}'), \hat{\Lambda} \hat{\times} \hat{\Lambda}') . \qquad (4.5.11)$$

with total isounit given by

$$\hat{1}_{tot} = (\hat{1}, \hat{1}_{\hat{\Lambda}}) , \qquad (4.5.12)$$

and inverse

$$(\hat{g}, \hat{\Lambda})^{-1} = (\hat{\Lambda}^{-1}(\hat{g}^{-1}), \hat{\Lambda}^{-1}) . \qquad (4.5.13)$$

[24] We should indicate that the conventional derivative d/dw needs a suitable isotopic formulation $\hat{\partial}/\hat{\partial}\hat{w}$ presented in Ch. I.6. The results, however, will be the same as those of Eq.s (4.5.9).

As we shall see in Vol. II, the above notions play an important role in the isotopies of the inhomogeneous space-time symmetries, such as Galilei's and Poincaré's symmetries.

Let $\hat{G}_1$ and $\hat{G}_2$ be two Lie-isotopic groups with respective isounits $\hat{1}_1$ and $\hat{1}_2$. The *direct isoproduct* $\hat{G}_1 \hat{\otimes} \hat{G}_2$ of $\hat{G}_1$ and $\hat{G}_2$ is the Lie-isotopic group of all ordered pairs $g = (\hat{g}_1, \hat{g}_2)$, $\hat{g}_1 \in \hat{G}_1$, $\hat{g}_2 \in \hat{G}_2$, with isomultiplication

$$g \hat{\times} g' = (\hat{g}_1, \hat{g}_2) \hat{\times} (\hat{g}'_1, \hat{g}'_2) = (\hat{g}_1 \hat{\times} \hat{g}'_1, \hat{g}_2 \hat{\times} \hat{g}'_2), \tag{4.5.14}$$

total isounit

$$\hat{1}_{tot} = (\hat{1}_1, \hat{1}_2), \tag{4.5.15}$$

and inverse

$$g^{-1} = (\hat{g}_1^{-\hat{1}}, \hat{g}_2^{-\hat{1}}). \tag{4.5.16}$$

Definition 4.5.2 [8,9]: *Let $\hat{G}$ be an N-dimensional isotransformation group of Class I with infinitesimal generators $\hat{X}_k$, k = 1, 2, ..., N. The "isodual image" $\hat{G}^d$ of $\hat{G}$ is the N-dimensional isogroup with infinitesimal generators $\hat{X}^d{}_k = -\hat{X}_k$, isodual isounit $\hat{1}^d = -\hat{1}$ and isodual parameters $\hat{w}^d = -\hat{w}$ over the isodual isofield $\hat{F}^d(\hat{a}^d,+,\hat{\times}^d)$ with "isodual isotransformation" in a suitable neighborhood of $\hat{1}^d$*

$$\hat{x}^{d\,\prime} = \hat{U}^d(\hat{w}^d) \hat{\times}^d \hat{x}^d = \{ e_{\hat{\xi}^d}^{\,i^d \hat{X}^d \hat{\times}^d \hat{w}^d} \} \hat{\times}^d \hat{x}^d = -\{ e_{\xi}^{\,i X T w} \} \hat{x}^d. \tag{4.5.17}$$

In particular, the above antiautomorphic conjugation can also be defined for conventional Lie groups, yielding the "isodual Lie group" G^d which is defined over the isodual field $F^d(a^d,+,\times^d)$ with generic "isodual transformations"

$$x^{d\,\prime} = U^d(w^d) \times^d x^d = \{ e_{\xi^d}^{\,i X^d \times^d w^d} \} \times^d x^d = -\{ e_{\xi}^{\,i X w} \} x^d. \tag{4.5.18}$$

In summary, any Lie group admits the following four realizations relevant for our analysis:

Lie groups G of conventional type;
Lie-Santilli isogroups $\hat{G}$;
Isodual Lie groups G^d; and
Isodual Lie-Santilli isogroups $\hat{G}^d$.

Realization G (G^d) is useful for the characterization of particles (antiparticles) in

vacuum within the context of the exterior problem, while realization $\hat{G}$ ($\hat{G}^d$) is useful for the characterization of particles (antiparticles) in physical media within the context of the interior dynamical problem.

It is hoped the reader can see from the above foundations that the entire conventional Lie's theory does indeed admit a consistent and nontrivial lifting into the covering Lie-isotopic formulation.

We now illustrate the primary results of this section with the isotopies and isodualities of the *rotational group* SO(3). Let $S\hat{O}(3)$ be the lifting of SO(3) of Class III on isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{R})$ with isometric and isounit (4.3.16). Let $\theta_k \in R(n,+,\times)$ be the conventional *Euler's angles* and $\hat{\theta}_k = \theta_k \hat{1} \in \hat{R}(\hat{n},+,\hat{\times})$ their isotopes. Then, a generic isotransformation on $\hat{E}(r,\hat{\delta},\hat{R})$ can be written

$$r' = \hat{\mathfrak{R}}(\hat{\theta}) \hat{\times} r = \tilde{\mathfrak{R}}(\theta)\, r, \qquad \hat{\mathfrak{R}} = \tilde{\mathfrak{R}} \hat{1}\,. \tag{4.5.19}$$

We then have the realization of isoexponentials (4.5.6)

$$\hat{\mathfrak{R}}(\hat{\theta}) = \{ e_{\xi}^{\hat{J}_1 \hat{\times} \hat{\theta}_1} \} \hat{\times} \{ e_{\xi}^{\hat{J}_2 \hat{\times} \hat{\theta}_2} \} \hat{\times} \{ e_{\xi}^{\hat{J}_3 \hat{\times} \hat{\theta}_3} \} =$$

$$= \{ e_{\xi}^{J_1 T \theta_1} \} \{ e_{\xi}^{J_2 T \theta_2} \} \{ e_{\xi}^{J_3 T \theta_3} \} \hat{1}\,. \tag{4.5.20}$$

where the J's are the (skew-symmetric) generators (4.3.13) of the adjoint representation of so(3), the θ's are the conventional Euler's angles, and T is the isotopic element (4.3.16) in realization (4.4.28b).

It is an instructive exercise to verity for structure (4.5.20): the validity of laws (4.5.2) ensuring its Lie-isotopic group structure; the validity of Theorem 4.5.1 ensuring its finite-dimensional (actually three-dimensional) character; Corollary 4.3.1A ensuring the correct isoexponentiation from the Lie-isotopic algebras to the corresponding Lie-isotopic groups; rule (4.5.9) on the inverse transition from isogroups (4.5.20) to the corresponding isoalgebras; and others.

It is finally instructive to verify the following classification of all possible isogroups (4.5.20)

$$S\hat{O}(3): \quad \begin{array}{l} SO(3) \text{ for } T = \text{diag.}\,(1,\ 1,\ 1); \\ SO(2.1) \text{ for } T = \text{diag.}\,(1,\ -1,\ 1); \\ S\hat{O}(3) \text{ for sign. } T = (+,\ +,\ +); \\ S\hat{O}(2.1) \text{ for sign.} T = (+,-,+) \\ SO^d(3) \text{ for } T = (-1,\ -1,\ -1); \\ SO^d(2.1) \text{ for } T = \text{diag.}\,(-1,\ +1,\ -1); \\ S\hat{O}^d(3) \text{ for sign. } T = (-,\ -,\ -); \end{array} \tag{4.5.21}$$

$$S\hat{O}^d(2.1) \text{ for sign. } T = (-, +, -).$$

thus illustrating the compatibility of the above classification with the corresponding one at the isoalgebra level, Eq.s (4.4.30)–(4.4.37), and the original one at the level of isoenvelope, Eq.s (4.3.18).

We now remain with the need to illustrate the nonlinear, nonlocal and noncanonical character of the Lie–Santilli isogroups for dimensions bigger than 1 (that of dimension 1 has been illustrated earlier with the isotopuic time evolutions).

The first and most fundamental example is, again, that of the isorotations $S\hat{O}(3)$ with isogroup structure (4.5.20). Assume for isotopic element the diagonal form of Class III

$$T = \text{diag.}\,(\,g_{11}, g_{22}, g_{33}\,), \qquad g_{kk}(t, r, \dot{r}, \ddot{r}, ...) \neq 0\,. \tag{4.5.21}$$

Then, simple calculations (studied in details in Vol. II, Ch. 6) yield the following *isorotation around the third isoaxis*

$$\hat{\mathfrak{R}}(\theta_3) = \begin{pmatrix} \cos[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & g_{22}(g_{11}g_{22})^{-\frac{1}{2}}\sin[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & 0 \\ -g_{11}(g_{11}g_{22})^{-\frac{1}{2}}\sin[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & \cos[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{1}\,. \tag{4.5.22}$$

But the elements g_{kk} generally depend on the local variables r (as well as their derivative. The non linearity of isotransforms (4.5.22) is then transparent. Equally transparent is their noncanonical character, while their isolocality requires a knowledge of the underlying isotopology and will not be illustrated at this time.

Note that isorotations (4.5.22) provide a realization of all eight different isogroups (4.5.21), as the reader si encouraged to verify.

The singular Lie–isotopic groups of Class IV are unexplored. It is hoped that experts in the field will indeed study them because, as we shall see in Vol. II, they constitute the symmetries of gravitational singularities. The Lie–isotopic theory of Class V is equally unexplored at this writing and equally significant, e.g., to study the deformation of crystal via discrete isogroup with continuously varying isounits.

4.6: THE FUNDAMENTAL THEOREM ON ISOSYMMETRIES AND THEIR ISODUALS

In this section we shall apply the isotopic methods for the construction of the

isosymmetries, i.e., the symmetries of the isoseparation in (nowhere degenerate, Hermitean) isospace $\hat{S}(\hat{x},\hat{g},\hat{F})$

$$(\hat{x} - \hat{y})^{\hat{2}} = [(x - y)^{\mu} \hat{g}_{\mu\nu}(t, x, \dot{x}, \ddot{x}, \psi, \psi^{\dagger}, \partial\psi, \partial\psi^{\dagger}, \mu,. \tau, n, ...) (x - y)^{\nu}] \hat{1} \quad (4.6.1a)$$

$$\det \hat{g} \neq 0, \quad \hat{g} = \hat{g}^{\dagger}, \quad (4.6.1b)$$

The objective is the form-invariant characterization of the most general known interior dynamical problems which are:

1) nonlinear in the coordinates x (as available, e.g., in conventional gravitation) as well as nonlinear in the velocities $\dot{x}$ (e.g., to represent the drag forces of missiles in atmosphere which are nowadays proportional to the tenth power of the velocity $\dot{x}^{10}$ and more), as well as in the accelerations $\ddot{x}$ (as requested by certain particular interior dynamical motions studied in Vol. II);
2) nonlocal-integral on some or all these variables to represent the extended character of the particles moving within physical media;
3) noncanonical as a necessary condition for interior dynamics, i.e., violation of the conditions of variational selfadjointness for the existence of a Hamiltonian [3];
4) Inhomogeneous, to represent experimental evidence on interior physical medium (e.g., local variation of the density μ); and
5) Anisotropic, also to represent experimental evidence on interior media (e.g., as occurring under an intrinsic angular momentum of the media themselves).

The above invariance problem was solved by this author in ref.s [6-9] in 1982-1983 via: paper [8] on a general theorem on isosymmetries reviewed below; paper [9] on the first construction of the isotopies $\hat{O}(3)$ of the rotational symmetry O(3); in paper [7] on the construction of the isotopies $\hat{O}(3.1)$ of the Lorentz symmetry O(3.1); and paper [6] on the operator formulations of the above results (isotopies of Wigner's theorem on symmetries). The inclusion of the isotranslations to reach the isotopies $\hat{P}(3.1)$ of the Poincaré symmetry P(3.1) was first done in memoir [16].[25]

These studies essentially permitted the formulation and proof of the following

[25] It should be noted that papers [8,9] were written prior to paper [6,7] of 1983, but they ended up to be published in 1985, some two years after the appearance of the latter because of rather questionable and unreasonable editorial processing by several journals reported in details in page 26 of ref. [8].

Theorem 4.6.1 - Fundamental Theorem on isosymmetries [8]: *Let* G *be an N-dimensional Lie symmetry group of an m-dimensional metric or pseudo-metric space* $S(x,g,F)$ *over a field* $F(a,+,\times)$ *of characteristic zero,*

$$G: \quad x' = A(w) \times x, \quad x'^{\dagger} = x^{\dagger} \times A^{\dagger}(w), \tag{4.6.2a}$$

$$(x'-y')^{\dagger} \times A^{\dagger}(w) \times g \times A(w) \times (x-y) \equiv (x-y)^{\dagger} g (x-y), \tag{4.6.2b}$$

$$A^{\dagger} \times g \times A = A \times g \times A^{\dagger} = g, \quad \mathrm{Det}\, A = \pm 1. \tag{4.6.2c}$$

Then, the infinitely possible isotopies $\hat{G}$ *of* G *characterized by the same generators and parameters of* G *and new isounits* $\hat{1}$ *(isotopic elements* T) *of Class III automatically leave invariant the isocomposition on the isospaces* $\hat{S}(\hat{x},\hat{g},\hat{F})$, $\hat{g} = Tg$, $\hat{1} = T^{-1}$,

$$\hat{G}: \hat{x}' = \hat{A}(\hat{w}) \hat{\times} \hat{x} = \tilde{A}(\hat{w}) \hat{\times} \hat{x}, \quad \hat{x}'^{\dagger} = \hat{x}^{\dagger} \hat{\times} \hat{A}^{\dagger}(\hat{w}) = x^{\dagger} \tilde{A}^{\dagger}(\hat{w}), \tag{4.6.3a}$$

$$(\hat{x}'-\hat{y}')^{\dagger} \hat{\times} \hat{A}^{\dagger}(\hat{w}) \times \hat{g} \times \hat{A}(\hat{w}) \hat{\times} (\hat{x}-\hat{y}) \equiv (\hat{x}-\hat{y})^{\dagger} \tilde{A}\, \hat{g}\, \tilde{A}\, (\hat{x}-\hat{y}) = (\hat{x}-\hat{y})^{\dagger} \hat{g} (\hat{x}-\hat{y}), \tag{4.6.3b}$$

$$\hat{A}^{\dagger} \hat{g} \hat{A} = \hat{A} \hat{g} \hat{A}^{\dagger} = \hat{1} \hat{g} \hat{1}, \quad \textit{or} \tag{4.6.3c}$$

$$\tilde{A}^{\dagger} \hat{g} \tilde{A} = \tilde{A} \hat{g} \tilde{A}^{\dagger} = \hat{g}, \qquad \mathrm{Det}(\hat{A}\hat{T}) = \mathrm{Det}\,\tilde{A} = \pm 1. \tag{4.6.3d}$$

For a detailed proof one may inspect papers [8,9]. The main aspect which is here important is that the original symmetry G is generally linear-local-canonical, while the isosymmetries $\hat{G}$ are generally nonlinear, nonlocal and noncanonical when projected in the original space, owing to the arbitrary functional dependence of the isometric $\hat{g} = \hat{T}(t, x, \dot{x}, \ddot{x}, \ldots)g$, although they are isolinear, isolocal and isocanonical in their proper isospace (Sect. I.4.2).

Note that *the trivial isotopy* $X_k \rightarrow \hat{X}'_k = X_k\hat{1}$ *is excluded in Theorem 4.6.1, because it does not provide the invariance of the generalized metric.* This is due to the fact that the isotransformations characterized by the isoexponentiation of X'_k coincide with the conventional ones

$$x' = \{ \hat{e}^{i \hat{X}' w} \} * x = \{ e^{i X \hat{1} \hat{T} w} \} x = \{ e^{i X w} \} x, \tag{4.6.4}$$

by losing in this way the crucial appearance of the isotopic element $\hat{T}$ in the exponent. This occurrence indicates the needs of using the Lie-isotopic theory in its entirety, and illustrates once more the reason for the preservation of the original basis under isotopies.

Note that the explicit construction of the isosymmetry $\hat{G}$ of any given separation (4.6.1) is quite simple because it is done via the knowledge of the original symmetry and of the deformed metric. The invariance of the

isoseparation is then guaranteed by Theorem 4.6.1.

As an illustration of Theorem 4.6.1, consider the rotational symmetry G = SO(3) of the separation in Euclidean space E(r,δ,R)

$$G = SO(3){:}\ \ r' = \mathfrak{R}(\theta)\, r, \quad \mathfrak{R}(\theta)\, \mathfrak{R}^{\dagger}(\theta) = \mathfrak{R}^{\dagger}(\theta)\, \mathfrak{R}(\theta) = I, \ \text{Det.}\, \mathfrak{R} = +1, \tag{4.6.5a}$$

$$r^2 = x^i\, \delta_{ij}\, x^j = = x^1 x^1 + x^2 x^2 + x^3 x^3 = \text{inv.}, \tag{4.6.5b}$$

Consider now the most general possible deformation of the above invariant of Class III which, as such, can always be diagonalized into the form

$$r^2 = x^i\, \hat{\delta}_{ij}\, x^j = = x^1 g_{11} x^1 + x^2 g_{22} x^2 + x^3 g_{33} x^3 =$$

$$= \pm x^1 b_1{}^2 x^1 \pm x^2 b_2{}^2 x^2 \pm x^3 b_3{}^2 x^3 = \text{inv.}, \tag{4.6.6a}$$

$$\hat{\delta} = \hat{T}\, \delta = \hat{T} = \text{diag.}\,(g_{11}, g_{22}, g_{33}) = \text{diag.}\,(\pm b_1{}^2, \pm b_2{}^2, \pm b_3{}^2), \tag{4.6.6b}$$

$$b_k = b_k(t, r, \dot{r}, \ddot{r}, \ldots) \neq 0\,. \tag{4.6.6c}$$

Ref. [9] computed via Theorem 4.6.1 the symmetries of all infinitely possible deformations (4.6.6b). They are given by the isotopes (4.5.20) (see Ch. II.6 for details)

$$\hat{G} = S\hat{O}(3){:}\ \ r' = \hat{\mathfrak{R}}(\theta)\, \hat{\times}\, r, \tag{4.6.7a}$$

$$\hat{\mathfrak{R}}(\theta) * \hat{\mathfrak{R}}^{\dagger}(\theta) = \hat{\mathfrak{R}}^{\dagger}(\theta)\, \hat{\times}\, \hat{\mathfrak{R}}(\theta) = \hat{1}, \qquad \text{Det.}(\hat{\mathfrak{R}}\, \hat{T}) = +1, \tag{4.6.7b}$$

$$\hat{\mathfrak{R}}(\theta) = \{\hat{e}^{J_1 \hat{\times} \theta_1}\}\, \hat{\times}\, \{\hat{e}^{J_2 \hat{\times} \theta_2}\}\, \hat{\times}\, \{\hat{e}^{J_3 \hat{\times} \theta_3}\} =$$

$$= \{e^{J_1 T \theta_1}\}\{e^{J_2 T \theta_2}\}\{e^{J_3 T \theta_3}\}\hat{1}\,. \tag{4.6.7c}$$

where all quantities are known: the generators J_k are in their adjoint representation (4.3.13), the parameters θ_k are the conventional Euler's angles, and the isotopic element T is that of deformation (4.6.6b).

The isosymmetry transformations can also be computed in the needed explicit form, because the convergence of isoexponentials (4.6.7c) is ensured by the original convergence plus the conditions for Class III (isotopic elements that are sufficient smooth, bounded, nowhere degenerate and Hermitean).

As an example, ref. [9] computed the following *isorotation* (4.5.222) around the third axis, i.e.,

$$\hat{\mathfrak{R}}(\theta_3) = \begin{pmatrix} \cos[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & g_{22}(g_{11}g_{22})^{-\frac{1}{2}} \sin[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & 0 \\ -g_{11}(g_{11}g_{22})^{-\frac{1}{2}} \sin[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & \cos[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{1}\,. \tag{4.6.8}$$

It is instructive to verify that isotransformations (4.6.7a) with realization (4.6.8) do indeed leave invariant generalized separation (4.6.6a). The following comments are now in order:

1) The $\hat{SO}(3)$ isoinvariance of generalized separation (4.6.6a) is ensured by the original invariance SO(3) of sphere (4.6.5b) for all infinitely possible deformations of the admitted Class III (Theorem 4.6.1);
2) The original SO(3) transformations (the ordinary rotations in Euclidean space) are linear, local and canonical, as well known. On the contrary, the covering $\hat{SO}(3)$ transformations (the isorotations (4.6.8)) are nonlinear, nonlocal and noncanonical, although they are isolinear, isolocal and isocanonical in the sense of Sect. I.4.2;
3) Owing to the general character of invariant (4.6.6a), Riemannian generalizations of the original Euclidean space are a particular case of isosymmetry (4.6.8) for $g_{kk} = g_{kk}(r)$, with the understanding that isosymmetries (4.6.8) are considerably more general than Riemann owing to their additional unrestricted dependence in the velocities, accelerations, etc.;
4) Isotransformations are already computed in the needed explicit form and there is no need of additional calculations. As an example, consider the lifting of the Euclidean metric δ into a Riemannian three-dimensional metric g(r), e.g., the space component of the Schwarzschild line element. Then the explicit symmetry of the latter is merely provided by plotting the g_{kk} values in (4.6.8);
5) The classification of all possible isosymmetries (4.6.8) recovers again classification (4.3.18) at the level of the isoenvelopes, classification (4.4.30)-(4.4.37) at the level of Lie-isotopic algebras, and classification (4.5.21) at the level of Lie-isotopic groups, according to the following invariances:[26]

$$\begin{array}{ll}
SO(3): & x^1 x^1 + x^2 x^2 + x^3 x^3 = \text{inv.}, \\
SO(2.1): & x^1 x^1 - x^2 x^2 + x^3 x^3 = \text{inv.}, \\
\hat{SO}(3): & x^1 b_1^2 x^1 + x^2 b_2^2 x^2 + x^3 b_3^2 x^3 = \text{inv.}, \\
\hat{SO}(2.1): & x^1 b_1^2 x^1 - x^2 b_2^2 x^2 + x^3 b_3^2 x^3 = \text{inv.} \\
SO^d(3): & -x^1 x^1 - x^2 x^2 - x^3 x^3 = \text{inv.}, \\
SO^d(2.1): & -x^1 x^1 + x^2 x^2 - x^3 x^3 = \text{inv.},
\end{array} \tag{4.6.9}$$

[26] Note that for hyperbolic invariants the trigonometric functions of (4.6.8) become hyperbolic functions, exactly as they should be).

$$\hat{SO}(3): \quad - x^1 b_1^2 x^1 - x^2 b_2^2 x^2 - x^3 b_3^2 x^3 = \text{inv.},$$
$$\hat{SO}^d(2.1): \quad - x^1 b_1^2 x^1 + x^2 b_2^2 x^2 - x^3 b_3^2 x^3 = \text{inv.},$$

In summary, Theorem 4.6.1 provides the invariance of all infinitely possible deformations of the Euclidean space under the sole condition that they are nowhere singular, Hermitean and well behaved. This includes the invariance not only of all possible ellipsoidical deformations of the sphere, but also of all possible hyperbolic deformations, thus admitting as particular cases the conventional rotational and Lorentz symmetries, all their infinitely possible isotopes, their isoduals and all the infinitely possible isodual isotopes.

As we shall see in Ch. II.8, one of the first applications of the Lie-isotopic theory is the construction of the general invariant of conventional exterior gravitation and the proof that it is locally isomorphic to the conventional Poincaré symmetry of the special relativity.

In Ch. II-8 we shall show that, starting from the familiar invariance of the separation in Minkowski space

$$P(3.1): \qquad x^\mu \eta_{\mu\nu} x^\nu = \text{inv.}, \quad \eta \in M(x,\eta,R), \tag{4.6.10}$$

Theorem 4.6.1 permits the construction of the general invariance for all possible Riemannian separations

$$\hat{P}(3.1): \quad x^\mu g_{\mu\nu}(x) x^\nu = \text{inv.}, \quad g \in \Re(x,g,R), \tag{4.6.11}$$

via the decomposition $g(x) = T(x)$ and the construction of the isosymmetries $\hat{P}(3.1)$ with respect to the generalized isounit $\hat{1} = [T(x)]^{-1}$. The invariance of the Riemannian separation (4.6.11) is then ensured by Theorem 4.6.1[27].

The isotopic unification of the Minkowski and Riemannian spaces of Sect. 3.3 will be carried over, in this way, to the unification of symmetries of the special and general relativities as a foundation for their isotopies for interior problems.

In turn, these results permit far reaching and basically novel advances not possible via conventional methods, such as a new quantum =version of gravity, a novel formulation of antimatter which begins for the first time at the classical level, the prediction of antigravity for antiparticles in the field of matter, and others.

The relevance of Theorem 4.6.1 is further illustrated by the fact that all isosymmetries of hadronic mechanics studied in Vol.s II and III are particular applications of Theorem 4.6.1.

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27 The attentive reader may have noted that isorotations (4.6.8) do already contain the isosymmetry for (2+1) dimensional Riemannian metrics.

The understanding is that signature changing deformations, e.g., $(+, +, +) \to (+, -, +)$, are of sole mathematical character because they cannot be reached in actual experiments. This is the reason that practical applications of the isotopies are restricted to Class I which ensures the preservation of the original signature.

In summary, Theorem 4.6.1 is "directly universal" for all infinitely possible deformations $g \to \hat{g} = \hat{T} g$ of Class III of any given well behaved, symmetric and real valued metric g. The "direct universality" of hadronic mechanics for the treatment of nonlinear-nonlocal-nonhamiltonian systems is then consequential, as we shall see.

4.7: ISOREPRESENTATION THEORY

Recall that the representation theory of Lie algebras has profound physical implications because it characterizes the contemporary notion of *point-like particles* for the exterior problem in vacuum

A primary objective of the representation theory of the covering Lie-isotopic algebras is that of characterizing a generalized notion of *extended, nonspherical and deformable particles* for interior problems within physical media called *isoparticles* [22]. The more general representation theory of Lie-admissible algebras characterizes a yet more general notion of particles called *genoparticles* [loc. cit.] which are studied in Ch. I.7. The corresponding antiparticles are characterized by the representation of isodual isoalgebras.

In this section we study the rudiments of the isorepresentations of Lie-isotopic algebras of Class I or II over an isofield $\hat{F}(\hat{\alpha},+,\hat{\times})$ of isocharacteristic zero. The representation theories of isoalgebras of Classes IV and V are unknown at this writing.

Consider a vector space U with elements a, b, c, ... and abstract product "ab" over a field $F(\alpha,+,\times)$. We say that U constitutes an *algebra* when it verifies the right and left scalar and distributive laws (Definition I.2.4.1). The algebra U is said to be *associative (nonassociative)* when ab is associative (nonassociative).

The *right and left multiplications* in U (see, e.g., ref. [34]) are given by the following linear transformations of U onto itself as a vector space

$$R_X : a \to a\,x, \quad \text{or} \quad a\,R_X = a_X, \tag{4.7.1a}$$

$$L_X : a \to x\,a, \quad \text{or} \quad L_X a = x\,a, \tag{4.7.1b}$$

for all $a, x \in U$, and verify the following general properties

$$(a\,\alpha)\,R_X = (a\,\alpha)\,x = a\,(\alpha\,x), \quad \text{or} \quad \alpha\,R_X = R_{\alpha X}, \tag{4.7.2a}$$

$$a\,R_{(x+y)} = a\,(x + y) = a\,R_x + a\,R_y = a(R_x + R_y),$$

$$\text{or } R_{(x+y)} = R_x + R_y, \tag{4.7.2b}$$

with evident similar properties for the left multiplications L_x.

When the algebra is associative, we have the additional properties

$$a\,(x\,y) = (a\,x)\,y, \text{ i.e.,} \tag{4.7.3a}$$

$$a\,R_{xy} = a\,R_x\,R_y, \text{ or } R_{xy} = R_x\,R_y, \tag{4.7.3b}$$

$$(x\,y)\,a = x\,(y\,a), \text{ i.e.} \tag{4.7.3c}$$

$$L_{xy}\,a = L_x\,L_y\,a, \quad \text{or } L_{xy} = L_x\,L_y. \tag{4.7.3d}$$

The above properties imply that the mapping $a \rightarrow R_a$ ($a \rightarrow L_a$) is a homomorphism (antihomomorphism) of A into the associative algebra V(A) of all linear transformations in A. Thus, they provide a *right representation* $a \rightarrow R_a$ or a *left representation* $a \rightarrow L_a$, respectively, of A, also called left or right $\mathrm{Hom}^A{}_F(V_p)$, for p = Right or Left. When both maps $a \rightarrow R_a$ and $a \rightarrow L_a$ are considered we have a *birepresentation.*

If the algebra A contains the identity I, we have a *one-to-one (or faithful) representation* because $R_a = R_b$ implies $IR_a = IR_b$ which can hold iff a = b. When the space V is the algebra A itself, we have the so-called *adjoint representation* also called *fundamental or regular representation.*

In the case of nonassociative algebras, the mapping $a \rightarrow R_a$ is no longer a homomorphism, and this illustrates the reason for the study of the representation theory of Lie algebras via that of the underlying universal enveloping associative algebra, as done in the mathematical literature, e.g., ref. [31] (but generally not in the physical literature).

Consider now an isoassociative algebra $\hat{A}$ with elements $\hat{a}, \hat{b}, \hat{c}$, ...over an isofield $\hat{F}(\hat{\alpha},+,\hat{\times})$ with isounit $\hat{1}$ and isoassociative product $\hat{a}\hat{\times}\hat{b}$. Introduce the *right and left isomultiplications*

$$\hat{R}_{\hat{x}}:\ \hat{a} \rightarrow \hat{a}\,\hat{\times}\,\hat{x}, \qquad \text{or} \quad \hat{a}\,\hat{\times}\,\hat{R}_{\hat{x}} = \hat{a}\,\hat{\times}\,\hat{x}, \tag{4.7.4a}$$

$$\hat{L}_{\hat{x}}:\ \hat{a} \rightarrow \hat{x}\,\hat{\times}\,\hat{a}, \qquad \text{or} \quad \hat{a}\,\hat{\times}\,\hat{L}_{\hat{x}} = \hat{x}\,\hat{\times}\,\hat{a}, \tag{4.7.4b}$$

for all $\hat{a} \in \hat{A}$. It is then easy to see that properties (4.7.2) and (4.7.3) are lifted into the forms

$$\hat{\alpha}\,\hat{\times}\,\hat{R}_{\hat{x}} = \hat{R}_{\hat{\alpha}}\,\hat{\times}\,\hat{x}, \quad \hat{R}_{(\hat{x}+\hat{y})} = \hat{R}_{\hat{x}} + \hat{R}_{\hat{y}}, \tag{4.7.5a}$$

$$\hat{R}_{\hat{x}} \hat{\times} \hat{y} = \hat{R}_{\hat{x}} \hat{\times} \hat{R}_{\hat{y}}, \quad \hat{1} \hat{\times} \hat{R}_{\hat{a}} = \hat{1} \hat{\times} \hat{R}_{\hat{b}} \quad \rightarrow \quad \hat{a} = \hat{b}, \tag{4.7.5b}$$

with similar properties for the left isomultiplications.

It is easy to see that the mapping $\hat{a} \rightarrow \hat{R}_{\hat{a}}$ characterizes a *right, faithful, isorepresentation* of $\hat{A}$ in the isoassociative algebra $\hat{V}(\hat{A})$ of isolinear transformations on $\hat{A}$ denoted $\mathrm{Hom}^{\hat{A}}_{\hat{F}}(\hat{V}_R)$, with similar results holding for the left isorepresentations.

The nontriviality of the isotopy is made clear by the following

Lemma 4.7.1 [4]: *Isorepresentations of isoassociative algebras* $\hat{A}$ *over an isofield* $\hat{F}(\hat{\alpha},+,\hat{\times})$ *are isolinear and isolocal in* $\hat{V}$ *but generally nonlinear and nonlocal when projected in* V.

Thus, *the transition from Lie algebras to Lie-isotopic algebras generally implies the transition from linear-local-canonical to nonlinear-nonlocal-noncanonical representations,* as desired. Recall that the contemporary notion of point-like particle in vacuum is essentially a manifestation of the linear-local-canonical character of the theory. The more general isoparticles will then result to be a manifestation of the covering nonlinear-nonlocal-noncanonical character of the isorepresentation theory.

A *module* of an algebra U over a field F, also called *U-module* [34] is a linear vector space V over $F(\alpha,+,\times)$ together with a mapping $U \times V \rightarrow V$ denoted with the symbol $(a, v) \rightarrow av$ which satisfies the distributive and scalar rules

$$a(v + t) = a v + a t, \qquad (a + b) v = a v + b v, \tag{4.7.6a}$$

$$\alpha (a, v) = (\alpha a, v) = (a, \alpha v), \tag{4.7.6b}$$

as well as all the axioms of U, for all $a, b \in U$, $v, t \in V$, and $\alpha \in F$.

The mappings $a \rightarrow R_V = av$ and $a \rightarrow L_V = va$ show that the space V is a left and a right U-module. When both left and right actions are considered, the U is a *bimodule.*

The notion of one-sided, left or right *isomodule* of an isoalgebra $\hat{U}$ over an isofield $\hat{F}$ was introduced in ref. [35] and it is given by a straighforward isotopy of the preceding structure. The more general notion of two-sided, left and right *isobimodule* was also introduced in ref. [35] as reviewed in Ch. I.7.

The isomodules are sufficient for the representation theory of Lie-isotopic algebras, but the more general isobimodules are necessary for the representations of the more general Lie-admissible algebras. The above structures permit a first characterization of the notion of particles of hadronic mechanics as follows:

1) Conventional particles, those characterized by linear-local-canonical representations of Lie symmetries on a one sided module;

2) Isoparticles, those characterized by nonlinear-nonlocal-noncanonical representations of Lie-Santilli isosymmetries on one-sided isomodules;

3) Genoparticles, those characterized by nonlinear-nonlocal-noncanonical representations of Lie-Santilli genosymmetries on two-sided isobimodules (Ch. I.7).

As we shall see in Vol. II and III, the above characterizations yield notions of particles for physical conditions of increasing complexities, such as for a particle when members of an atomic structure (conventional particle), when member of a hadronic structure (isoparticle) and when in the core of a collapsing star (genoparticle).

Definition 4.7.1: *Let* $\hat{\xi}$ *be the universal enveloping isoassociative algebra of a Lie-isotopic algebra* $\hat{L} \approx \hat{\xi}^-$ *of Class I. Then, the one-sided, right or left, "isorepresentations"* $\mathrm{Hom}^{\hat{\xi}}{}_{\hat{F}}(\hat{V}_p)$, p = right *or* Left, *of* $\hat{\xi}$ *on a corresponding, one-sided isomodule* $\hat{V}$ *over an isofield* $\hat{F}(\alpha,+,\hat{\times})$ *are characterized by*

$$\hat{R}_{\hat{a} \hat{\times} \hat{b}} = \hat{R}_{\hat{a}} \hat{\times} \hat{R}_{\hat{b}} , \tag{4.7.7a}$$

$$\hat{R}_{\hat{e}} = \hat{1} . \tag{4.7.7b}$$

The "isodual isorepresentations" of $\hat{L}^d$ *on* $\hat{V}^d$ *over* $\hat{F}^d(\hat{a}^d,+,\hat{\times}^d)$ *are the isodual images of* $\mathrm{Hom}^{\hat{\xi}}{}_{\hat{F}}(\hat{V}_p)$ *characterized by*

$$\hat{R}(\hat{w})_{\hat{a}} \rightarrow \hat{R}^d(\hat{w}^d)_{\hat{a}^d} = -\hat{R}(-\hat{w})_{\hat{a}} . \tag{4.7.8a}$$

$$\hat{R}_{\hat{e}} = \hat{1} \rightarrow \hat{R}^d{}_{\hat{e}^d} = \hat{1}^d = -\hat{1} . \tag{4.7.8b}$$

Conditions (4.7.7b) and (4.7.8b) ensure the invertibility of the elements, in the sense that

$$\hat{R}_{a * a^{-\hat{1}}} = \hat{R}_{\hat{a}} \hat{\times} \hat{R}_{\hat{a}^{-\hat{1}}} = \hat{R}_{\hat{e}} = \hat{1} . \tag{4.7.9a}$$

$$\hat{R}_{\hat{a}^{-\hat{1}}} = (\hat{R}_{\hat{a}})^{-\hat{1}} . \tag{4.7.9a}$$

It should be indicated that isorepresentations (4.7.7) exhaust all isolinear and isolocal cases, but they are not expected to be unique. In fact, an additional class of nonlinear representations emerges in the conventional case [31-33] and a similar occurrence is expected in the isotopic case.

The matrix form of isorepresentations are also given by a simple isotopy of conventional matrix forms [31-33]. Let $\hat{e}_k$, k = 1, 2, ..., N be an isobasis of $\hat{A}$ which is *isoorthonormal*, i.e.,

$$(\hat{e}_i \hat{,} \hat{e}_j) = \hat{\delta}_{ij} = \hat{1}\, \delta_{ij} \tag{4.7.10}$$

where $(\hat{,}\ )$ is the isoscalar product on the isomodule and $\hat{\delta}_{ij}$ is called the *isokronecker delta*. The desired matrix form of the isorepresentation is then given by

$$\hat{R}_{\hat{a}} \hat{\times} \hat{e}_k = \hat{D}_{ik}(\hat{a}) \hat{\times} \hat{e}_i, \tag{4.7.11a}$$

i.e.,

$$\hat{D}_{ij}(\hat{a}) := (\hat{R}_{\hat{a}} \hat{\times} \hat{e}_j \hat{,} \hat{e}_i). \tag{4.7.11a}$$

Note that the matrix for of the product $\hat{R}_{\hat{a} \hat{\times} \hat{b}}$ is given by

$$\hat{D}_{ij}(\hat{a} \hat{\times} \hat{b}) = \sum_{r,s} \hat{D}_{ir}(\hat{a})\, \hat{T}_{rs}\, \hat{D}_{sj}(\hat{b}) \tag{4.7.12}$$

From the above properties it is easy to see the following

Lemma 4.7.2: *The dimension of the representation of a Lie algebra does not change under isotopies.*

We now study the "degrees of freedom" of isorepresentations. Let $\hat{a} \to \hat{R}_{\hat{a}}$ be an isorepresentation of an algebra $\hat{A}$ over $\hat{F}(\hat{\alpha},+,\hat{\times})$ on an isomodule $\hat{V}$. Let

$$\hat{S}: \hat{V} \to \hat{V}', \tag{4.7.13}$$

be a (bounded, sufficiently smooth and regular) isomap of $\hat{V}$ into $\hat{V}'$. Then the isomap

$$\Gamma: \hat{a} \to \hat{R}'_{\hat{a}} = \hat{S} \hat{\times} \hat{R}_a \hat{\times} \hat{S}^{-1}, \tag{4.7.14}$$

characterizes the image of the isorepresentation $\hat{R}_{\hat{a}}$ in $\hat{V}'$ because

$$\hat{R}'_{\hat{a} \hat{\times} \hat{b}} = \hat{S} \hat{\times} \hat{R}_{\hat{a}} \hat{\times} \hat{S} \hat{\times} \hat{S}^{-1} \hat{\times} \hat{R}_{\hat{b}} \hat{\times} \hat{S}^{-1} = \hat{R}'_{\hat{a}} \hat{\times} \hat{R}'_{\hat{b}}, \quad \hat{R}'_{\hat{e}} = \hat{1}. \tag{4.7.15}$$

Recall that a conventional representation R_a of an associative algebra A on a module V and R'_a on another module V′ over $F(\alpha,+,\times)$ are said to be *equivalent* when there is an invertible map S: V → V′ such that

$$R_a = S\, R_a\, S^{-1}, \tag{4.7.22}$$

and they are said to be *unitarily equivalent* for the particular case

$$R_a = S\, R_a\, S^{\dagger}, \quad S\, S^{\dagger} = S^{\dagger} S = I, \quad S^{-1} = S^{\dagger}. \tag{4.7.16}$$

Definition 4.7.2: *An isorepresentations $\hat{R}_{\hat{a}}$ of an isoassociative algebra $\hat{A}$ on an isomodule $\hat{V}$ and a second isorepresentation $\hat{R}'_a$ on $\hat{V}'$ over the same isofield $\hat{F}(\hat{\alpha},+,\hat{\times})$ are said to be "isoequivalent" when there exist an sufficiently smooth invertible isomap $\hat{S}$: $\hat{V} \rightarrow \hat{V}'$ such that for all elements $\hat{a} \in \hat{A}$*

$$\hat{R}'_{\hat{a}} \equiv \hat{S} \hat{\times} \hat{R}_a \hat{\times} \hat{S}^{-1}, \tag{4.7.17}$$

and they are said to be "isounitarily equivalent" for the particular case

$$\hat{R}'_{\hat{a}} = \hat{S} \hat{\times} \hat{R}_{\hat{a}} \hat{\times} \hat{S}^{\dagger}, \quad \hat{S} \hat{\times} \hat{S}^{\dagger} = \hat{S}^{\dagger} \hat{\times} \hat{S} = \hat{1}, \quad \hat{S}^{\dagger} = \hat{S}^{-1}. \tag{4.7.18}$$

It is then easy to prove the following

Lemma 4.7.3 : *Let* D *be a (finite-dimensional) representation of a Lie algebra* L *and* $\hat{D}$ *the corresponding isorepresentation of the Class I isotope* $\hat{L}$ *of* L, *in which case* $\hat{L}$ *is isomorphic to* L, $\hat{L} \approx$ L, *and the dimensions of* $\hat{D}$ *and* D *are the same. Then* $\hat{D}$ *and* D *are not equivalent or unitarily equivalent.*

As in the conventional case (see, e.g., [33]), the notion of isoequivalence of isorepresentations is reflexive, symmetric and transitive. In fact, every isorepresentation is isoequivalent to itself; if an isorepresentation $\hat{R}_{\hat{a}}$ is isoequivalent to $\hat{R}'_{\hat{a}}$, then $\hat{R}'_{\hat{a}}$ is isoequivalent to $\hat{R}_{\hat{a}}$; etc. Thus, the set of all isorepresentations can be divided into *isoequivalence classes.*

In the conventional Lie theory only one matrix representation per each equivalence class is considered [33]. This is due to the fact that *the matrices of two equivalent representations can be made to coincide with a suitable selection of the basis.* In fact, the basis e_i for V and $e'_i = Se_i$ for V′ yield the same matrix representations,

$$D(a)\, e_k = \textstyle\sum_i D_{ik}\, e_i \;\rightarrow\; D'(a)\, e'_k = \sum_i D'_{ik}\, e'_i = S \sum D_{ik}\, e_i , \tag{4.7. 19}$$

Under isotopy we evidently have the corresponding image, in the sense that we can indeed select the isobasis $\hat{e}_i$ on $\hat{V}$ and $\hat{e}'_i = \hat{S}\hat{\times}\hat{e}_i$ on $\hat{V}'$, thus reaching the similar results

$$\hat{D}(\hat{a}) \hat{\times} \hat{e}_k = \textstyle\sum_{r,s} \hat{D}_{rk}\, \hat{T}_{rs}\, \hat{e}_s \;\rightarrow$$

$$\rightarrow\; \hat{D}'(\hat{a}) \hat{\times} \hat{e}'_k = \textstyle\sum_{r,s} \hat{D}'_{rk}\, \hat{T}_{rs}\, \hat{e}'_s = \hat{S} \hat{\times} \sum_{r,s} \hat{D}_{rk}\, \hat{T}_{rs}\, \hat{e}_s . \tag{4.7. 20}$$

However, isoequivalent but different isorepresentations play an important role in

the Lie-isotopic theory, particularly for physical applications, as illustrated below in this section.

Recall that for a conventional N-dimensional Lie algebra **L** with generators X_i, the structure constants $C_{ij}{}^k$ characterize the adjoint representation of X_i with matrix elements

$$(X_i)_j{}^k = -C_{ij}{}^k . \qquad (4.7.21)$$

The repetition of the conventional proof via the use of the Isotopic Second and Third Theorem (Sect. I.4.4) then leads to the following

Lemma 4.7.4: *Let $\hat{L}$ be a Lie-isotopic algebra with generators $\hat{X}_i$ and structure functions $\hat{C}_{ij}{}^k(t, x, \dot{x}, \ddot{x}, ...) = C_{ij}{}^k\hat{1}$, Eq.s (I.4.4.7). Then, up to isoequivalence, the "isoadjoint isorepresentation' of $\hat{L}$ is characterized by the elements of the isomatrix*

$$(\hat{X}_i)_j{}^k = -\hat{C}_{ij}{}^k(t, x, \dot{x}, \ddot{x}, ...) . \qquad (4.7.22)$$

Additional types of adjoint representations will be identified shortly.

Note the constancy of the elements of the adjoint representation in the conventional case, as compared to an arbitrary functional dependence of the corresponding elements under isotopy.

Consider an isolinear space $\hat{\mathcal{H}}$ equipped with an isoscalar product $(\hat{x}\hat{,}\hat{y})$. As we shall see in Ch. I.6, an operator $\hat{X}$ of an isoenvelope $\hat{\xi}$ is called *isohermitean* when

$$(\hat{X}^{\dagger} \hat{\times} \hat{x} \hat{,} \hat{y}) \equiv (\hat{x} \hat{,} \hat{X} \hat{\times} \hat{y}) . \qquad (4.7.23)$$

Consider now an isobasis $\hat{e}_i$ which is isonormalized with respect to the product $(...\hat{,}...)$, i.e., satisfying Eq.s (4.7.15).

Definition 4.7.3: *Let $\hat{D}$ be an isorepresentation of a Lie-isotopic group $\hat{G}$ with respect to basis (4.7.15) on an isolinear space $\hat{\mathcal{H}}$. Then the "isohermitean conjugate" $\hat{D}^{\dagger}$ of D is given by*

$$\hat{D}_{ij}{}^{\dagger}(\hat{a}) = \overline{\hat{D}_{ji}}(\hat{a}^{-1}), \quad \hat{a} \in \hat{G} . \qquad (4.7.24)$$

where the upper bar denotes complex conjugate. The isorepresentation is called "isounitary" when it coincides with its isohermitean conjugate,

$$\hat{D} \equiv \hat{D}^{\dagger} . \qquad (4.\hat{7}.25)$$

An inspection of the structure of the isorepresentations leads to the following classification.

Definition 4.7.4 [54]: *Let* X_k, k = 1, 2, .., n, *represent a maximal commuting set of generators of a Lie algebra* L *(such as* J^2 *and* J_3 *for the* so(3) *algebra) and let* $S_k{}^\circ \in R(n,+,\times)$ *be its spectrum of eigenvalues with respect to a given basis* $|\, b >$,

$$X_k \,|\, b > = S_k{}^\circ \,|\, b >, \qquad (4.7.26)$$

(such as L(L + 1) *and* M = L, L − 1, ..., −L, L = 0, 1, 2, ... *for* so(3)) *characterizing a set of representations* D *of* L. *Let* $\hat{L}$, $\hat{X}_k$ *and* $|\, \hat{b} >$ *be the corresponding isotopes of Class I, and let* $\hat{S}_k(\hat{T}) \in \hat{R}(\hat{n},+,\hat{\times})$ *be the corresponding new spectrum of eigenvalues,*

$$\hat{X}_k \,\hat{\times}\, |\, \hat{b} > \; = \; \hat{X}_k \,\hat{T}\, |\, \hat{b} > \; = \; \hat{S}_k(\hat{T}) \,\hat{\times}\, |\, \hat{b} > \; = \; S_k(T) \,|\, \hat{b} > , \qquad (4.7.27)$$

Then, the isorepresentation $\hat{D}$ *of* $\hat{L}$ *is said to be:*
A) "regular" when the isotopic spectrum $S_k(T)$ *is entirely factorizable into the form*

$$S_k(T) \; = \; S^\circ{}_k \, f_k(\Delta), \quad \text{no sum}, \quad \Delta \; = \; \det T \, ; \qquad (4.7.28)$$

where $f_k(\Delta)$ *are smooth functions of* Δ *such that* $f_k(1) = 1$, k = 1, 2, ..., n;
B) "irregular" when the above factorization does not exist for at least one element of the spectrum $S_k(T)$; and
C) "standard" when the isotopic and conventional spectra coincide, $S_k \equiv S^\circ{}_k$, k = 1, 2, ..., n, *but the two representations* $\hat{D}$ *and* D *are not equivalent.*

We learn in this way that the spectrum of eigenvalues of a Lie representation can be preserved under a particular type of isotopy called standard, but the structure of the representation is generalized. This property is relevant for physical applications because, as we shall illustrate below and study in details in Vol. II and III, isotopic techniques permit the preservation of conventional quantum numbers under new functional degrees of freedom in the representations.

In turn, the latter permit physical applications which are prohibited in conventional theories, such as the exact representation of the still unknown total magnetic moments for few-body nuclei via the representation of the deformability of the charge distribution of protons and neutrons when members of a nuclear structure with consequential alteration of their intrinsic magnetic moments.

One explicit form of the regular and standard isoadjoint isorepresentations

can be easily constructed from the corresponding representations via a rule here called *Klimyk's rule* [52] (although the rule does not apply for irregular and other isorepresentations). Let D_k denote the adjoint representation of a given Lie algebra L. Then one adjoint representation of the isotope $\hat{L}$ of L (up to isoequivalence) is given by

$$D_k = \hat{D}_k \hat{T}, \text{ i.e., } \hat{D}_k = D_k \hat{1}. \tag{4.7.30}$$

In fact, rule (4.7.30) transforms ordinary commutators into the isocommutators according to

$$[D_i, D_j] = D_i D_j - D_j D_i = \hat{D}_i T \hat{D}_j \hat{T} - \hat{D}_j T \hat{D}_i \hat{T} =$$
$$= C_{ij}{}^k D_k = \hat{C}_{ij}{}^k \hat{T} \hat{D}_k \hat{T}, \tag{4.7.31}$$

The removal of the common $\hat{T}$ factor on the r.h.s. then yields the isocommutators,

$$[\hat{D}_i \hat{,} \hat{D}_j] = \hat{D}_i T \hat{D}_j - \hat{D}_j T \hat{D}_i = [D_i, D_j] \hat{1} = \hat{C}_{ij}{}^k \hat{T} \hat{D}_k . \tag{4.7.32}$$

The preservation of the spectrum of eigenvalues $S°_\alpha$ of a conventional adjoint representation D_α under the Klimyk rule is also evident because

$$D_\alpha | b > = S°_\alpha | b > = \hat{D}_\alpha \hat{\times} | b > = \hat{S}°_{\alpha\alpha} (\hat{T}) \hat{\times} | b > = S°_\alpha | b >, \quad (4.7.32)$$

and the same holds for the isocasimir invariants, e.g.,

$$D^2 | b > = S | b > = \sum_k D_k (T) T D_k (T) T | b > = S | b >, \tag{4.7.33}$$

thus confirming the "regular" character of the isorepresentation $\hat{D}$.

By noting that rule (4.7.31) has an additional "degree of freedom constituted by a non-null multiplicative constant N, we have the following

Lemma 4.7.5 - Klimyk's rule [52] : *Let* D *be the adjoint representation of a Lie algebra* L *and let* $\hat{L}$ *be a Class I isotope of* L*. Then, a regular adjoint isorepresentation of* $\hat{L}$ *up to isoequivalence is given by*

$$D = N \hat{D} \hat{T}, \quad N \in \hat{F}, \ N \neq N, \tag{4.7.34}$$

and the standard adjoint isorepresentation occurring for N = 1, under which the original spectrum of L *is preserved*

It should be stressed that rules (4.7.34) *are not* equivalence transformations, i.e., *there exist no matrix* U *such that*

$$\hat{D}_k = D_k \hat{1} = U D_k U^{-1}, \tag{4.7.35}$$

for all k = 1, 2, ..., N. Thus, adjoint representations and isorepresentations are not equivalent. Also, there exists no known rule for the construction of the irregular isorepresentations from conventional ones.

it is understood that the above differences between representations and isorepresentations characterize the desired mathematical differences between particles and isoparticles.

It is an instructive exercise for the interested reader to work out the definitions of *isoreducibility* and *isoirreducibility, isotensor product* and other known aspects of the conventional Lie's theory. For additional mathematical studies we refer the interested reader to ref. [24].

We now illustrate the results of this section with specific examples. Consider the adjoint representation of the **su(2)** Lie algebra on the complex Euclidean, two-dimensional, space E(z,δ,C). It is given by the celebrated *Pauli's matrices* we encountered in the representation of quaternions, Eq.s (2.7.6), and in the illustration of Theorem 4.4.1,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.7.36}$$

which satisfy the associative rules

$$\sigma_n \sigma_m = 2 i \epsilon_{nmk} \sigma_k + \delta_{nm}, \quad n, m = 1, 2, 3, \tag{4.7.37}$$

where ϵ_{nmk} is the conventional totally antisymmetric tensor of rank three, and the Lie rules

$$[\sigma_n, \sigma_m] = \sigma_n \sigma_m - \sigma_m \sigma_n = 2 i \epsilon_{nmk} \sigma_k, \tag{4.7.38}$$

with Casimir invariant $\sigma^2 = \sum_{k=1,2,3} \sigma_k{}^2$, maximal commuting set $X = \{\sigma_k, \sigma^2\}$, and eigenvalues on a two-dimensional orthogonal basis $|b>$

$$\sigma_3 |b> = \pm |b>, \quad \sum_{k=1,2,3} \sigma_k{}^2 |b> = 3|b>, \tag{4.7.39}$$

The isotopies of Pauli's matrices were outlined at the 1993 *International Third Wigner Symposium* at Oxford University [45] and then studied in detail in ref. [54]. They are reviewed in detail in Ch. II.6 where we also construct the irreducible isorepresentations of the Lie-isotopic algebra $\hat{\mathbf{su}}(2)$. In this section it is

sufficient to indicate that the isotopy here considered begins with the lifting of E(z,δ,R) into the complex isoeuclidean space

$$\hat{E}(\hat{z},\hat{\delta},\hat{R}): \quad \hat{\delta} = \hat{T}\,\delta, \quad T = \mathrm{diag.}\,(g_{11}, g_{22}), \quad \hat{1} = \mathrm{diag.}\,(g_{11}^{-1}, g_{22}^{-1})\,. \quad (4.7.40)$$

The Lie-isotopic group $S\hat{U}(2)$ is then the invariance of the generalized expression

$$\hat{z}^{\dagger}\,\hat{\delta}\,\hat{z} = \bar{z}_1\, g_{11}\, z_1 + \bar{z}_2\, g_{22}\, z_2\,, \qquad (4.7.41a)$$

$$\hat{z}' = \hat{U} \hat{\times} \hat{z}, \quad \hat{U} \hat{\times} \hat{U}^{\hat{\dagger}} = \hat{U}^{\dagger} \hat{*} \hat{U} = \hat{1}, \quad \det(\hat{U}\,\hat{T}) = 1, \qquad (4.742b).$$

$$\hat{U} = \hat{e}^{\,i\,\hat{\sigma}_k * \hat{\theta}_k} = \{ e^{\,i\,\hat{\sigma}_k\, T\,\theta_k} \}\,\hat{1}\,, \quad \mathrm{tr.}\,(\hat{\sigma}\,\hat{T}) = 0\,, \qquad (4.7.43c)$$

with Lie-isotopic algebra for the isoadjoint isorepresentation

$$[\,\hat{\sigma}_n \hat{,} \hat{\sigma}_m\,] = \hat{\sigma}_n\,\hat{T}\,\hat{\sigma}_m - \hat{\sigma}_m\,\hat{T}\,\hat{\sigma}_n = 2\,i\,\hat{C}_{nmk}(t, z, \bar{z}, \ldots)\,\hat{T}\,\hat{\sigma}_k\,, \qquad (4.7.44)$$

where the $\hat{C}$'s are the structure functions of **$\hat{su}(2)$** as identified below. The following adjoint isorepresentations of $\hat{su}(2)$ were then constructed in ref.s [45,54]:

A) Regular isopauli matrices, they are given from rule (4.7.35) by

$$\hat{\sigma}_1 = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & g_{11} \\ g_{22} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & -i\,g_{11} \\ +i\,g_{22} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \Delta^{-\frac{1}{2}} \begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} \quad (4.7.45a)$$

$$P = \Delta^{-\frac{1}{2}}\,\mathrm{dig.}\,(g_{22}, g_{111}) = \Delta^{\frac{1}{2}}\,T^{-1}, \quad K = \Delta^{\frac{1}{2}}, \quad \Delta = \det Q = g_{11} g_{22} > 0, \qquad (4.7.45b)$$

with the isoassociative rules

$$\hat{\sigma}_i\,\hat{T}\,\hat{\sigma}_j = 2\,\Delta^{\frac{1}{2}}\,i\,\epsilon_{ijk}\,\hat{\sigma}_k + \Delta^{\frac{1}{2}}\,\hat{1}\,\delta_{ij}, \qquad (4.7.46)$$

and, consequential isocommutator rules

$$[\,\hat{\sigma}_n \hat{,} \hat{\sigma}_m\,] = \hat{\sigma}_n\,\hat{T}\,\hat{\sigma}_m - \hat{\sigma}_m\,\hat{T}\,\hat{\sigma}_n = 2\,\Delta^{\frac{1}{2}}\,i\,\epsilon_{nmk}\,\hat{\sigma}_k\,, \qquad (4.7.47)$$

Note the identity in this case of the structure functions with the conventional structure constants of su(2) up to the multiplicative term $\Delta^{\frac{1}{2}}$, thus confirming the local isomorphism $\hat{su}(2) \approx su(2)$. The isoeigenvalues are generalized and are given by

$$\hat{\sigma}_3 \hat{\times} |\,\hat{b}_i^{\,2}> = \pm\,\Delta^{\frac{1}{2}}\,|\,\hat{b}_i^{\,2}>, \qquad (4.7.48a)$$

$$\hat{\sigma}^{\hat{2}} \hat{\times} |\,\hat{b}_i^{\,2}> = \textstyle\sum_k \hat{\sigma}_k \hat{\times} \hat{\sigma}_k \hat{\times} |\,\hat{b}> = 3\,\Delta\,|\,\hat{b}_i^{\,2}>, \; i = 1, 2 \qquad (4.7.48b)$$

thus confirming the "regular" character of the isotopy here considered (i.e., the factorizability of the isotopic contribution in the spectrum of eigenvalue). The isonormalized isobasis is then given by a trivial extension of the conventional basis, $|\hat{b}> = T^{-\frac{1}{2}}|b>$.

It is instructive to verify that isorepresentation (4.7.45a) is indeed derivable from Klimyk's rule.

B) Irregular isopauli matrices, they must be constructed via the full use of the isorepresentation theory resulting in expressions of the type

$$\hat{\sigma}_1{}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \ \hat{\sigma}_2{}' = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2, \ \hat{\sigma}_3{}' = \begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} = \Delta \hat{1} \sigma_3, \quad (4.7.49)$$

with isocommutation rules

$$[\hat{\sigma}'_1 \hat{,} \hat{\sigma}'_2] = 2\, i\, \hat{\sigma}'_3, \qquad [\hat{\sigma}'_2 \hat{,} \hat{\sigma}'_3] = 2\, i\, \Delta\, \hat{\sigma}'_1, [\hat{\sigma}'_1 \hat{,} \hat{\sigma}_1{}'] = 2\, i\, \Delta\, \hat{\sigma}'_2, \quad (4.7.50)$$

which evidently do not alter the local isomorphism $\hat{SU}_Q(2) \approx SU(2)$. The new isoeigenvalue equations are given by

$$\hat{\sigma}'_3 \hat{\times} |\hat{b}_i{}^2> = \pm \Delta |\hat{b}_i{}^2>, \quad \hat{\sigma}''^{\hat{2}} \hat{\times} |\hat{b}_i{}^2> = \Delta\, (\Delta + 2) |\hat{b}_i{}^2>, \quad (4.7.51)$$

which confirm the "irregular" character under consideration (i.e., lack of factorizability of the isotopic contributions in all elements of the spectrum). As one can see, isorepresentations (4.7.52) are far from trivial because they imply the lifting of the notion of spin $\frac{1}{2}$ into a local quantity

$$s = \tfrac{1}{2} \quad \rightarrow \quad \hat{s} \ = \ \tfrac{1}{2}\, \Delta(t, z, \overline{z}, \psi, \psi\dagger, \ldots\ldots) . \quad (4.7.52)$$

as expected for a particle in hyperdense interior conditions (e.g., a proton in the core of a collapsing star).

It is instructive to verify that isorepresentation (4.5.49) *is not* derivable from Klimyk's rule.

C) Standard isopauli matrices, which occur when $K = 1$, resulting in the expressions

$$\hat{\sigma}''_1 = \begin{pmatrix} 0 & g_{22}{}^{-1} \\ g_{11}{}^{-1} & 0 \end{pmatrix}, \hat{\sigma}''_2 = \begin{pmatrix} 0 & -ig_{22}{}^{-1} \\ ig_{11}{}^{-1} & 0 \end{pmatrix}, \hat{\sigma}''_3 = \begin{pmatrix} g_{11}{}^{-1} & 0 \\ 0 & -g_{22}{}^{-1} \end{pmatrix}, \quad (4.7.53)$$

possessing isocommutation rules with conventional structure constants,

$$[\hat{\sigma}''_n \hat{,} \hat{\sigma}''_m] = \hat{\sigma}''_n \hat{T} \hat{\sigma}''_m - \hat{\sigma}''_m \hat{T} \hat{\sigma}''_n = 2\, i\, \epsilon_{nmk}\, \hat{\sigma}''_k , \qquad (4.7.57)$$

and admitting conventional eigenvalues

$$\hat{\sigma}''_3 \hat{\times} |\hat{b}> = \pm |\hat{b} , \quad \hat{\sigma}''^{\hat{2}} \hat{\times} |\hat{b}> = 3 |\hat{b}> . \qquad (4.7.54)$$

Yet, isorepresentations (4.7.56) exhibit the "hidden variables" g_{kk} in their very structure. Also, the above matrices are not unitarily equivalent to the conventional Pauli's matrices, thus establishing the "standard" character of the isorepresentation.

It is instructive to verify that isorepresentations (4.7.53) is indeed derivable from Klimyk's rule. This illustrates Definition 4.7.4. 4.

We now study the degrees of freedom of the above isorepresentations. Those of the standard isorepresentations are trivially expressed by the arbitrariness of the factor K. The degrees of freedom of the other isorepresentations are less trivial.

D) Isoequivalent irregular isopauli matrices, which are illustrated

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & g_{22}^{-\frac{1}{2}} \\ g_{11}^{-\frac{1}{2}} & 0 \end{pmatrix}, \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -ig_{22}^{-\frac{1}{2}} \\ i\, g_{11}^{-\frac{1}{2}} & 0 \end{pmatrix}, \ \hat{\sigma}_3 = \begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & -g_{22}^{-1} \end{pmatrix}, \qquad (4.7.55)$$

with isocommutation rules

$$[\hat{\sigma}_1 \hat{,} \hat{\sigma}_2] = 2\, i\, \Delta\, \hat{\sigma}_3 , \ [\hat{\sigma}_2 \hat{,} \hat{\sigma}_3] = 2\, i\, \hat{\sigma}_1 , \ [\hat{\sigma}_3 \hat{,} \hat{\sigma}_1] = 2\, i\, \hat{\sigma}_2 , \qquad (4.7.56)$$

and isoeigenvalues

$$\hat{\sigma}_3'' \hat{\times} |\hat{b}_i^2> = \pm |\hat{b}_i^2 , \quad \hat{\sigma}^{\hat{2}} \hat{\times} |\hat{b}_i^2> = (1 + 2\dot{\Delta}) |\hat{b}_i^2> , \qquad (4.7.57)$$

where, as one can see, the eigenvalue of the third component is conventional, but that of the magnitude is generalized with a nonfactorizable isotopic contribution, thus confirming the "irregular" character of the isorepresentation. Again, the above isorepresentation is not derivable from Klimyk's rule.

E) Isoequivalent standard isopauli matrices, which are given by particularizations of the standard and irregular isorepresentations for the case

$$\Delta = g_{11}\, g_{22} = 1 \qquad (4.7.58)$$

which holds under the identification

$$g_{11} = g_{22}^{-1} := \lambda \neq 0, \tag{4.7.59}$$

where λ is a real value and nowhere null but arbitrary functions of local quantities, resulting in expressions of the type

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}, \tag{4.7.60a}$$

$$\hat{\sigma}_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \hat{\sigma}_2' = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2, \quad \hat{\sigma}_3' = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}, \tag{4.7.60b}$$

$$\hat{\sigma}''_1 = \begin{pmatrix} 0 & \lambda^{\frac{1}{2}} \\ \lambda^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad \hat{\sigma}''_2 = \begin{pmatrix} 0 & -i\lambda^{\frac{1}{2}} \\ i\lambda^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad \hat{\sigma}''_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}. \tag{4.7.60c}$$

They also satisfy isocommutation rules with conventional structure constants and possess conventional eigenvalues, yet remain inequivalent to the conventional Pauli's matrices as one can verify.

The physical implications of the isotopies of Pauli's matrices, as well as of the isorepresentation theory at large, will be studied in the applications of Vol. III. In essence, the appearance of the "hidden parameter" (actually the "hidden function") λ under conventional values of spin $s = \frac{1}{2}$ has a num,ber of novel and intriguing applications, that is, applications not possible with quantum mechanics, such as: the reconstruction of the exact isospin symmetry in nuclear physics under weak and electromagnetic interactions via equal proton and neutron masses in isospace represented by λ^2; the representation of total magnetic moments of few-body nuclei via a deformation of that of the individual nucleons conjectured since the early stages of nuclear physics but not treated via quantum mechanics; the characterization of a generalized notion of quark called *isoquark* which is indistinguishable with conventional quarks (because the quantum number are the same), yet possessing an exact confinement because of the incoherence between the interior and exterior Hilbert spaces; and others.

In summary, the above example indicates that the SU(2)-spin symmetry, admits an isotopic image $S\hat{U}(2)$ which is isomorphic to the original symmetry, $S\hat{U}(2) \approx SU(2)$, because of the axiom-preserving character of the isotopies. Yet the isotopic $\hat{\mathbf{su}}(2)$ algebra and its isorepresentations are not unitarily equivalent to the original ones, and the spectra of eigenvalues are generally altered, thus illustrating the nontriviality of the isotopies.

APPENDIX 4.A: ELEMENTS OF ABSTRACT ALGEBRAS AND ISOALGEBRAS

The hadronic generalization of quantum mechanics was born thanks, specifically, to studies in abstract algebras [1,2]. A few rudimentary notions in that field appear therefore recommendable as an introduction to the content of this chapter which are here essentially derived from Sect. II.5, of ref. [21]. These notions are important to understand later on in Ch. I.7 and in Vol. II the emergence, apparently for the first time in physics, of *Jordan algebras* in the structure of the Lie-admissible time evolution for open irreversible systems.

Let us recall from Sect. I.2.4 that a (finite-dimensional) *linear algebra* U, or *algebra* for short (see, e.g., ref. [34]) is a linear vector space V over a field $F(\alpha,+,\times)$ (hereon assumed to be of characteristics zero (Sect. I.2.3)) equipped with a multiplication ab verifying the following axioms

$$\alpha(a\,b) = (\alpha\,a)b = a(\alpha\,b),\ (a\,b)\beta = a(b\beta) = (a\beta)b, \tag{4.A.1a}$$

$$a(b+c) = a\,b + a\,c,\ (a+b)c = a\,c + b\,c, \tag{4.A.1b}$$

called *right and left scalar and distributive laws,* respectively, which must hold for all elements a, b, c $\in$ U, and $\alpha, \beta \in$ F.

The reader should keep in mind that *the above axioms must be verified by all products to characterize an algebra as commonly understood* [34]. In particular, the distributive law is the basic axiom which prevented the lifting of the operation of addition as shown in Sect. I.2.3.

Among the existing large number of algebras [34], an understanding of hadronic mechanics requires a knowledge of the following primary algebras:

1) **Associative algebras** A, characterized by the additional axiom (besides laws (4.A.1))

$$a(b\,c) = (a\,b)c \tag{4.A.2}$$

for all a, b, c $\in$ A, called the *associative law.* Algebras violating the above law are called *nonassociative.* All the following algebras are nonassociative:

2) **Lie algebras** L which are characterized by the additional axioms

$$a\,b + b\,a = 0, \tag{4.A.3a}$$

$$a(b\,c) + b(c\,a) + c(a\,b) = 0. \tag{4.A.3b}$$

A familiar realization of the Lie product is given by

$$[a, b]_A = a\,b - b\,a, \tag{4.A.4}$$

with the classical counterpart being given by the familiar Poisson brackets among functions A, B in cotangent bundle (phase space) $T^*E(r,\delta,\Re)$

$$[A,B]_{Poisson} = \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k}\frac{\partial A}{\partial p_k}. \tag{4.A.5}$$

3) **Commutative Jordan algebras** J, characterized by the additional axioms

$$a\,b - b\,a = 0, \tag{4.A.6a}$$

$$(a\,b)\,a^2 = a\,(b\,a^2), \tag{4.A.6b}$$

A realization of the special commutative Jordan product is given by

$$\{a, b\}_A = a\,b + b\,a. \tag{4.CA7}$$

where a b is associative.

The *noncommutative Jordan algebras* are algebras U which verify Jordan's axiom (4A.6b) but not (4.A.6a).

Intriguingly, no realization of the commutative Jordan product in classical mechanics is known at this writing. As an example, the brackets

$$\{A, B\} = \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k} + \frac{\partial B}{\partial r^k}\frac{\partial A}{\partial p_k}, \tag{4.A.8a}$$

evidently verify axiom (4.A.6a), but violate axiom (4.A.6b).

4) **General Lie-admissible algebras** U [1,2,38] which are characterized by a product ab verifying laws (4.A.1), which is such that the attached product $[a, b]_U = a\,b - b\,a$ is Lie. This implies, besides (4.A.1), the unique axiom

$$(a, b, c) + (b, c, a) + (c, a, b) = (c, b, a) + (b, a, c) + (a, c, b), \tag{4.A.9}$$

where

$$(a, b, c) = a\,(b\,c) - (a\,b)\,c, \tag{4.A.10}$$

is called the *associator*.

Note that *Lie algebras are a particular case of the Lie-admissible algebras.* In fact, given an algebra L with product ab = $[a, b]_A$, the attached algebra L^- has the product

$$[a, b]_U = 2\,[a, b]_A\,, \qquad (4.A.11)$$

and, thus, L is Lie-admissible.

Therefore, the classification of the Lie Lie-admissible algebras contains all possible Lie algebras. Also, Lie algebras enter in the Lie-admissible algebras in a two-fold way: first, in their classification and, second, as the attached antisymmetric algebras. Finally, *associative algebras are trivially Lie-admissible.*

The first realization of general Lie-admissible algebras U in classical mechanics was identified by the author in memoir [1] and, in its simplest possible form, it is given by the following product for functions A(r,p) and B(r,p) in $T^*E(r,\delta,\mathfrak{R})$

$$U: \quad (A, B) = \frac{\partial A}{\partial r^k}\,\frac{\partial B}{\partial p_k}\,, \qquad (4.A.12)$$

namely, *the general, nonassociative Lie-admissible algebras are at the foundations of the structure of the conventional Poisson brackets* (4.A.5) which can be written

$$[A, B]_{Poisson} = [A, B]_U = (A, B) - (B, A), \qquad (4.A.13)$$

The first operator realization of the general Lie-admissible algebras was also given by the author in the subsequent memoir [2] Sect. 4.14, and can be written

$$U: \quad (a, b)_A = a\,r\,b - b\,s\,a, \qquad (4.A.14)$$

$$r, s \ \ \text{fixed} \in A, \quad r \neq s, \quad r, s \neq 0$$

where ar, rb, etc., are associative. In fact, the antisymmetric product attached to U is a particular form of a Lie algebra (see below)

5) **Flexible Lie-admissible algebras** U [1,2,38], which are characterized by the axioms in addition to (5.1)

$$(a, b, a) = 0, \qquad (4.A.15a)$$

$$(a, b, c) + (b, c, a) + (c, a, b) = 0, \qquad (4.A.15b)$$

where condition (4.C.14a), called the *flexibility law* [38].is a simple generalization

of the anticommutative law, as well as a weaker form of associativity. A first realization of the flexible Lie-admissible product was identified by this author back in 1967 [55]

$$(a, b)_U = \lambda\, a\, b - \mu\, b\, a, \quad \lambda, \mu \in F \tag{4.A.16}$$

where the products λa, ab, etc. are associative. It is instructive to verify that the algebras characterized by the above product is a realization of the noncommutative Jordan algebras.

As we shall see in App. I.7.A, *a certain class of the so-called q-deformations* [56] *are a particular case of product (4.A.16) and, as such, they are flexible, Lie-admissible and Jordan-admissible, as well as noncommutative Jordan algebras.*

No classical realization of flexible Lie-admissible algebras has been identified until now, to our best knowledge. As an example, the brackets on $T^*E(r,\delta,\Re)$

$$(A, B) = \lambda \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k} - \mu \frac{\partial B}{\partial r^k} \frac{\partial A}{\partial p_k} \tag{4.A.17}$$

are Lie-admissible, but violate the flexibility law.

6) **General Jordan-admissible algebras** U [1,2,38], which are characterized by a product ab verifying laws (4.C.1), such that the attached symmetric product $\{a, b\}_U = a\,b + b\,a$ is Jordan, i.e., verifies the axiom

$$(a^2, b, a) + (a, b, a^2) + (b, a^2, a) + (a, a^2, b) = 0. \tag{4.A.18}$$

Again, *associative and Jordan algebras are trivially Jordan- admissible.* Also, Jordan algebras enter in the Jordan-admissible algebras in a two-fold way, in the classification of the latter, as well as the attached symmetric algebras.

It is important for the operator formulation of the isotopies of Vol. II to point out the following important

Proposition 4.A.1 [2]: *The general Lie-admissible product (4.A.14) is, jointly, Lie-admissible and Jordan-admissible.*

But Lie-admissible product (4.A.11) characterizes the brackets of the most possible general time evolution of hadronic mechanics. The Jordan algebra therefore enter as the attached form U^+ of the Lie-admissible algebras U of hadronic operators.

By comparison, there is no "Jordan content" in quantum mechanics, because the algebra of its time evolution is a Lie algebra L, whose attached

symmetric part is identically null, $L^+ \equiv 0$.

As we shall see in Vol. II, the emergence of a nontrivial "Jordan content" has far reaching physical implications, such as the capability of constructing a quark theory with an "exact confinement", i,.e., with a transition probability for free quarks which is explicitly computed and rigorously proved to be identically null under any possible physical condition.

Intriguingly, this emergence of a "Jordan content" at the operator level has no known counterpart in classical mechanics. In fact, the classical Lie-admissible product (4.A.12) is only Lie-admissible and not jointly Jordan-admissible.

7) **Flexible Jordan-admissible algebras** U [1,2,38), which, in addition to axioms (4.A.1), are characterized by the axioms

$$a(ba) = (ab)a, \tag{4.A.19a}$$

$$a^2(ba) + a^2(ab) = (a^2 b)a + (a^2 a)b. \tag{4.A.19b}$$

The flexible Lie-admissible product (4.A.16) is also a flexible Jordan-admissible product, but the classical product (4.C.17) is only Lie-admissible, and not flexible Lie-admissible or Jordan-admissible.

8) **Alternative algebras** U, which are algebras characterized by the additional axioms encountered in Sect 2.2,

$$(a, a, b) = 0 \text{ and } (a, b, b) = 0, \ \forall\ a, b, c \in U \tag{4.A.20}$$

called right and left alternative laws. A realization of alternative algebras is given by the octonions (Sect. I.2.8).

9) **Power associative algebras** U, characterized by the additional law

$$a^n a^m = a^{n+m}, \quad \forall\ a \in U, \ n, m \text{ integers} \tag{4.A.21}$$

which constitutes the axiomatization of an important physical notion. In fact, algebras currently used used in physics are power associative.

For additional algebras we refer the reader to ref.s [34,38] and quoted literature.

We now pass to the study of the isotopies of the above notions.

Definition 4.A.1 [1]: *An "isoalgebra", or simply an "isotope"* $\hat{U}$ *of an algebra* U *with elements* a,b,c,... *and product* ab *over a field* F, *is the same vector space* U *but defined over the isofield* $\hat{F}$, *equipped with a new product* a*b, *called "isotopic product", which is such to verify all original axioms of* U.

Thus, by definition, the isotopic lifting of an algebra does not alter the type of algebra considered.

It is important for these studies to review the isotopies of the primary algebras listed above.

Given an associative algebra A with product ab over a field F, its simplest possible *isotope* $\hat{A}$, called *associative-isotopic* or *isoassociative algebra* [1] is given by

$$\hat{A}_1: \quad a * b = \alpha\, a\, b, \qquad \alpha \in F \text{, fixed and} \neq 0, \tag{4.A.22}$$

and called a *scalar isotopy.* The preservation of the original associativity is trivial in this case. This is evidently the case of the q-deformations [56].

A second less trivial isotopy is the fundamental one of the Lie-isotopic theory, and it is characterized by the basic product of this chapter [2]

$$\hat{A}_2: \quad a * b = a\, T\, b, \tag{4.A.23}$$

where T is an nonsingular (invertible) and Hermitean elements not necessarily belonging to the original algebra A.

The third known isotopy of A is given by [2]

$$\hat{A}_3: \quad a * b = w\, a\, w\, b\, w, \qquad w^2 = w\, w = w \neq 0, \tag{4.A.24}$$

Additional isotopies are given by the combinations of the preceding ones, such as

$$\hat{A}_4: \quad a * b = w\, a\, w\, T\, w\, b\, w, \quad w^2 = w\, w = w \neq 0 \tag{4.A.25a}$$

$$\hat{A}_5: \; a * b = \alpha\, w\, a\, w\, T\, w\, b\, w, \; \alpha \in F, \quad w^2 = w, \quad a, w, T \neq 0. \tag{4.A.25b}$$

It is believed that the above isotopies exhaust all possible isotopies of an associative algebra over a field of characteristic zero, although this property has not been rigorously proved to this writing.

We now pass to the study of the *isotopes* $\hat{L}$ of a Lie algebra L with product ab over a field F, which are the same vector space L but equipped with a *Lie-isotopic product* [1] $a \odot b$ over the isofield $\hat{F}$ which verifies the left and right scalar and distributive laws (4.A.1), and the axioms

$$a \odot b + b \odot a = 0, \tag{4.A.26a}$$

$$a \odot (b \odot c) + b \odot (c \odot a) + c \odot (a \odot b) = 0, \tag{4.A.26b}$$

Namely, the abstract axioms of the Lie algebras remain the same by assumption.

The simplest possible *realization of the Lie-isotopic product* is that attached to isotopes $\hat{A}_1$,

$$\hat{L}_1: \ [a, b]_{\hat{A}_1} = a \odot b - b \odot a = \alpha (a b - b a) = \alpha [a, b]_A, \ \alpha \in F, \alpha \neq 0, \quad (4.A.27)$$

and it is also called a *scalar isotopy.* It is generally the first lifting of Lie algebras one can encounter in the operator formulation of the theory.

The second independent realization of the Lie-isotopic algebras is that characterized by the isotope $\hat{A}_2$ which is that of primary use in hadronic mechanics [1,2]

$$\hat{L}_2: \quad [a, b]_{\hat{A}_2} = a \odot b - b \odot a = a T b - b T a, \quad (4.A.28)$$

The third, independent isotopy is that attached to $\hat{A}_3$ [2]

$$\hat{L}_3: \quad [a,b]_{\hat{A}_3} = w a w b w - w b w a w, \quad w^2 = w w \neq 0. \quad (4.A.29)$$

A fourth isotope is that attached to $\hat{A}_4$, i.e.,

$$\hat{L}_4: \quad [a, b]_{\hat{A}_4} = w a w T w b w - w b w T w a w, \quad w^2 = w, \quad w, T \neq 0. \quad (4.A.30)$$

A fifth and final (abstract) isotope is that characterized by $\hat{A}_5$, i.e.

$$\hat{L}_5: \ [a, b]_{\hat{A}_5} = \alpha [a, b]_{\hat{A}_4}. \quad (4.A.31)$$

Again, it is believed that the above five isotopes exhaust all possible abstract Lie algebra isotopies (over a field of characteristics zero), although this property has not been proved to date on rigorous grounds.

Note that the Lie algebra attached to the general Lie-admissible product (4.A.12) are not conventional, but isotopic. In fact, we can write

$$[a, b]_U = (a, b)_A - (b, a)_A = a r b - b s a - b r a + a s b = \quad (4.A.32a)$$

$$= a T b - b T a = a * b - b * a, \quad (4.A.32b)$$

$$r \neq s, \quad r, s, T \neq 0, \quad T = r + s \neq 0.$$

As a matter of fact, the author first encountered the Lie-isotopic algebras by studying precisely the Lie content of the more general Lie-admissible algebras [1]

The following property can be easily proved from properties of type (5.30).

Lemma 4.A.1 [1]: *An abstract Lie-isotopic algebra* $\hat{L}$ *attached to a general, nonassociative, Lie-admissible algebra* U, $\hat{L} \approx U^-$, *can always be isomorphically*

rewritten as the algebra attached to an isoassociative algebra $\hat{A}$, $\hat{L} \approx \hat{A}^-$, and vice-versa, i.e.

$$\hat{L} \approx U^- \approx \hat{A}^-. \tag{4.A.33}$$

The above property has the important consequence that *the construction of the abstract Lie-isotopic theory does not necessarily require a nonassociative enveloping algebra because it can always be done via the use of an isoassociative envelope.* In turn, this focuses again the importance of knowing all possible isotopes of an associative algebra, e.g., from the viewpoint of the representation theory.

The most general possible, classical, *local-differential* [28] realization of Lie-isotopic algebras via functions A(a) and B(a) in $T^*E(r,\delta,\Re)$ with local chart

$$a = (a^\mu) = (r, p) = (r^i, p_i), \quad i, = 1, 2,..., n, \quad \mu = 1, 2,....., 2n, \tag{4.A.34}$$

is provided by the *Birkhoffian brackets* [1,4] also called generalized Poisson brackets

$$[A , B]_{\text{Birkhoff}} = [A, B]_U = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu} , \tag{4.A.35}$$

where $\Omega^{\mu\nu}$, called the *Lie-isotopic tensor* , is the contravariant form of (the exact, symplectic, *Birkhoff's tensor*

$$\Omega^{\mu\nu} = (|\Omega_{\alpha\beta}|^{-1})^{\mu\nu}, \tag{4.A.36a}$$

$$\Omega_{\mu\nu} = \frac{\partial R_\nu(a)}{\partial a^\mu} - \frac{\partial R_\mu(a)}{\partial a^\nu}, \tag{4.A.36b}$$

where the R's are the so-called *Birkhoff's functions.* The symplectic character of the covariant tensor ensures the Lie-isotopic character of brackets (see the geometric treatment of the next chapter).

Recall that, unlike the conventional, abstract, Lie brackets (4.A.4), the conventional Poisson brackets (4.A.5) characterize a Lie algebra attached to a *nonassociative* Lie-admissible algebra U, Eq.s (4.A.13). It is then evident that the covering Birkhoff's brackets (4.A.35) are also attached to a nonassociative Lie-admissible algebra, although of a more general type (see ref. [4] for brevity).

For other classical Lie-isotopic brackets, such as *Dirac's generalized brackets* for systems with subsidiary constraints see the locally quoted

[28] The nonlocal-integral realizations will be presented in the next chapter after studying the underlying nonlocal-integral geometries.

references.

Note the lack of identification of the underlying generalized unit in Birkhoff's brackets (4.A.35). This is precisely the aspect which has requested the isotopies of conventional geometries of the next chapter.

Realizations of the abstract isotopes $\hat{U}$ of the Lie-admissible algebras can be easily constructed via the above techniques. For instance, an isotope of the general Lie-admissible product (4.A.14) is given by

$$\hat{U}: \quad (a \hat{,} b) = w\,a\,w\,r\,w\,b\,w \; - \; w\,b\,w\,s\,w\,a\,w, \qquad w^2 = w, \quad w, r, s \neq 0, \; r \neq s. \tag{4.A.37}$$

An isotope of the classical realization (4.A.11) is then given by

$$\hat{U}: \qquad (A \hat{,} B) = \frac{\partial A}{\partial a^\mu} \, S^{\mu\nu}(t,a) \frac{\partial B}{\partial a^\nu} \;, \tag{4.A.38}$$

where the tensor $S^{\mu\nu}$, called the *Lie-admissible tensor,* is restricted by the conditions of admitting Birkhoff's tensor as the attached antisymmetric tensor, i.e.,

$$S^{\mu\nu} - S^{\nu\mu} \equiv \Omega^{\mu\nu}, \tag{4.A.39}$$

Brackets (4.A.38] constitutes the basic product of the classical Lie-admissible studies of ref.s [10–14].

Historical notes on the origin of the isotopies are provided in ref. [3]. The broader genotopies will; be studied in Ch. I.7.

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5: ISOGEOMETRIES AND THEIR ISODUALS

5.1: INTRODUCTION

5.1.A: Foundational elements. Since the time of his graduate studies in theoretical physics at the University of Turin, Italy, in the 1960's, this author has been interested in *the geometric representation of dynamical systems within physical media,* such as a space-ship during re-entry in our atmosphere or, along conceptually similar lines, a neutron in the core of a neutron star.

Subsequent studies conducted at the Department of Mathematics of Harvard's University in the late 1970's, revealed that the conventional Euclidean, Minkowskian, symplectic, Riemannian and other geometries (see, e.g., ref.s [1–4], respectively and quoted literature) are not effective for the problem considered, e.g., because they are not capable of representing the *free* fall of a leaf in atmosphere as a *geodesic* trajectory. Numerous additional insufficiencies also emerged for an effective geometrization of physical media.

The main guiding principle which resulted from these studies is that *physical media alter the geometry of empty space.*

As an example, the geodesic representation of the free fall of a leaf in atmosphere requires a *necessary* alteration of the Riemannian geometry, evidently because the latter geometry can only characterize a geodesic *in empty space* which is different than that within our atmosphere, while the *geodesic* character of the presentation is necessary because the fall is *free.*

The inclusion of resistive forces does not ensure the preservation of conventional geometries because in their most general possible form they are *essentially nonselfadjoint,* that is, outside the representational capabilities of a first-order lagrangian in the frame of the experimenter (Ch. I.1). The further admission of the physical evidence that the trajectory depends on the *shape* of the leaf, thus implying nonlocal-integral terms (Sect. I.1.1.1), establishes the inapplicability beyond scientific doubts of conventional local-differential geometries from their topological foundations, let alone their first-order lagrangian character or geodesic capabilities.

As we shall see during the course of our analysis, this basic geometric

principle appears to be verified at all levels, including the classical nonrelativistic, relativistic and gravitational levels, with corresponding operator counterparts at the particle level.

The above studies brought into focus the historical distinction between *exterior dynamical problems in vacuum*, and *interior dynamical problems within physical media* considered since the Preface of this volume. Conventional local-differential geometries can be proved to be *exactly valid* for exterior problems, but they result to be only *approximately valid* for interior problems.

EXTERIOR AND INTERIOR PROBLEMS

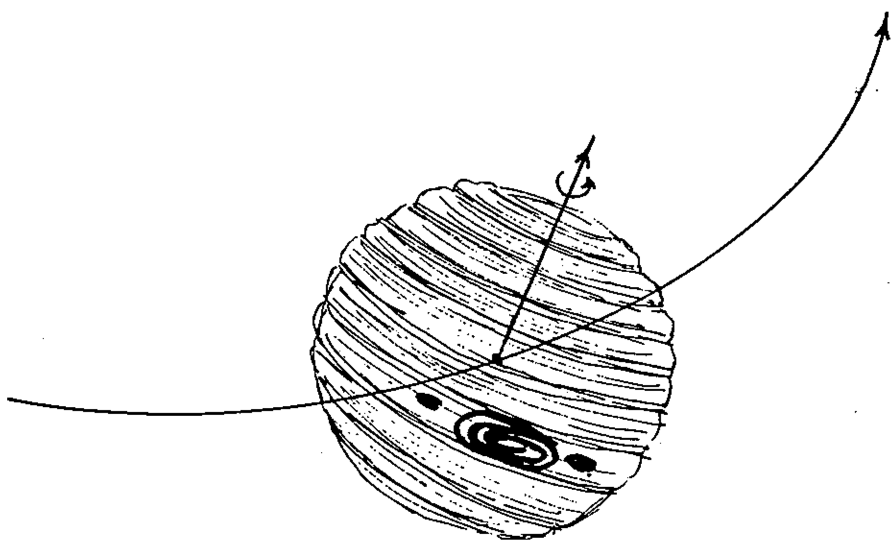

FIGURE 5.1.1: The physical and mathematical inequivalence of exterior and interior problems can be identified via direct visual observations, e.g., a telescopic view of Jupiter. In its orbit around the Sun, Jupiter can be well approximated as a *massive point* along Galileo's conception, because its size, shape and structure do not affect its trajectory in the solar system. This implies the exact validity of the Euclidean, Minkowskian and Riemannian geometries for the corresponding Newtonian, relativistic and gravitational representations. Visual observation of the Jupiter's structure reveals an interior scenario profoundly different then the exterior one, such as the existence of *vortices with continuously varying angular momenta* and, more generally, an aggregate of constituents which is globally stable, yet each constituent is in a generally unstable orbit with irreversible internal processes. The lack of exact applicability for Jupiter's structure

of the geometries so effective for its exterior motion in vacuum is then beyond credible doubts, and so is the need for the construction of more general geometries for the direct representation of the interior problem *beginning at the purely classical level and prior to any operator version.* It is sufficient to indicate in this respect that the Euclidean, Minkowskian and Riemannian geometries all admit, locally, the exact rotational symmetry, with resulting *necessary stability of the orbits and conserved angular momenta.* The very reason for the effectiveness of conventional geometries for the exterior problem (representation of the *stability* of Jupiter's orbit in the Solar system) then become their insufficiencies for an effective representation of interior conditions (the *instabilities* of vortices with continuously varying angular momenta). Numerous, additional, independent, topological, analytic and geometries reasons will be identified during the course of our analysis.

This is due to the physical evidence at the foundations of these volumes according to which bodies moving in vacuum can be well approximated as *massive points,* much along Galileo's original conception, thus implying the validity of local–differential geometries, because the size and shape of the bodies do not affect their trajectories. On the contrary, when moving within physical media the same bodies require a representation of their extended, generally nonspherical and deformable shapes, thus requiring suitable generalized geometries, because sizes and shapes now directly affect the trajectories (see Figure 5.1.1 for more details).

The originators of contemporary geometries were fully aware of the distinction between the exterior and interior problems. As an example, Schwarzschild wrote *two* articles in 1916 [9] in which the distinction is stated beginning from the titles of the papers. In particular, Schwarzschild's first article is dedicated to the *exterior* gravitational problem with emphasis on the *exact* character of his celebrated solution, while the second (little known) article is dedicated to the *interior* gravitational problem with emphasis on the *approximate* character of the solution).

The distinction between the exterior and interior problems was kept in the early well written treatises in gravitations (see, e.g., the *title* of Ch. VI of the treatise by Bergmann [10] – with Einstein's preface –, or the *titles* of Sect.s 11.6, p. 439 and Sect. 11.7, p. 444, of the treatise by Møller [11]).

By no means the insufficiency of the Riemannian geometry for interior problems are treated in these volumes for the first time, because they are known in the literature. For instance, the insufficiencies are at times called the (E.) *Cartan legacy,* because expressing Cartan's indication of the inability of the Riemannian geometry to recover under current limiting procedures *all* generally nonconservative Newtonian systems of our physical reality.

As a concrete example, missiles in atmosphere have nowadays reached such speeds to imply drag forces up to expressions of the type $F = -\gamma\dot{x}^{10}$, which are beyond any realistic treatment via the Riemannian geometry under any conceivable limit.

Unfortunately, with the passing of time the distinction between the exterior and interior problems was progressively lost, up to the current trend of its virtually complete elimination in the contemporary literature. This is done via the (often tacit) reduction of interior problems to ideal collections of dimensionless elementary particles which, as such, recover the exterior conditions in vacuum.

The scientific reality is that interior, nonconservative and irreversible physical events, as majestically shown by the direct observation of Jupiter's structure (Fig. I.5.1.1), simply cannot be reduced to an ideal collection of dimensionless elementary particles in stable orbits, because of the *No-reductions Theorems* indicated in Ch. I.1.

Also, we do not possess today an unambiguous operator formulation of gravity which is an evident pre-requisite for the reduction. Thus, interior gravitational problems must first be represented *classically* as they actually are in the physical reality, that is, with nonconservative irreversible interior effects. Their reduction to particle descriptions can only be studied thereafter, provided that they are capable of representing visual evidence of the interior problem, such as Jupiter's vortices with *continuously varying angular momenta*.

At the extreme, the insistence on applying conventional geometries for interior conditions leads to excessive approximations, such as *the often tacit acceptance the "perpetual motion" within a physical environment*, as necessary from the local rotational invariance of conventional geometries.

The isotopies have been selected to broaden existing geometries over other possible methods because they permit the achievement of the desired advances for the interior problem while preserving the original axioms of the exterior problem. In turn, this permits a rather remarkable unity of geometric and physical thought in which both, the exterior and interior problems emerge as *different realizations* of the same abstract axioms.

Also, the isotopies permit a clear separation between the original local-differential exterior structure, and the nonlinear-nonlocal interior effects which is evidently important for experimental verifications. In fact, as we shall see in Vol. III, experiments to test the prediction of the new geometries for interior conditions require a the identification and separation of interior effects from known exterior ones.

The isotopies are finally preferred over other approaches because they permit numerous rather intriguing advances studied throughout our analysis which do not appear to be possible with other approaches, such as the *deformations* of existing geometries.[29]

[29] The term *deformations* of a given geometry with metric m and unit I is referred to any (well behaved) deformation of the metric, $m \rightarrow \hat{m} = \hat{T}m$, while keeping the original unit and, thus, the original field, unchanged. The term *isotopies* of the same geometry is instead referred to the deformation of the metric $m \rightarrow \hat{m} = \hat{T}m$ while jointly deforming the unit of the inverse amount, $I \rightarrow \hat{1} = \hat{T}^{-1}$, thus requiring a reformulation of the basic

The analysis of this chapter is organized as follows. In Sections 5.1.B, 5.1.C and 5.1.D we present the conceptual foundations of the isogeometries for flat, symplectic and curved spaces, respectively. In Sect. 5.1.E we present the conceptual foundations of the *isodual isometries* for our *classical* representation of antimatter in a way fully parallel to the current representation of matter. In the subsequent sections of this chapter we study the isotopies of various conventional geometries. Finally, we outline in App. 5.C the *isotrigonometric* and *isohyperbolic functions* which are important for the conduction of calculations on isogeometries. Continuity aspects on isomanifolds are deferred to the next chapters.

The presentation will be as elementary as possible and intended for graduate students in theoretical and experimental physics, with primary emphasis on *physical* profiles. The achievement of the needed *mathematical rigor* is recommended for study by interested *mathematicians* in *mathematical* journals.

For recent presentations of the Euclidean, Minkowskian, symplectic and Riemannian geometries we refer the interested reader to monographs [1–4], respectively, and literature quoted therein. In this chapter we mainly use the monograph by Lovelock and Rund [4] of which we preserve the notations and symbol for clarity in the comparison of the results.

The isotopies at the foundations of this chapter were first submitted by this author in ref.s [5] of 1978 (see also monograph [6] for comprehensive treatments and applications in classical mechanics). The isotopies of metric or pseudo-metric *spaces* has been reviewed in Ch. I.3 jointly with the related original literature. The first isotopies of conventional Euclidean and Minkowskian *geometries* were constructed by this author paper [12] of 1983 and of the Riemannian *geometry* in memoirs [15,16] of 1988. These isogeometries were then studied in more details in ref.s [17–23]. The notion of isodual map was first proposed by this author in papers [13–14] of 1985 and first applied to the isodualities of the various conventional geometries with corresponding representation of antimatter in monographs [19,20] of 1991.

The independent studies on isogeometries are the following. Aringazin [24] first proved the "direct universality" of the isominkowskian geometry for all possible deformations of the Minkowski metric. Lopez [25] studied certain implications of the interior isoriemannian geometry for the exterior problem. Kadeisvili [26] wrote the first comprehensive review of the isogeometries and their classification with emphasis on the isoriemannian geometry. Sourlas and Tsagas published monograph [27] with emphasis on the isosymplectic geometry. Other independent contributions will be indicated later on.

The reader should be aware that we shall solely consider in this chapter contributions on *isogeometries*, that is, *geometries centrally dependent on the generalization of the unit.*[30] It is evident that the literature *indirectly* related to

field with respect to $\hat{1}$.

the isogeometries is vast indeed. We are here referring, e.g., to several forms of noneuclidean geometries [28] and their possible relativistic extensions *based on the conventional unit,* which we cannot possibly review here for brevity which we cannot possibly review for brevity.

5.1.B: Isotopies of flat geometries. As well known, the Euclidean [1] and Minkowskian [2] geometries provide a geometrization of the homogeneity and isotropy of empty space. As such, they are *exactly valid* for the *nonrelativistic and relativistic exterior problems in vacuum,* respectively.

Our central problem here is the identification of covering geometries which permit a direct, classical geometrization of the inhomogeneity and anisotropy of physical media for *nonrelativistic and relativistic interior dynamical problems,* first reached in ref. [12].

An important illustration, particularly for applications, is the identification of the image of the *light cone* for interior conditions, in which case the speed of light is *locally varying* thus implying the loss of the very "cone".

Even though evidently not unique, the isotopies of flat geometries are particularly suited to achieve: 1) the desired, direct, interior geometrization; 2) the representation of the most general possible nonlinear–nonlocal–nonhamiltonian interior systems; while 3) preserving the original axioms, and therefore admitting the original geometries as particular cases.

Moreover, the isotopies of the Euclidean and Minkowskian geometries permit the most general known dependence not only on local coordinates x, but also on the velocities $\dot{x}$, accelerations $\ddot{x}$, and other variables. Despite that, the isotopies here considered preserve the original axioms of flatness, thus resulting in fundamentally novel geometries in which, for instance, the notions of angles and trigonometric functions can still be defined, although in a predictable generalized way.

As we shall see in Vol.s II and III, the applications and experimental verifications of these new geometries are considerable, and include nuclear physics, particle physics, astrophysics, superconductivity and unexpected other fields, such as conchology.

5.1.C: Isotopies of the symplectic geometry. As well known, the symplectic geometry provides the geometrization of Lie algebras and, as such, is strictly local–differential, thus being inapplicable for the geometrization of nonlocal–integral systems.

Our primary objective in this chapter is therefore the identification of a covering of the symplectic geometry which is the geometric counterpart of the Lie–isotopic theory in its most general possible nonlinear–nonlocal–

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[30] The paucity of contributions in the field is also due to truly unreasonable editorial obstructions for papers on isogeometries submitted by various authors to a number of journals, which have prevented several papers in the field from seeing the light.

nonhamiltonian formulation.

The technical problem we shall address was indicated in Sect. 1.4, and can be treated now in more details. In essence, the Lie–isotopic algebras in their abstract formulation as presented in the preceding chapter showed since the original formulation [5] their natural capability to admit the most general possible nonlinear–nonlocal–nonhamiltonian systems owing to the arbitrariness o the functional dependence of the isotopic element $\hat{T}$ in the isoproduct

$$[A \hat{,} B] = A\, \hat{T}(t, x, \dot{x}, \ddot{x}, \ldots)\, B - B\, \hat{T}(t, x, \dot{x}, \ddot{x}, \ldots)\, A\,. \tag{5.1.1}$$

The geometry of the first studies [5] was the *conventional* symplectic geometry, although realized in its most general possible exact version, with noncanonical, symplectic, exact two–form on a 2n–dimensional manifold M(x,R) over the reals R(n,+,×) (see App. 5.A for an outline and monograph [6] for a more detailed treatment)

$$\Theta = d\,\Phi = d\,[\, R_i(x)\, dx^i\,] = \Omega_{ij}(x)\ dx^i \wedge dx^j\,. \tag{5.1.2}$$

The covariant symplectic tensor

$$\Omega_{ij} = \partial_i R_j - \partial_j R_i\,,\ \ \partial_i = \partial / \partial x^i\,,\ i, j = 1, 2, \ldots, 2n\,. \tag{5.1.3}$$

and corresponding contravariant form

$$\Omega^{ij} = (\,|\,\Omega_{pq}\,|^{-1}\,)^{ij}\,, \tag{5.1.4}$$

are manifestly noncanonical and therefore result to be a direct realization of the *Lie–Santilli Isotopic Theorems* (Sect. I.4.5). In fact, the generalized brackets

$$[A \hat{,} B] = \frac{\partial A}{\partial x^i}\ \Omega^{ij}(x)\ \frac{\partial B}{\partial x^j}\,, \tag{5.1.5}$$

are isotopic, as ensured by the *Poincaré lemma* (see Sect.I.5.2 and App. I.5.A)[31]

$$d\,\Theta = d\,(\,d\,\Phi\,) \equiv 0. \tag{5.1.6}$$

This permitted a step–by–step generalization of classical Hamiltonian mechanics into a new discipline submitted in ref. [5] under the name of *Birkhoffian mechanics,* and subsequently elaborated in monograph [6].

However, brackets (5.1.5) are strictly local–differential, thus preventing a

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[31] The Lie–isotopic algebras were originally formulated precisely on these grounds, that is, by showing that the transition from Lie's theorems to their isotopic coverings implies the transition from the Poisson to generalized Lie brackets.

treatment of nonlocal-integral systems. In fact, the theorem of "direct universality" of Birkhoffian mechanics and of the related conventional symplectic geometry (ref. [6], p. 54 and ff., and Theorem 5.A.1 of App. I.5.A) was specifically formulated for all possible nonlinear and nonhamiltonian systems, under the conditions that they are local-differential and verify the needed regularity and smoothness conditions.

Comparison of brackets (5.1.1) and (5.1.5) clearly reveals important structural differences. As well known [4], the symplectic tensor Ω_{ij} , and, consequently, the Lie-isotopic tensor Ω^{ij}, can only have a dependence on the local coordinates, $\Omega^{ij}(x)$, while the isotopic element T can have an arbitrary functional dependence, $\hat{T}(t, x, \dot{x}, \ddot{x}, ...)$.

The above disparity between algebras and geometries persisted for a decade. Its solution required the author to conduct, again, a step-by-step generalization, this time, of Birkhoffian mechanics into the so-called *Birkhoff-isotopic (or isobirkhoffian) mechanics* and of its underlying geometry into the so-called *symplectic-isotopic (isosymplectic) geometry,* as a necessary condition to achieve a complete equivalence between isoalgebras, isogeometries and isomechanics.

As we shall see, the solution was provided by the full implementation of the same methods that had originated the Lie-isotopic theory: the systematic isotopic lifting of the entire structure of the symplectic geometry, including fields, vector spaces, exterior calculus, and the like.

Note that the isosymplectic geometry is the *only* geometric counterpart of the Lie-Santilli isotheory.

5.1.D: Isotopies of the Riemannian geometry. The need for a broadening of the Riemannian geometry for a geodesic characterization of free fall within physical media has been indicated at the beginning of this chapter.

As we shall see , the isotopies of the Riemannian geometry, first proposed by this author in memoir [16] of 1988 under the name of *isoriemannian geometry,* do indeed permit the achievement of the desired objective. Unexpectedly, the geodesics within physical media represented via isoriemannian spaces, called *isogeodesics,* resulted to be *identical* to those in the absence of physical media, such as a straight line for a free fall. The actual trajectory of an object (say, a leave), in free fall in atmosphere emerged only in the *projection* of the isogeodesic in a *conventional* Riemannian space.

The achievement of the above covering notion of isogeodesic is evidently at the foundation of the isotopic relativities studied in Vol. II.

In addition, the isoriemannian geometry resulted to be essential for a number of other aspects of interior gravitational problems.

A central characteristic in the transition from flat to curved geometries is that *the metric acquires a nonlinear dependence on the local coordinates (only), while preserving the local-differential and (first-order) Lagrangian characters of the flat geometries.* In fact, when compared to the constant Minkowskian metric

η, the Riemannian metric g(x) is a 4×4 matrix whose elements have a nonlinear dependence on the space–time coordinates x, although the original local-differential and Lagrangian characters of the Minkowskian geometry are preserved.

The above characteristics have proved to be exactly valid for exterior gravitational problems in vacuum, but they are insufficient for interior gravitational problems.

For instance, the interior of gravitational collapse, including black holes, big bang and all that, is not an aggregate of a large but finite number of ideal isolated points, but is instead composed of extended and hyperdense hadrons in condition of total mutual penetration as well as of compression in large numbers into a small region of space. These conditions imply the clear emerge of the most general known systems which are

> nonlinear not only in the coordinates x, but also in the velocities $\dot{x}$ and, expectedly, in the accelerations $\ddot{x}$, as well as in the wavefunctions ψ and their derivatives $\partial\psi$, $\partial\partial\psi$.

> nonlocal–integral in all needed quantities; and

> nonlagrangian, in the sense of not being solely representable via a first-order Lagrangian in the coordinates of the experimenter.

In short, the very use of the *Riemannian geometry itself* for the study of black holes, big bang and gravitational collapse implies the suppression of internal effects which are nonlinear and nonlocal in the velocities, wavefunctions and their derivatives. The insufficient character of the .geometry then implies predictable insufficiencies in the physical results.

Another geometric problem addressed in this chapter is the identification of a generalization of the Riemannian geometry which, on one side, allows metrics with unrestricted functional dependence $\hat{g} = \hat{g}(x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, ...)$ while, on the other side, preserves the original Riemannian and (3+1)–dimensional characters.

The first condition is evidently needed for a more realistic representation of interior gravitational problems, while the latter condition is needed to achieve a geometry unity in the treatment of both the exterior and interior problems as different *realizations* of the same abstract geometric structure.

As we shall see, the isotopies also permit the achievement of the latter objective. As a matter of fact, some of the most intriguing and far reaching implications of the isotopies occur precisely in gravitation.

As an illustration, readers familiar with conventional treatments may wonder how can a *classical* metric depend also on *quantum mechanical* quantities such as the wavefunctions.

This occurrence is extraneous to the conventional formulation of the Riemannian geometry, but it is possible under isotopies. In fact, the isoriemannian metrics can always be subjected to the isotopic factorization of the Minkowski metric η

$$\hat{g}(x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, ...) = \hat{T}(x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, ...)\, \eta\,, \tag{5.1.7}$$

The *gravitational isounit*

$$\hat{1} = \hat{1}(x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, ...) = \hat{T}^{-1}, \tag{5.1.8}$$

can then be assumed as the *basic isounit of operator theories* resulting in a *novel operator form of gravitation for both the exterior and interior problem without any need of the Hamiltonian.*

The above *isotopic quantization of gravity* studied in detail in Vol. II resolves the historical difficulty in the quantization of Einstein's gravitation given by the fact that, on one side, quantum formulations can only occur with a well defined Hamiltonian, while Einstein's gravitation (in vacuum) notoriously possess an identically null Hamiltonian.

Numerous other novel advances in gravitation are also permitted by the isotopies, as studied in Vol.s II and III, such as the "identification" (rather than the "unification") of gravitation with the electromagnetic field originating the structure of the elementary constituents of a given body, a theory on the "origin" (rather than on the "description") of gravitation, and others.

5.1.E: Isodual geometries and isogeometries for antimatter. Yet another objective of our studies discussed earlier in this volume, is the achievement of a representation of antimatter which is completely equivalent to the current representation of matter, thus initiating at the purely *classical* level and then persisting at the operator level.

This requires the identification of *a novel antiautomorphic map which is applicable to both classical; and quantum formulations.* The conventional *charge conjugation* is basically insufficient for this task because it is solely applicable to operator formulations and becomes effective only at the level of *second* quantization.

The antiautomorphic map selected by this author for the achievement of the above objectives is given by *isoduality* [13,14]

$$\hat{1} \rightarrow \hat{1}^d = -\hat{1}\,, \tag{5.1.9}$$

with corresponding conjugation of the isoreal numbers into their isoduals (Sect. I.2.2) and of all remaining aspects. By recalling that the trivial unit I is a particular case of the isounit, the isoduality therefore applies to all conventional and isotopic geometries, whether flat or curved.

The point here important is that the isodualities imply the emergence of new geometries, submitted by this author in memoir [15] of 1988 under the names of *isodual geometries* and *isodual isogeometries,* for the characterization of

antimatter in a way totally parallel to that of matter.

The physical characteristics of our current description of *matter* are defined in terms of fields or isofields with a *positive-definite norm*, and therefore imply a *positive energy with a motion forward in time*,

$$E > 0, \ t > 0, \qquad E, \ t \in R(n,+,\times) . \tag{5.1.10}$$

The isodual physical characterization of antimatter is defined instead on isodual fields and isofields which have a *negative-definite norm* (Ch. I.2), thus implying *negative energies with motion backward in time*,

$$E^d < 0, t^d < 0, \ | E^d |^d < 0, | t^d |^d < 0, \ \ E^d, \ t^d \in R^d(n^d,+,\times^d) . \tag{5.1.11}$$

The above occurrences have permitted the identification of an new universe, called *isodual universe*, which coexists and is interconnected with our own universe because of the finite transition probability between the positive- and negative-energy solutions of conventional field equations.

It should be recalled that the concepts of negative time and negative energies for antiparticles are rather old, and actually date back to the early stages of the discovery of antiparticles (Stueckelberg and others). What is new is their referral; to *negative-definite units*, with consequential systematic treatment via a body of formulations specifically conceived for that purpose, such as isodual numbers, isodual algebras, isodual geometries, etc.

As we shall see in Vol. II, the referral of negative energies and times to corresponding negative-definite units permit the resolution of the historical difficulties of the negative-energy solutions of Dirac's equation.

In the final analysis, *positive* energies and times referred to *positive* units are fully equivalent to *negative* energies and times referred to *negative* units.

The above isodual characterization of antimatter permits intriguing predictions, such as the existence of antigravity for an *elementary* antiparticle in the field of Earth, and other novel features studied in Volumes II and III which are becoming known as the "new physics of antimatter".

5.2: ISOEUCLIDEAN GEOMETRY AND ITS ISODUAL

5.2.A: Introduction. In this section we shall study: 1) the isotopies of the conventional Euclidean geometry of Class I and their isoduals; 2) the new geometric features which occur when projecting the isotopic realization of the Euclidean geometry in the conventional Euclidean space; and 3) the novel geometric advances permitted by the isotopies of Classes II–V.

The geometric isotopies here studied were introduced by this author [12] under the name of *isoeuclidean geometry* as a particular case of the *isominkowskian geometry* of the next section. Subsequent studies have indicated that they characterize a *new geometry* because they preserve the original axioms of the *flat* Euclidean geometry, but also embody at the same time *curvature* and other features belonging to *different* geometries.

The above main results can be anticipated from these introductory words. In fact, the isotopies preserve by assumption the original geometric axioms, and therefore permit the preservation of the conventional features of the Euclidean geometry, such as the definition of angles, the notion of straight, perpendicular and parallel lines, etc. At the same time, the isometric possesses the most general possible functional dependence, $\hat{\delta} = \hat{\delta}(t, r, \dot{r}, \ddot{r})$, thus including as particular cases the Riemannian, Finslerian, Labacevskiian or any other possible *noneuclidean* geometry in the same dimension.

The ultimate meaning of the n-dimensional isoeuclidean geometry of Class I which will emerge from our studies is that of unifying all possible geometries with the same dimension and signature. When the restriction of Class I is removed, the isoeuclidean geometry unifies all possible geometries of the same dimension irrespective of their signature.[32]

The reader should finally recall that, while conventional geometries have a unique formulation, isogeometries have a *dual* formulation, the first in isospace over isofields and the second via the projection in the original space over conventional fields.

5.1.B: Basic properties of isoeuclidean geometry. Let us begin by studying first the axiom-preserving content of the isoeuclidean geometry, with particular attention to the image under isotopies of flatness, while curvature and other noneuclidean aspects will be considered later on.

By conception and construction, the reader should expect no deviation from the abstract axioms of the Euclidean geometry *under the conditions that the isounit is positive-definite and the isogeometry is computed in isospace over isofields.*

However, when the isoeuclidean geometry is projected in the conventional Euclidean space, new geometric features are expected to occur and the same is the case when the basic unit can be negative-definite (Class II), or of indefinite signature (Class III), or singular (Class IV), or arbitrary, e.g., a step function (Class V).

As we shall see, the results are the same irrespective of whether one considers the abstract approach by Euclide and Hilbert or the coordinate approach by Descartes.

[32] This result does not surprise the attentive reader because it is expected as the geometric counterpart of the unification of all simple, compact and noncompact Lie algebras of the same dimension with the Lie-Santilli algebra studied in Ch. I.4.

Consider the conventional three-dimensional *Euclidean vector space* $V(r,\odot,R(n,+,\times)$ with elements r (vectors), their composition $r\odot r'$ (scalar product) over the field R of real numbers n equipped with the conventional addition + and multiplication × and respective additive unit 0 and multiplicative unit I.

Our first objective is to reconstruct $V(r,\odot,R)$ under isotopies, that is, when defined over an isofield $\hat{R}(\hat{n},+,\hat{\times})$ of isonumbers $\hat{n} = n\times\hat{1}$, equipped with the isosum $\hat{n} + \hat{n}' = (n+n')\times\hat{1}$ with isomultiplication $\hat{n}\hat{\times}\hat{n}' = \hat{n}\times\hat{T}\times\hat{n}' = (n\times n')\times\hat{1}$, equipped with the conventional additive unit $\hat{0} \equiv 0$ and a multiplicative isounit $\hat{1} = \hat{T}^{-1}$ which is a positive-definite quantity *outside the original field* R (e.g., $\hat{1}$ is an integral).

Let us begin with the study of the isotopies of the line.

Definition 5.2.1: *An "isoline" is the image of the ordinary line on the reals under the lifting* $R(n+,\times) \rightarrow \hat{R}(\hat{n},+,\hat{\times})$.

Coordinates on the isoline can be introduced as in the ordinary case, although they are now isonumbers, that is, ordinary numbers multiplied by the isounits,

$$\hat{x} = x \times \hat{1}\,, \tag{5.2.1}$$

on the isofield $\hat{R}$, and are thus called *isocoordinates.* One can first set up the *isoorigin* $\hat{0} = 0 \times \hat{1}$. Then the *isopoint* on the isoline are arbitrary, positive or negative isonumbers $\hat{x}$. The isodistance among two isopoints is given by the isonorm on $\hat{R}$ (Ch. I.2)

$$\hat{D} = \hat{\uparrow}(\hat{x} - \hat{x}')\hat{\uparrow} = [\,(x - x')\times\hat{T}\times(x - x')\,]^{1/2}\times\hat{1}, \tag{5.2.2}$$

and, as such, it is an isonumber.

One of the important implications of the isotopies of the straight line is that, even though the axioms are the same, the values of the distance among two points is different for lines and isolines with the same points x and x'. In fact, $\hat{D}/D \neq I$ and $\neq \hat{1}$.

This seemingly innocuous occurrence has a number of intriguing mathematical implications and physical applications, such as it permits the *mathematical* conception of a new propulsion called *geometric propulsion,* here introduced apparently for the first time which, as we shall see better later on, is essentially based on *the motion of a point from one isocoordinates to another via the alteration of the underlying geometry, rather than the actual displacement of the point itself.*

The study of the remaining properties of an isostraight line is left to the interested reader.

Ve now introduce the isotopies of the three-dimensional Euclidean vector space $V(r,+,\odot,R(n,+,\times)$.[33] We introduce in V a system of *Cartesian coordinates,*

name a system in which *all axes have the same (dimensionless) unit* +1 and are perpendicular to each other.[34] In this way, the Euclidean vector admit the familiar components along the three axes r = {x, y, z}.

We shall continue to use our main notation whereby quantities with the "hat" are computed in isospace and quantities without are computed in their projection in the original space. The symbol $+ \equiv \hat{+}$ will be used without a "hat" under isotopies to recall the fundamental assumption of Ch. I.2 that the lifting of the sum implies the divergence of the exponentiation and other undesirable features.

Definition 5.2.2: *The isotopies of Class I of the three-dimensional Euclidean vector space* $V(r,+,\odot,R(n,+,\times))$, $r = \{x, y, z\}$, *called the three-dimensional "isoeuclidean isovector space", are given by the same original set of contravariant vectors reformulated as "isovectors"* $\hat{r} = r\times\hat{1} = \{\hat{x}, \hat{y}, \hat{z}\} = \{x\times\hat{1}, y\times\hat{1}, z\times\hat{1}\}$ *on isospaces* $\hat{V}(\hat{r},+,\hat{\odot},\hat{R}(\hat{n},+,\hat{\times}))$ *over the isofield* $\hat{R}(\hat{n},+,\hat{\times})$ *with isounit* $\hat{1} > 0$ of Class I *(Sect. I.2.2) equipped with the original sum + and an isoproduct* $\hat{\odot} = \odot\hat{T}\odot$, $\hat{1} = \hat{T}^{-1} = \hat{1}^{\dagger} > \hat{0}$, *verifying the following properties for all possible* $\hat{r}$, $\hat{r}' \in \hat{V}$ and $\hat{n}$, $\hat{n}'$, $\hat{n}'' \in \hat{R}$:

1) $\hat{r} + \hat{r}' = \hat{r}' + \hat{r}$,
2) $(\hat{r} + \hat{r}') + \hat{r}'' = \hat{r} + (\hat{r}' + \hat{r}'')$,
3) *the set* $\hat{V}$ *includes the element* $\hat{0}$ *such that* $\hat{r} + \hat{0} \equiv \hat{r}$,
4) *for every element* $\hat{r}$ *there is an element* $-\hat{r}$ *such that* $\hat{r} + (-\hat{r}) = \hat{0}$,
5) $(\hat{n} + \hat{n}')\hat{\times}\hat{r} = \hat{n}\hat{\times}\hat{r} + \hat{n}'\hat{\times}\hat{r}$,
6) $\hat{n}\hat{\times}(\hat{r} + \hat{r}') = \hat{n}\hat{\times}\hat{r} + \hat{n}\hat{\times}\hat{r}'$,
7) $\hat{n}\hat{\times}\hat{n}'\hat{\times}\hat{r} = (\hat{n}\hat{\times}\hat{n}')\hat{\times}\hat{r} = \hat{n}\hat{\times}(\hat{n}'\hat{\times}\hat{r})$,
8) $\hat{1}\hat{\times}\hat{r} = \hat{r}\hat{\times}\hat{1} \equiv \hat{r}$,
9) *the isoproduct is an isonumber, i.e.,* $\hat{r}\hat{\odot}\hat{r}' = \hat{n} = n\times\hat{1} \in \hat{R}$,
10) $(\hat{n}\hat{\times}\hat{r})\hat{\odot}\hat{r}' = \hat{n}\hat{\times}(\hat{r}\hat{\odot}\hat{r}')$
11) $\hat{r}\hat{\odot}(\hat{r}' + \hat{r}'') = \hat{r}\hat{\odot}\hat{r}' + \hat{r}\hat{\odot}\hat{r}''$,
12) $\hat{r}\hat{\odot}\hat{r} = \hat{0}$ iff $\hat{r} = 0$,
13) $\hat{r}\hat{\odot}\hat{r}' \leq \hat{r}\hat{\odot}\hat{r}'' + \hat{r}''\hat{\odot}\hat{r}'$,
14) $\hat{r}\hat{\odot}\hat{r}' \neq \hat{r}\hat{\times}\hat{r}'$.

The "isoeuclidean metric space", or "isoeuclidean space" for short, is the isospace $\hat{E}(\hat{r},+,\hat{\odot},\hat{R}(n,+,\times))$ *equipped with the "isodistance"*

$$\hat{D} = (\hat{r}\hat{\odot}\hat{r}')^{\hat{\frac{1}{2}}} = (r\odot T\odot r')^{1/2}\times\hat{1} \in \hat{R}. \tag{5.2.3}$$

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[33] Note the different products in a vector space, the product $\times \in R$ for numbers and the product $\odot \in V$ for vectors. Such a difference evidently persists under isotopies.

[34] Note that *noncartesian coordinate systems* also exist in the literature in which different axes have different units, but they are all referred to the *same field* and related basic unit. By comparison, the *isocartesian coordinates systems* have different units for different axes whose tensorial product is assumed as the basic unit of the underlying field. As a result, *noncartesian and isocartesian coordinate systems are inequivalent.*

The "isoeuclidean geometry" of Class I is the geometry of the isoeuclidean spaces. Unless explicitly stated, the terms "isoeuclidean geometry' are specifically referred to the isogeometries of Classes I.

As one can see, Hamilton's original conception of "vectors" is merely reinterpreted as *isovectors,* that is, vectors belonging to a space in which the original scalar product is deformed by a given amount, $\odot \rightarrow \hat{\odot} = \odot \hat{T} \odot$, where $\hat{T}$ is fixed for all possible r, while jointly the basic unit is deformed by an amount which is the *inverse* of the deformation of the scalar product, $I \rightarrow \hat{1} = \hat{T}^{-1}$. As we shall see, this dual lifting permits the preservation of all original axioms. The basic quantity of the Euclidean geometry which remains invariant under lifting is therefore the quantity:

$$\text{Length} \times \text{Unit} \equiv \text{Isolength} \times \text{Isounit}\,. \tag{5.2.4}$$

The realization of the isoeuclidean spaces of Class I primarily studied until now (fall 1994) is that characterized by diagonal isotopic elements and isounits. The proof that Definition 5.2.2. permits the preservation of the Euclidean axioms therefore exists only for the above particular form which is assumed hereon.

We therefore study the three-dimensional *isoeuclidean geometry* on the isospace of the same dimension with diagonal Class I isotopic elements and isounits, which can be written

$$\hat{E}(\hat{r},\hat{\delta},\hat{R}): \ \hat{r} = \{\, \hat{r}^k \,\} \equiv \{\, r^k \times \hat{1} \,\}, \quad \hat{r}_k = \hat{\delta}_{ki}\, \hat{r}^i \times \hat{1} \neq r_k \times \hat{1}, \tag{5.2.5a}$$

$$\hat{\delta} = \hat{\delta}^\dagger = \hat{T} \times \delta = (\, \hat{T}_i{}^k \times \delta_{kj} \,) = (\, \hat{\delta}_{ij} \,), \quad \delta = \text{diag.}\,(1, 1, 1), \tag{5.2.5b}$$

$$\hat{T} = \hat{T}(t, r, \dot{r}, \ddot{r}, \ldots) = \text{diag.}\,(\, b_1{}^2, b_2{}^2, b_3{}^2 \,) = \hat{T}^\dagger > 0\,, \ b_k > 0\,, \tag{5.2.5c}$$

$$\hat{1} = \hat{T}^{-1} = \text{diag.}\,(\, b_1{}^{-2}, b_2{}^{-2}, b_3{}^{-2} \,), \ \hat{\delta}^{ij} = \hat{1}^i{}_k \times \delta^{kj}, \ \hat{\delta}_{ik} \times \hat{\delta}^{kj} = \delta_i{}^j, \tag{5.2.5d}$$

$$\hat{r}^{\hat{2}} = (\, r^i\, \hat{\delta}_{ij}\, r^j \,) \times \hat{1} =$$

$$(\, x\, b_1{}^2\, x + y\, b_2{}^2\, y + x\, b_3{}^2\, z \,) \times \hat{1} \in \hat{R}(\hat{n},\hat{+},\hat{*}), \ i, j, k = 1, 2, 3. \tag{5.2.5e}$$

The most important differences between the Euclidean and isoeuclidean spaces are the following. The Euclidean space has the single and unique basic unit I = diag. (1, 1) which is the unit of the SO(3) symmetry, and which essentially implies *the same dimensionless unit* +1 *for all axes,* $I_k = +1$, k = x, y, z. On the contrary, the isoeuclidean spaces have the infinite family of generally different isounits $\hat{1}$ = diag. $(b_1{}^{-2}, b_2{}^{-2})$ which are the isounits of the basic $S\hat{O}(3)$ symmetry (see Vol. II), and which implies *infinitely many possible, dimensionless units for each axes which are different among themselves and different than* +1, $\hat{1}_k = b_k{}^{-2}$, $\hat{1}_k \neq \hat{1}_y \neq \hat{1}_z \neq +1$.

The above differences have a number of intriguing mathematical implications studied below and physical applications studied in Vol. II. To begin, the Euclidean "space" is unique per each dimension, while there exist infinitely many possible isoeuclidean "spaces" per each dimension, although they all admit a single and unique abstract treatment for the same class.

Recall that the coordinates of an isopoints are isoscalars, i.e., $\hat{x} = x \times \hat{1}$, $\hat{y} = y \times \hat{1}$, $\hat{z} = z \times \hat{1}$. The isoseparation $\hat{r}^{\hat{2}}$ is therefore the results of *two* sequential isooperations, the first is given by the isosquare yielding the isoscalar structure,

$$\hat{r}^{\hat{2}} = \hat{r} \hat{\times} \hat{r} = r_k \times \hat{1} \times \hat{T} \times r^k \times \hat{1} = (r_k \times r^k) \times \hat{1}, \tag{5.2.6}$$

while the second operation is the isocontraction on the index k which must be done via the isometric (because we are now in isospace),

$$\hat{r}^{\hat{2}} = (r_k \times r^k) \times \hat{1} = (r^i \times \hat{\delta}_{ij} \times r^j) \times \hat{1}, \tag{5.2.7}$$

thus yielding expression (5.2.5e)

We mention that the isometric $\hat{\delta}$ could also be written as an 2×2 *isomatrix*, that is, *a 2×2 matrix whose elements are isoscalars*

$$\hat{\delta}_{ij} = \hat{T}_i^{\ k} \times \delta_{kj} \times \hat{1}. \tag{5.2.8}$$

In this case however the product of its elements among themselves and with any other quantity must be isotopic, thus reproducing again the fundamental isoinvariant (5.2.5e). For this reason, the isometric elements will be hereon considered to be ordinary scalars $\hat{\delta}_{ij}$ and their product with any quantity Q, $\hat{\delta}_{ij} \times Q$, an ordinary product.

Note that *the isoseparation coincides with the conventional separation for all possible scalar forms of the isounit.* In fact, in this case we have

$$\hat{r}^{\hat{2}} = (r_k \times r^k) \times \hat{1} = (r^i \times \hat{T} \times \delta_{ij} \times r^j) \times \hat{1} = r^2 \times \hat{T} \times \hat{T}^{-1} \equiv r^2. \tag{5.2.9}$$

This simple property illustrates the "hidden" character of geometric isotopies, as well as provides a reason why they have remained undetected until recently. The above "hidden" character will persist in Vol. II when studying hadronic mechanics.

Note also that the possible assumption of the basic invariant Length/Unit, rather than invariant (5.2.4), would imply a geometry different than the isogeometry because characterized by the dual lifting $\delta \to \hat{\delta} = \hat{1} \times \delta$ and $I \to \hat{1}$. In this case the liftings are no ,longer "hidden" because property (5.2.9) no longer holds.

The *isodistance* between two points $\hat{P}_1(\hat{x}_1, \hat{y}_1, \hat{z}_1)$ and $\hat{P}_2(\hat{x}_2, \hat{y}_2, \hat{z}_2)$ of the isoeuclidean geometry is the isoscalar

$$\hat{D}_{12} = \hat{|}\,(\hat{r}_1 - \hat{r}_2)\,\hat{|} =$$

$$= [\,(x_1 - x_2)^2\, b_1{}^2 + (y_1 - y_2)^2\, b_2{}^2 + (z_1 - z_2)\, b_3{}^2]^{1/2} \times \hat{1} \in \hat{R}\,, \qquad (5.2.9)$$

where $\hat{r}_1$ and $\hat{r}_2$ are the isovectors from the origin to $\hat{P}_1$ and $\hat{P}_2$, respectively.

A primary implication of the notion of isodistance is that of *altering* the conventional Euclidean distance among two points according to the following:

Proposition 5.2.1: *Let* d_{12} *be the conventional Euclidean distance between two points* $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$, *and let* $\hat{D}_{12} = D_{12} \times \hat{1}$ *be the corresponding isoeuclidean distance among two isopoints* $\hat{P}_1(\hat{x}_1, \hat{y}_1, \hat{z}_1)$ *and* $\hat{P}_2(\hat{x}_2, \hat{y}_2, \hat{z}_2)$, $\hat{x}_k = x_k \times \hat{1}$, $\hat{y}_k = y_k \times \hat{1}$, $\hat{z}_k = z_k \times \hat{1}$ *with the same coordinates* x_k, y_k, z_k, $k = 1, 2$, *of the original points. Then*

$$D_{12} > d_{12} \quad \text{for} \quad \det \hat{1} < 1\,, \qquad (5.2.10a)$$

$$D_{12} < d_{12} \quad \text{for} \quad \det \hat{1} > 1\,. \qquad (5.2.10b)$$

The above property has a number of intriguing implications. First, the *same object has different sizes and shapes in the Euclidean and isoeuclidean geometries*, as illustrated in Fig. 5.2.1.

THE ISOBOX

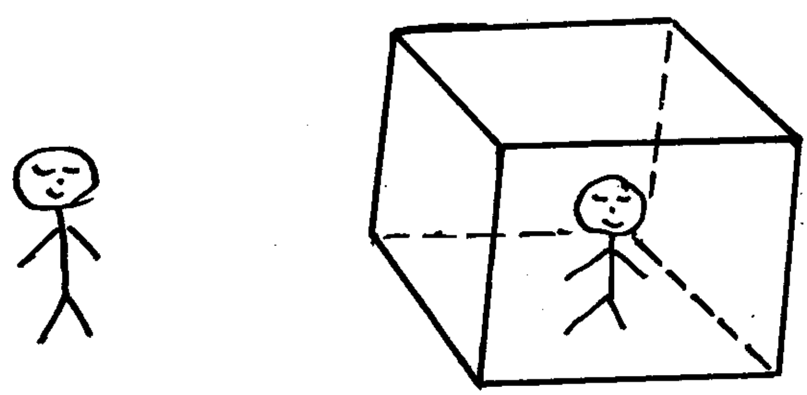


FIGURE 5.2.1. Consider a cube with sides of a given length d which is inspected from an outside observer in Euclidean space with geometric unit I = diag. (1, 1, 1), and individual units per each axis $I_k = +1$, $k = x, y, z$. Suppose that there is a

second observer in the interior of the cube belonging to an isoeuclidean space with isounit $\hat{1} = \mathrm{diag.}\,(b_1^{-2}, b_2^{-2}, b_3^{-2})$, and individual isounits per each axis $\hat{1}_k = b_k^{-2}$, k = z, y, z. It is easy to see that the *same* object can have for the interior: 1) a volume arbitrarily *smaller* or *bigger* than d^3 depending on whether det $\hat{1} > 1$ or < 1, respectively (Proposition 5.2.1) ; 2) a shape *different* than a cube; as well as 3) a shape and volume varying in time. In fact, the *same* k–side has *different* values depending on whether referred to the unit $I_k = +1$ or $\hat{1}_k = b_k^{-2}$, resulting in a volume for the interior observer which is arbitrarily smaller or bigger then that of the external observer. Also, equal sides for the outside observer are generally different among themselves when referred to *different units for different axes*, $\hat{1}_x \neq \hat{1}_y \neq \hat{1}_z \neq +1$, resulting in different shapes. Finally, the length of the sides is constant in time for the outside observer, while it may vary in time for the interior observer because the individual isounits may depend on time, $\hat{1}_k = \hat{1}_k(t, \ldots)$.

As a result of the above peculiar characteristics, far away stars which have a large distance from Earth when represented in Euclidean space, can have a distance as small or as large as desired when represented in isoeuclidean space. This notion is illustrated with the following self–evident property.

Definition 5.2.3: *The "geometric propulsion" is the mathematical displacement from a point* $P_1(x_1, y_1, z_1)$ *to a point* $P_2(x_2, y_2, z_2)$, *here assumed to be on the same straight line from the origin* 0 *in Euclidean space realized via such an isotopy of the underlying Euclidean geometry for which the isodistance* $D_{01}\times\hat{1}$ *between* 0 *and* P_1 *is such that* D_{01} *is equal to the distance* d_{02} *between* 0 *and* P_2 *(see Fig. 5.2.2 for details), i.e.*

$$\hat{D}_{01} = D_{01} \times \hat{1} \;\equiv\; d_{02} \times \hat{1}. \qquad (5.2.11)$$

The equation of an *isostraight isoline* is given by one of the following forms

$$\hat{a}\,\hat{\times}\,\hat{x} + \hat{b}\,\hat{\times}\,y + \hat{c}\,\hat{\times}\,\hat{z} + \hat{d} = (a\,x + b\,y + c\,z + d)\times\hat{1} = 0, \qquad (5.2.12a)$$

$$\begin{cases} \hat{x} - \hat{x}_1 - \hat{p}\,\hat{\times}\,\hat{a}_1 = (x - x_1 - p\,a_1)\times\hat{1} = 0, & (5.2.12c) \\ \hat{y} - \hat{y}_1 - \hat{p}\,\hat{\times}\,\hat{a}_2 = (y - y_1 - p\,a_2)\times\hat{1} = 0, & \\ \hat{z} - \hat{z}_1 - \hat{p}\,\hat{\times}\,\hat{a}_3 = (z - z_1 - p\,a_3)\times\hat{1} = 0, & (5.2.12d) \end{cases}$$

where $\hat{a}, \hat{b}, \hat{c}, \hat{d} \in \hat{R}$, a, b, c, d $\in$ R, p is an (ordinary) real parameter, and at least one of the isonumbers $\hat{a}$, $\hat{b}$ and $\hat{c}$ is not null. The isoline is called isostraight because its derivatives are constant.

Notice the importance for the consistency of the isoeuclidean geometry that the isocoordinates are isoscalars, i.e., are elements $\hat{x} = x\times\hat{1}$, $\hat{y} = y\times\hat{1}$, $\hat{z} = z\times\hat{1}$ of the isofield $\hat{R}$. In fact, the use for isocoordinates of of conventional scalars x, y, z

would prohibit a consistent definition of isostraight isoline.

GEOMETRIC PROPULSION

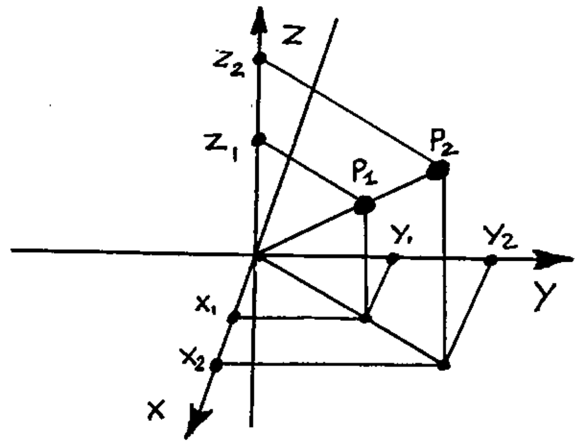


FIGURE 5.2.2: A schematic view of the *geometric propulsion* apparently introduced here for the first time. It is based on the idea of *realizing motion between two points via an isotopic alteration of the underlying geometry,* rather than the conventional displacement. Consider the Euclidean plane and two points P_1 and P_2 on a straight line from the origin 0 as in the figure. Let d_{01} and d_{02} be the conventional distances of the points P_1 and P_2 from the origin

$$d_{01} = (x_1^2 + y_2^2 + z_1^2)^{1/2}, \quad d_{02} = (x_2^2 + y_2^2 + z_2^2)^{1/2}, \quad d_{02} > D_{01}. \quad (1)$$

The geometric propulsion of the point P_1 to P_2 occurs under the following steps: 1) the geometry underlying the point P_1 is lifted isotopically with resulting isodistance (5.2.9); 2) the isotopy is chosen according to law (.5.2.11), i.e.,

$$\hat{D}_{01} = (x_1^2 b_1^2 + y_1^2 b_2^2) \times \hat{1} = d_{02} \times \hat{1}, \quad (2)$$

with simplest possible solution

$$b_1^2 = x_2^2 / x_1^2, \quad b_2^2 = y_2^2 / y_1^2, \quad b_3^2 = z_2^2 / z_1^2 \quad \text{Det.}\hat{1} > 1; \quad (3)$$

and 3) the geometry is then returned to the original Euclidean form. Under the above assumptions, the projection of the isopoint $\hat{P}_1(\hat{x}_1, \hat{y}_1, \hat{z}_1)$ in the Euclidean plane coincides with $P_2(x_2, y_2, z_2)$. The activation and subsequent de-activation of the isotopy then yield the motion from P_1 to P_2. It should be stressed that the above geometric propulsion is a *purely mathematical notion,* here defined for a *point.* The possibility of its future realizations is studied in Vol.s II and III and essentially deal with the identification of means which can alter the basic units of space (and time). The reader should keep in mind the most intriguing property of the geometric propulsion, that of being permitted by the *Euclidean* axioms themselves, only realized in a way more general than the usual one. Thus, an outside observer will simply see the motion from the point P_1 to P_2 *without*

any visible change of the geometry or visible means for the displacement. As we shall see in the next section, even more intriguing properties emerge when introducing the geometric propulsion in space-time.

In isovector notation, the isoline can be represented by

$$\hat{r}_k - \hat{r}_{1k} + \hat{n} \hat{\times} \hat{a} = (r_k - r_{1k} - n\, a_k) \times \hat{1}, \quad k = 1, 2, 3, \qquad (5.2.13)$$

where $\hat{r} = \{\hat{r}^k\} = \{\hat{x}, \hat{y}, \hat{z}\}$, $\hat{r}_1$ is fixed point on the isoline, $\hat{a} = \{\hat{a}_1, \hat{a}_2, \hat{a}_3\}$ is the direction of the isoline itself and n is an arbitrary parameter.

It is important to understand that while isoline (.5.2.13) is isostraight in isoeuclidean spaces, it is generally *curved* when projected in the conventional space. This property can be best inspected via an "old trick" of the isotopies, the reduction of the *isospace* $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ to a *conventional* space $\bar{E}(\bar{r},\delta,R)$ possessing the same invariant. This is readily possible for the values $\bar{r}_k = r_k b_k$ under which

$$r^i \times \hat{\delta}_{ij} \times r^j = \bar{r}^i \times \delta_{ij} \times \bar{r}^j . \qquad (5.2.14)$$

It then follows that the projection of the isostraight isoline (5.2.12a) in a flat space is given by

$$a\, x\, b_1(t, r, ...) + b\, y\, b_2(t, r, ...) + c\, z\, b_3(t, r, ...) + d = 0, \qquad (5.2.15)$$

which is evidently curved.

We therefore expect the existence of the inverse property, that is, given an arbitrary well behaved surface (5.2.15) in Euclidean space, there always exists an isotopy under which said surface is mapped into the isostraight isoline in isospace.

An *isopoint* in $\hat{V}(\hat{r},+,\hat{\delta},\hat{R})$ is a point $\hat{P}(\hat{x}, \hat{y}, \hat{z})$ with isocoordinates $\hat{x}, \hat{y}, \hat{z}$. Consider now two isovectors from the isoorigin $\hat{0}$ to the isopoints $\hat{P}_1$ and $\hat{P}_2$. An *isosegment* is the portion of an isostraight line between two isopoints.

In other conventional generalizations of the Euclidean metrics $\delta \to \hat{\delta}(t, r, \dot{r}, \ddot{r}, ...)$ the notion of angle is generally lost (as it is the case for the Riemannian geometry) because of the emergence of the curvature. A peculiarity of the isoeuclidean geometry is that, despite the most general possible functional dependence of the isometric, a generalized notion of angle can still be introduced. It is called the *isoangle,* denoted with the symbol $\hat{\alpha}$, and characterized by the expression on the isoplane $\hat{z} = \hat{0}$ for simplicity studied in detail in App. 5.C

$$\text{isocos}\, \hat{\alpha} = \frac{x_1 b_1^2 x_2 + y_1 b_2^2 y_2}{(x_1 b_1^2 x_1 + y_1 b_2^2 y_1)^{\frac{1}{2}} (x_2 b_1^2 x_2 + y_2 b_2^2 y_2)^{\frac{1}{2}}} . \qquad (5.2.16)$$

which, as one can see, is an ordinary scalar (rather than an isoscalar) because the

isounits cancels out in the ratio.

As shown in App. 5.C, and studied in more details in Ch. II.6 on the isorepresentation of Lie–Santilli isorotation group $\hat{O}(2)$, the explicit form of $\hat{\alpha}$ is given by $\hat{\alpha} = b_1 b_2 \alpha$, where α is the original angle prior to the isotopies.[35] This implies that the *angular isotopic element* and *angular isounits*, for the case of realization (5.2.5), are given respectively by $\hat{T}_{\hat{\alpha}} = b_1 b_2$, $\hat{1}_{\hat{\alpha}} = b_1^{-1} b_2^{-1}$.

The mechanism of isotopies of angles is therefore that a given angle α is lifted in the amount $\alpha \rightarrow \hat{\alpha} = \hat{T}_{\hat{\alpha}} \alpha$, but the unit is lifted by the inverse amount, $I \rightarrow \hat{1}_{\alpha} = \hat{T}^{-1}$, thus allowing the preservation of trigonometric axioms (App. 5.C).

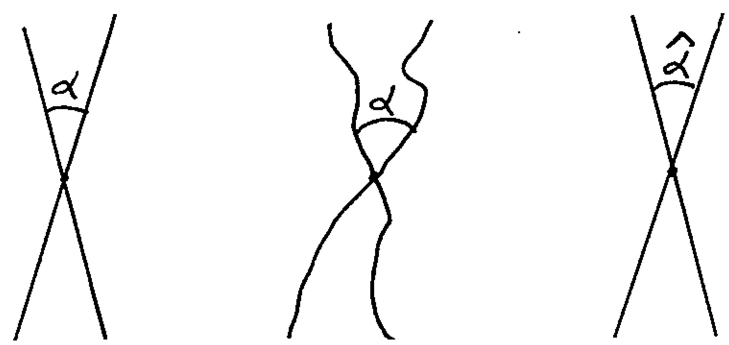
We shall say that two isovectors originating from $\hat{0}$ to the isopoints $\hat{P}_1$ and $\hat{P}_2$ in the isoplane $\hat{z} = \hat{0}$ are *isoperpendicular* when their intersecting isoangle is $\hat{\alpha} = 90°$, which can hold iff

$$x_1 \, b_1^2 x_2 + y_1 \, b_2^2 y_2 \equiv 0 , \tag{5.2.17}$$

and they are *isoparallel* when their intersecting isoangle is null, $\hat{\alpha} = \hat{}$, which can hold iff

$$x_1 \, b_1^2 y_2 - y_1 \, b_2^2 x_2 \equiv 0 . \tag{5.2.18}$$

RECONSTRUCTION OF ANGLES IN THE ISOPLANE



(A) (B) (C)

FIGURE 5.2.3. Diagram (A) depicts the origin of the notion of angle in the conventional Euclidean plane from two straight intersecting lines, which can be analytically expressed via the familiar expression

[35] Note that the definition of isoangle for *nondiagonal* isometrics is unknown at this writing.

$$\cos\alpha = \frac{x_1 x_2 + y_1 y_2}{(x_1 x_1 + y_1 y_2)^{\frac{1}{2}}(x_2 x_2 + y_2 y_2)^{\frac{1}{2}}} \quad . \tag{1}$$

In the transition to the Isoeuclidean plane, straight lines are generally mapped into curves when defined on the original plane over the conventional field R, as depicted in Diagram (B), thus implying the general loss of the notion of angle as typical in the transition from the Euclidean to the Riemannian geometry. The isotopies permit the reconstruction of the notion of angle, but only in the isoplane over isofields, because in the latter case the original straight lines are mapped into isostraight isolines as depicted in Diagram (C). In the latter case, the original angle α is lifted into the expression $\hat{\alpha} = b_1 b_2 \alpha$ called *isoangle* which is derived from the underlying Lie-Santilli isosymmetry of the isoplane $\hat{SO}(2)$ studied in detail in Vol. II. Expression (1) is lifted into expression (5.2.16) which evidently does not characterize $\cos\alpha$ any more, it is assumed as the definition of the $\mathrm{isocos}\hat{\alpha}$, and studied in App. 5.C.

The above two conditions establish the existence of simple yet, unique and unambiguous isotopic images of the Euclidean axioms of perpendicularity and parallelism. It is then easy to prove the following properties.

Theorem 5.2.1. *The isotopies map perpendicular lines into isoperpendicular isolines and parallel lines into isoparallel isolines.*

By using these results, it is possible to prove that the isoeuclidean geometry with diagonal Class I isounits is expressible via the following main assumptions (see, e.g., ref. [43], Ch. 2, for a recent study of the conventional Euclidean axioms).

Isoaxiom I: *There exists one and only one isostraight isoline from one isopoint to another isopoint.*

Isoaxiom II: *An isosegment can be prolonged continuously into an isostraight line from each end.*

Isoaxioms III: *For any given center and isoradius there is one and only one isosphere.*

Isoaxioms IV: *All isoright isoangles are equivalent.*

Isoaxioms V: *For each given isosegment between two isopoints there exist only two isoparallel lines, one per each isopoint, which are perpendicular to that isosegment.*

The lifting of the additional axioms of the Euclidean geometry [loc. cit.] is left to the interested reader. Additional axiomatic properties are studied in App.

5.C.

Isotopies characterized by *nondiagonal* isotopic elements are vastly unknown at this writing. We merely indicate that they imply a structural alteration of the original geometry more profound than that of diagonal isotopic elements. As an illustration, consider the isoeuclidean space as in Eq.s (5.2.5) but with isotopic element

$$\hat{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad (5.2.19)$$

for which det $\hat{T} = 1$. Thus, $\hat{T}$ is nondiagonal but still of Class I.

It is easy to see that in this case the basic isoinvariant (5.2.5d) becomes identically null, i.e.,

$$\hat{r}^{\hat{2}} = (r^i \times \hat{\delta}_{ij} \times r^j) \times \hat{1} = (x\,x + y\,z - z\,y) \times \hat{1} \equiv (x\,x) \times \hat{1}, \quad (5.2.20)$$

namely, the isotopy is regular, thus invertible, yet the isospace is degenerate and reduced from three to one dimension.

We learn in this way that *nondiagonal isotopies can reduce the number of effective dimensions of the original space.* This is the case of isoinvariant (5.2.19) for which the original dimension is three, with coordinates x, y, z, while the resulting dimension is one and represented by x, while the coordinates y and z remain outside the geometry.

One can see in this way additional peculiarities of the "isobox" of Fig. 5.2.1. In fact, the alteration of volume and shape of the cube and their variation in time for the internal observer should be complemented with the additional possibility that the number of dimensions themselves are changed in the interior. In fact, for isounit (5.2.19) the external observer perceives a three–dimensional cube, while the internal observer perceives it as a one–dimensional segment.

The invertible isotopies which alter the dimensionality of the original space are here called *degenerate Class I isotopies.* Rather than being a mere mathematical curiosity, the latter isotopies have emerge as having intriguing possibilities of applications studied in Vol.s II and III, such as a quantitative representation of the synthesis of neutrons from protons and electrons only as occurring in the interior of stars, which is possible under a nondiagonal Class I isotopy of the four–dimensional space–time of the electron down to two–dimensions.

Note that no study is available at this writing on the isoaxioms of the isoeuclidean geometry for nondiagonal isounits of Class I.

5.1.C: Basic properties of the isodual isoeuclidean geometry. The isoeuclidean geometry studied above is used in these volumes for the characterization of *matter.* In order to characterize *antimatter,* we need an

antiautomorphic map. This map cannot possibly be charge conjugation because it is applicable only to operator formulations on Hilbert spaces.

An antiautomorphic map which is indeed applicable to the Euclidean and isoeuclidean geometries was identified by this author in 1988 [13,14] under the name of *isoduality*, and it is given by

$$\hat{1} \;\rightarrow\; \hat{1}^d = -\hat{1}. \qquad (5.2.21)$$

Definition 5.2.4: *The "isodual isoeuclidean spaces" are given by the isoduals of the original isovectors* $\hat{r}^d = -\hat{r} = \{\hat{x}^d, \hat{y}^d, \hat{z}^d\} = \{-\hat{x}, -\hat{y}, -\hat{z}\}$, *called "isodual isovectors", defined on the isodual isospaces* $\hat{V}^d(\hat{r}^d,+,\hat{\odot}^d, \hat{R}^d(\hat{n}^d,+,\hat{\times}^d))$ *over the isodual isofield* $\hat{R}^d(\hat{n}^d,+,\times^d)$ *with isodual isounit* $\hat{1}^d = -\hat{1} < 0$ of Class II *(Sect. I.2.2) equipped with the original sum + and an isodual isoproduct* $\hat{\odot}^d = \odot^d\hat{T}^d\odot^d = \odot\hat{T}^d\odot = -\odot\hat{T}\odot$, $\hat{1}^d = (\hat{T}^d)^{-1} < \hat{0}$, *verifying the isodual images of properties 1)–14) of Definition 5.2.2 here omitted for brevity. The "isodual isoeuclidean geometry" is the geometry of the isodual isoeuclidean spaces of Class II. The "isoeuclidean geometries of Classes III, IV and V" are the geometries of isospaces with isounits of the corresponding classes. Unless explicitly stated, the terms "isodual isoeuclidean geometry" are referred to that of Class II.*

The construction of the isodual geometry via map (5.2.21) is straighforward. the *isodual isostraight isoline* is the infinite set $\hat{R}^d(\hat{n}^d,+,\hat{x}^d)$ with *isodual isopoints* given by the elements $\hat{n}^d = n\times\hat{1}^d = -\hat{n}$. The *isodual isoeuclidean isospace* can be written for the case of diagonal isodual isounits

$$\hat{E}^d(\hat{r}^d,\hat{\delta}^d,\hat{R}^d): \;\; \hat{r}^d = \{\, \hat{r}^k \times \hat{1}^d \,\} \equiv \{-\hat{r}^k\}, \quad \hat{r}_k{}^d = \delta^d{}_{ki} \times^d \hat{r}^{di} = -\hat{r}_k, \quad (5.2.22a)$$

$$\hat{\delta}^d = \hat{T}^d \times \delta = -\hat{\delta}, \; \delta = \mathrm{diag.}(1, 1, 1), \; \delta^d = \delta^{d\dagger}, \;\; \hat{1}^d = \hat{T}^{d-1} = -\hat{1}, \quad (5.2.22b)$$

$$\hat{T}^d = \hat{T}^d(t, r, \dot{r}, \ddot{r}, \ldots) = \mathrm{diag.}(-b_1{}^2, -b_2{}^2, -b_3{}^2) = -\hat{T} < 0, \; b_k{}^d <, \quad (5.2.22c)$$

$$\hat{r}^{d\hat{2}d} = (\, r^{di}\, \delta^d{}_{ij}\, r^{dj}\,) \times \hat{1}^d = (-xb_1{}^2x - yb_2{}^2y - zb_3{}^2z) \times \hat{1}^d \in \hat{R}^d. \quad (5.2.22d)$$

Note that the above isospace admits as a particular case the novel *isodual Euclidean space* which occurs for $\hat{1}^d = I^d = -I$. Note also the two sequential steps for the characterization of the isodual isoinvariant,

$$\hat{r}^{\hat{2}d} = \hat{r}^d{}_k\, \hat{\times}^d\, \hat{r}^{dk} = r_k \times \hat{1}^d \times \hat{T}^d \times r_k \times \hat{1}^d = (\, r_k \times r^k\,) \times \hat{1}^d =$$

$$(\, r^i \times \hat{\delta}^d{}_{ij}\; r^j\,) \times \hat{1}^d = (-r^i\, \overline{\hat{\delta}}_{ij} \times r^j\,) \times \hat{1}^d, \qquad (5.2.23)$$

thus reproducing isodual isoinvariant (5.2.22d).

The *isodual isodistance* between two isodual isopoints $\hat{P}_1{}^d(\hat{x}_1{}^d, \hat{y}_1{}^d, \hat{z}_1{}^d)$ and $\hat{P}_2{}^d(\hat{x}_2{}^d, \hat{y}_2{}^d, \hat{z}_2{}^d)$ is the *negative-definite* isodual isoscalar

$$\hat{D}^d{}_{12} = \uparrow(\hat{r}_1 - \hat{r}_2)|^d =$$

$$= [\,(x_1 - x_2)^2\, b_1{}^2 + (y_1 - y_2)^2\, b_2{}^2 + (z_1 - z_2)^2\, b_3{}^2\,]^{1/2} \times \hat{1}^d = - \hat{D}_{12} \in \hat{R}^d\,. \quad (5.2.24)$$

Proposition 5.2.4: *The basic invariants of the Euclidean or isoeuclidean geometries are "isoselfdual", i.e.., invariant under isodualities, i.e.,*

$$r^2 \equiv r^{d2d}\,, \quad \text{and} \quad \hat{r}^{\hat{2}} \equiv \hat{r}^{d2d}\,. \qquad (5.2.25)$$

As we shall see, the above mathematically elementary property has rather important physical implications for the representation of antimatter.

The *isodual isostraight line* can be expressed by

$$\hat{a}^d \hat{\times}^d \hat{x}^d + \hat{b}^d \hat{\times}^d \hat{y}^d + \hat{c}^d = (a \times x + b + y + c) \times \hat{1}^d = 0\,. \quad (5.2.26)$$

where $\hat{a}^d$, $\hat{b}^d$, $\hat{c}^d \in \hat{R}^d$. A given straight line can therefore be also interpreted as belonging to an isospace *as well as* to its isodual. As we shall see, this additional elementary property is evidently extendable to curves and results to have significant application in theoretical biology. In fact it indicates that, even though an object is *perceived* as belonging to our Euclidean geometry, and it *appear* to evolve with our time it may eventually belong to a structurally more general geometry with an inverted direction of time [30].

The *isodual angle* is the angle between two intersecting isodual straight lines in the isodual Euclidean plane, and it is simply given by $\alpha^d = -\alpha$. The *isodual isoangle* is the angle between two intersecting isodual isostraight isolines, and can be written

$$\hat{\alpha}^d = b_1{}^d\, b_2{}^d\, \alpha^d = - \hat{\alpha}\,. \qquad (5.2.\ 27)$$

We leave for brevity to the interested reader the definition of *isoperpendicular and isoparallel isodual isostraight isolines*, and the isodualities of the remaining properties of the isoeuclidean geometry.

The characterization of antimatter via the isodual geometry is made possible by the property of Proposition 5.2.4 and those of the following:

Proposition 5.2.5: *The maps from Euclidean* $E(r,\delta,R)$ *to the isodual Euclidean space* $E^d(r^d,\delta^d,R^d)$ *and from the isoeuclidean* $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ *to the isodual isoeuclidean space* $\hat{E}^d(\hat{r}^d,\hat{\delta}^d,\hat{R}^d)$ *are antiautomorphic and imply the change of the sign of all positive-definite quantities.*

The above properties confirm the possibility for isoduality to provide a

classical representation of antimatter, as studied in detail in Vol. II. At this moment we limit ourselves to indicate that the map from matter to antimatter is indeed antiautomorphic (Proposition 5.2.5), yet it preserves conventional invariants (Proposition 5.2.4). this ensures that the same physical laws hold for both matter and antimatter although realized in isodual forms.

In particular, the charge of a particle changes sign under isoduality. Jointly, the energy which is positive in Euclidean and isoeuclidean geometries is turned into negative values for the corresponding isodual geometry, as requested for a consistent representation of antiparticles, while our the forward time of matter is reversed under isoduality.

The study of isodual isoeuclidean geometry with nondiagonal isotopic elements is left to the interested reader.

5.2.D: Operations on isovectors and their isoduals. We consider now the operations of isovectors in the isoeuclidean geometry with diagonal isounit, Eq.s (5.2.5). Recall from Ch. I.3 that the basis of a vector space is not changed under isotopy (up to possible renormalization factors). Let e_k, $k = 1, 2, 3$, be the unit vectors of a three–dimensional Euclidean space $E(r,\delta,R)$ directed along the x, y, z axes, and let

$$\hat{e}_k = e_k \times \hat{1}, \tag{5.2.28}$$

be the corresponding *isobasis* in $\hat{E}(\hat{r},\hat{\delta},\hat{R})$. Then, a *isovector* $\hat{V}$ can be expressed in isospace

$$\hat{V} = x\,\hat{e}_1 + y\,\hat{e}_2 + z\,\hat{e}_3 . \tag{5.2.29}$$

This is another way of expressing the fact that the isovector $\hat{V}$ is isostraight in $\hat{E}(\hat{r},\hat{\delta},\hat{R})$, although its projection in $E(r,\delta,R)$ is curved. As expected, *the operations on vectors are preserved under isotopies.* In fact, the familiar scalar product of two vectors $V_1 = \{x_1, y_1, z_1\}$ and $V_2 = \{x_2, y_2, z_2\}$

$$V_1 \odot V_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 , \tag{5.2.30}$$

is now lifted into the expression called *isoscalar product*

$$\hat{V}_1 \hat{\odot} \hat{V}_2 = (x_1 b_1^2 x_2 + y_1 b_2^2 y_2 + z_1 b_3^2 z_2) \times \hat{1} \in \hat{R}(\hat{n},+,*) . \tag{5.2.31}$$

Note that the isoscalar product preserves the original axioms, i.e.,

$$\hat{V}_1 \hat{\odot} \hat{V}_2 = \hat{V}_2 \hat{\odot} \hat{V}_1 , \quad \hat{V}_1 \hat{\odot} (\hat{V}_2 + \hat{V}_3) = \hat{V}_1 \hat{\odot} \hat{V}_2 + \hat{V}_1 \hat{\odot} \hat{V}_3 . \tag{5.2.32}$$

Moreover, the *isonorm* on $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ is expressible in terms of the isoscalar product

via the rule

$$\hat{\uparrow}\hat{V}\hat{\uparrow} = (\hat{V}\,\hat{\circ}\,\hat{V})^{\frac{1}{2}} \times \hat{1} \in \hat{R}(\hat{n},+,\hat{\times}). \qquad (5.2.33)$$

Thus, the isocosinus of the isoangle formed by two intersecting isovectors can be written as the isotopy of the conventional case

$$\text{isocos}\,\hat{\alpha} = \frac{\hat{V}_1\,\hat{\circ}\,\hat{V}_2}{\hat{\uparrow}\hat{V}_1\hat{\uparrow}\,\hat{\times}\,\hat{\uparrow}\hat{V}_2\hat{\uparrow}}. \qquad (5.2.34)$$

Also, one can introduce the *directional isocosinuses* of a vector

$$\text{isocos}\,\hat{\alpha} = \hat{V}_1 / \hat{\uparrow}\hat{V}\hat{\uparrow}\,,\ \text{isocos}\,\hat{\beta} = \hat{V}_2 / \hat{\uparrow}\hat{V}\hat{\uparrow}\,,\ \text{isocos}\,\hat{\gamma} = \hat{V}_3 / \hat{\uparrow}\hat{V}\hat{\uparrow}. \qquad (5.2.35)$$

Then, we have again the correct lifting of the corresponding conventional identity

$$b_1{}^2\,\text{isocos}^2\,\hat{\alpha} + b_2{}^2\,\text{isocos}^2\,\hat{\beta} + b_3{}^2\,\text{isocos}^2\,\hat{\gamma} = 1\,. \qquad (5.2.36)$$

Similarly the vectorial product $V_1 \wedge V_2$ is lifted in the expression called *isovectorial product*

$$\hat{V}_3 = \hat{V}_1\,\hat{\wedge}\,\hat{V}_2\,,\ \hat{V}_{3k} = \epsilon_{kij}(\,b_i\,x_{i1}\,)(\,b_j\,x_{2j}\,)\,,\ i, j, k = 1, 2, 3\,. \qquad (5.2.37)$$

which satisfies the basic axioms of a vector product

$$\hat{V}_1\,\hat{\wedge}\,\hat{V}_2 = \hat{V}_2\,\hat{\wedge}\,\hat{V}_1\,,\quad \hat{V}_1\,\hat{\wedge}\,(\,\hat{V}_2 + \hat{V}_3\,) = \hat{V}_1\,\hat{\wedge}\,\hat{V}_2 + \hat{V}_1\,\hat{\wedge}\,\hat{V}_3\,. \qquad (5.2.38)$$

Other operations on isovectors can be constructed accordingly.

The operations on isodual isovectors $\hat{V}^d = -\hat{V}$ on isodual spaces $\hat{E}^d(\hat{r}^d,\hat{\delta}^d,\hat{R}^d)$ with diagonal isodual isounits are easily derivable via the isodual map. As an example, the *isodual isoscalar product* is given by

$$\hat{V}_1{}^d\,\hat{\circ}^d\,\hat{V}_2 = (\,-x_1\,b_1{}^2\,x_2 - y_1\,b_2{}^2\,y_2 - z_1\,b_3{}^2 z_2\,) \times \hat{1}^d \in \hat{R}^d\,, \qquad (5.2.39)$$

and it is manifestly isoselfdual.

Similarly, the *isodual isovector product* is given by

$$\hat{V}_3{}^d = \hat{V}_1{}^d\,\hat{\wedge}^d\,\hat{V}_2\,,\ \hat{V}_{3k}{}^d = \epsilon_{kij}(\,b_i{}^d\,x_{i1}{}^d\,) \times^d (\,b_j{}^d\,x_{2j}{}^d\,) = -\hat{V}_{3k}. \qquad (5.2.40)$$

It is instructive for the interested reader to verify the preservation of *Lagrange's identity* under isotopies among four isovectors $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ in $\hat{E}(\hat{r},\hat{\delta},\hat{R})$

$$(\hat{A}\,\hat{\wedge}\,\hat{B})\,\hat{\circ}\,(\hat{C}\,\hat{\wedge}\,\hat{D}) = (\hat{A}\,\hat{\circ}\,C)\,\hat{\times}\,(B\,\hat{\circ}\,\hat{D}) - (\hat{B}\,\hat{\circ}\,C)\,\hat{\times}\,(\hat{A}\,\hat{\circ}\,\hat{D}). \qquad (5.2.41)$$

Other properties can be easily derived by the interested reader via similar procedures.

5.2.C: Representation of hadrons as isospheres and of anti-hadrons as isodual isospheres. We now pass to the study of one of the most fundamental geometric notions of hadronic mechanics, *the representation of hadrons as isospheres in isoeuclidean spaces* with corresponding antiautomorphic image for anti–hadrons. In this section we shall solely study geometrical aspects and defer all physical considerations and verifications to Vol.s II and III.

The *isosurfaces* on the three–dimensional isoeuclidean space (5.2.5) are given by a straighforward isotopic image of ordinary curves and, as such, are reducible to algebraic equations in the coordinates of order higher than the first.

While in ordinary Euclidean space we have a large number of different surfaces, in the isoeuclidean space we have the dominance of the following notion:

Definition 5.2.5 [15]: *The "isosphere" in the three–dimensional isoeuclidean space $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ with diagonal isounit is the isotopic image of the ordinary sphere with equation*

$$\hat{r}^{\hat{2}} = \hat{x}^{\hat{2}} + \hat{y}^{\hat{2}} + \hat{z}^{\hat{2}} = \hat{R}^{\hat{2}}, \qquad (5.2.42)$$

i.e.,

$$[\, x\, b_1^{\,2}(t, r, \dot{r}, \ddot{r}, ...)\, x \;+\; y\, b_2^{\,2}(t, r, \dot{r}, \ddot{r}, ...)\, y \;+\; z\, b_3^{\,2}(t, r, \dot{r}, \ddot{r}, ...)\, z\,] \times \hat{1} =$$

$$= \hat{R}^{2} = R^2 \times \hat{1} \in \hat{R}, \qquad (5.2.43)$$

where R *is an ordinary scalar. The isosphere is said to be of Class I, II, III, IV or V depending on the corresponding class of the isounit. The "isodual isosphere" is the isosphere in the isodual isospace $\hat{E}^d(\hat{r}^d,\hat{\delta}^d,\hat{R}^d)$ with equation*

$$\hat{r}^{d\hat{2}d} = \hat{x}^{d\hat{2}d} + \hat{y}^{d\hat{2}d} + \hat{z}^{d\hat{2}d} = \hat{R}^{d\hat{2}d}, \qquad (5.2.44)$$

i.e.,

$$[\, x^d\, b_1^{\,2d}(t, r, \dot{r}, \ddot{r}, ...)\, x^d \;+\; y^d\, b_2^{\,2d}(t, r, \dot{r}, \ddot{r}, ...)\, y^d \;+\; z^d\, b_3^{\,2d}(t, r, \dot{r}, \ddot{r}, ...)\, z^d\,] \times \hat{1} =$$

$$= \quad [-\, x\, b_1^{\,2}(t, r, \dot{r}, \ddot{r}, ...)\, x \;-\; y\, b_2^{\,2}(t, r, \dot{r}, \ddot{r}, ...)\, y \;-\; z\, b_3^{\,2}(t, r, \dot{r}, \ddot{r}, ...)\, z\,] \times \hat{1}^d =$$

$$= \hat{R}^{d2d} = R^{2d} \times \hat{1}^d \;\in\; \hat{R}^d, \qquad (5.2.45)$$

The ordinary sphere, hereon written in the form[36]

$$r^2 = (x^2 + y^2 + z^2) \times I = R^2 \times I \in R \qquad (5.2.46)$$

is a trivial particular case of the isosphere of Class I. The "isodual sphere" is the image of the sphere under duality with equation

$$r^{d2d} = - x^2 - y^2 - z^2) \times I^d = R^2 \times I^d, \qquad (5.2.47)$$

and it is also a particular case of the isodual isosphere of Class II.

The verification of the perfect spheridicity of the isosphere in isospace is important. Recall that, by central assumption, the Euclidean space and related Cartesian coordinates admit *the same unit for all three axes,* which is geometrically expressed by the unit I = diag. (1, 1, 1) of the basic SO(3) symmetry, and we shall write

$$I_x = I_y = I_z = + I . \qquad (5.2.48)$$

Recall that, also by central assumption, the isoeuclidean space and related isocartesian coordinates admit *different units for different axes* which can be expressed via the isounit $\hat{1}$ = diag. $(b_1^{-2}, b_2^{-2}, b_3^{-2})$ of the basic isosymmetry $\hat{SO}(3)$ (Ch. II.6), and we shall write

$$\hat{1}_x = b_1^{-2} \neq \hat{1}_y = b_2^{-2} \neq \hat{1}_z = b_3^{-2} \neq + I . \qquad (5.2.49)$$

Recall finally that the original geometric characteristics are preserved under the above an isotopy, e.g., a straight line is mapped into an isostraight isoline.

It is then easy to see that the perfect sphere in Euclidean space is mapped into a surface with perfect spheridicity in isospace. In fact, the semiaxes of the original sphere $S_k = +1$, k = x, y, z, are lifted under isotopy to the values $\hat{S}_k = b_k^2$, k = x, y, z, thus yielding an ellipsoid. Jointly, the unit of each deformed semiaxis is lifted by the *inverse* amount, thus restoring the perfect spheridicity in isospace, with the understanding that the *diameter of the original sphere is changed.*

The above occurrence is directly expressed by the basic invariant (5.2.4) of the isotopic lifting of the Euclidean space realized according to the rules

$$\delta = \text{diag.}(1, 1, 1) \rightarrow \hat{\delta} = \text{diag.}(b_1^2, b_2^2, b_3^2), \qquad (5.2.50)$$

$$I = \text{diag.}(1, 1, 1) \rightarrow \hat{1} = \text{diag.}(b_1^{-2}, b_2^{-2}, b_3^{-2}), \qquad (5.2.51)$$

---

[36] Note that formulation (5.2.46) requires the redefinition of the ordinary field of real numbers with respect to the unit I = diag. (1, 1, 1). Such a reformulation is necessary here to have the ordinary sphere admitted as a particular case of the isosphere.

THE ISOSPHERE IN ISOEUCLIDEAN SPACE

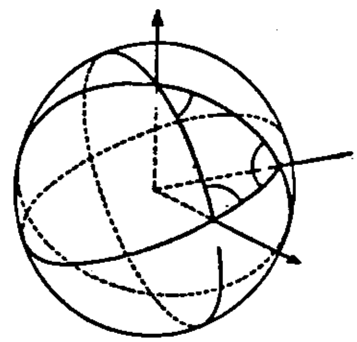


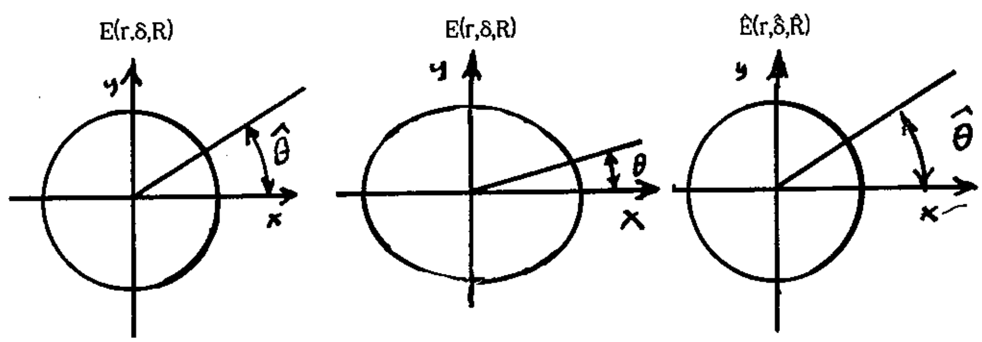
FIGURE 5.2.4: A schematic view of the perfect sphere in isospace over isofield introduced by this author [15] under the name of *isosphere*. Besides the argument presented in the text, the perfect spheridicity can be best proved via the use of the isospherical coordinates and isotrigonometric functions of App. 5.C. For the simpler case case of the *isocircle* of radius one in the isoplane $(\hat{x}, \hat{y})$ with diagonal isotopic element we have the *isopolar coordinates*

$$x = \text{isocos}\,\hat{\theta} = b_1^{-1}\cos\hat{\theta}, \quad y = \text{isosin}\,\hat{\theta} = b_2^{-1}\sin\hat{\theta}, \quad \hat{\theta} = b_1 b_2 \theta, \quad (1)$$

where θ is the original angle of the circle prior to the isotopy. In this case the equation for the isocircle is reduced to the conventional form

$$r^{\hat{2}} = x\, b_1^{2} x + y\, b_2^{2} y = b_1^{2}\,\text{isocos}^2\hat{\theta} + b_2^{2}\,\text{isosin}^2\hat{\theta} =$$
$$= \cos^2\hat{\theta} + \sin^2\hat{\theta} = 1, \qquad (2)$$

which can be schematically represented as follows



namely, the circle is first deformed into an ellipsoid in the original space and then reconstructed as a perfect circle in isospace. Note that the projection of the isocircle in

the original space can also be represented with the coordinates $\bar{r}_k = r_k b_k$, $k = 1, 2$, on the conventional space $\bar{E}(\bar{r},\delta,R)$ with the self-evident identities

$$\hat{r}^2 = x\, b_1{}^2 x + y\, b_2{}^2 y \equiv \bar{x}\,\bar{x} + \bar{y}\,\bar{y} = \bar{r}^2 . \qquad (3)$$

Similar results hold for the case of the isosphere via the use of the isospherical coordinates, as well as for the isodual isosphere, as well as for the isosphere of Class III, the latter one requiring the use of the *isohyperbolic functions* of App. 5.C. Applications of the isosphere of Class IV ≅with singular isounit∞ are indicated in Sect. 5.2.E.

or, equivalently, by the basic invariant under isotopies:

$$\delta \times I \;\rightarrow\; \hat{\delta} \times \hat{1} = \hat{T} \times \delta \times \hat{1} \equiv \delta \times I . \qquad (5.2.52)$$

We can now begin to understand the representation of a hadron as an isosphere in isospace. Recall that in contemporary particle physics hadrons are represented as *perfectly spherical and perfectly rigid objects*, evidently as necessary conditions not to violate a pillar of quantum mechanics, the rotational symmetry SO(3).

The representation of a hadron as an isosphere then includes the perfectly spherical and rigid cases as trivial subcases and permits the additional representation of all the infinitely possible signature-preserving deformations of the sphere in such a way to preserve the basic rotational symmetry, as studied in details in Vol. II. In different terms, the geometric representation of a hadron as an isosphere permits a single, unified, rotationally invariant characterization of all possible actual, nonspherical shapes of a hadrons and all their infinitely possible deformations due to collisions or external fields.

It is easy to prove the following:

Proposition 5.2.5: *The maps from the sphere to the isodual sphere*

$$r^2 = (x^2 + y^2 + z^2) \times I = R^2 \times I \;\rightarrow\; r^{d2d} = -x^2 - y^2 - z^2) \times I^d = R^2 \times I^d , \quad (5.2.53)$$

and from the isosphere to the isodual isospheres

$$\hat{r}^2 = (r^i \times \hat{\delta}_{ij} \times r^j) \times \hat{1} = R^2 \times \hat{1} \;\rightarrow\; \hat{r}^{d2d} = (r^{di} \times \hat{\delta}^d{}_{ij} \times r^{dj}) \times \hat{1}^d = R^2 \times \hat{1}^d , \quad (5.2.54)$$

are antiautomorphic, thus implying the reversal of the sign of all the original positive-definite characteristics.

The above property evidently permits the characterization of anti-hadrons as isodual isospheres. Note that in contemporary particle physics both hadrons and anti-hadrons are treated with the *same* geometry and are thus represented with the *same* sphere, resulting in the same classical characteristics which is basically insufficient for their distinction.

Proposition 5.2.6: *The sphere and the isosphere are isoselfdual*

$$r^2 = (x^2 + y^2 + z^2) \times I \equiv r^{d2d} = -x^2 - y^2 - z^2) \times I^d, \qquad (5.2.55)$$

$$\hat{r}^2 = (r^i \times \hat{\delta}_{ij} \times r^j) \times \hat{1} = R^2 \times \hat{1} \equiv \hat{r}^{d2d} = (r^{di} \times \hat{\delta}^d{}_{ij} \times r^{dj}) \times \hat{1}^d = R^2 \times \hat{1}^d, \qquad (5.2.56)$$

The above property is important for the observability of antihadrons in our space. In fact, it establishes that a given sphere cannot be claimed to belong to a particle or to an antiparticle without additional information, e.g., on charge, energy, etc. Additional properties of the isosphere will be studied in the next section and in Vol.s II, and III.

The following mathematical properties of the isosphere are self-evident.

Theorem 5.2.5 [15]: *The isosphere of Class III unifies all the following quadrics of the conventional Euclidean space*

1) All ordinary sphere

$$SO(3): \; x^1 x^1 + x^2 + x^2 + x^3 x^3 = R^2, \qquad (5.2.57)$$

2) All elliptic paraboloids *(paraboloids with one sheet)*

$$SO(2.1): \; x^1 x^1 - x^2 x^2 + x^3 x^3 = R^2, \qquad (5.2.58)$$

3) All prolate or oblate ellipsoids

$$\hat{SO}(3): \; x^1 b_1{}^2 x^1 + x^2 b_2{}^2 x^2 + x^3 b_3{}^2 x^3 = R^2, \qquad (5.2.59)$$

4) All isotopic deformations of the elliptic paraboloids

$$\hat{SO}(2.1): \; x^1 b_1{}^2 x^1 - x^2 b_2{}^2 x^2 + x^3 b_3{}^2 x^3 = R^2, \qquad (5.2.60)$$

5) All isodual sphere

$$SO^d(3): \; -x^1 x^1 - x^2 x^2 - x^3 x^3 = -R^2, \qquad (5.2.61)$$

6) All hyperbolic paraboloid

$$SO^d(2.1): \; -x^1 x^1 + x^2 x^2 - x^3 x^3 = -R^2, \qquad (5.2.62)$$

7) All isodual ellipsoids

$$\hat{SO}^d(3): \; -x^1 b_1{}^2 x^1 - x^2 b_2{}^2 x^2 - x^3 b_3{}^2 x^3 = -R^2, \qquad (5.2.63)$$

8) All isoduals deformations of the hyperbolic paraboloid

$$\hat{SO}^d(2.1): \quad -x^1 b_1{}^2 x^1 + x^2 b_2{}^2 x^2 - x^3 b_3{}^2 x^3 = -R^2, \tag{5.2.64}$$

The isosphere of Class IV unifies all the preceding surfaces plus

9) All possible cones in Euclidean space, *i.e.,*

$$SO(2.1): \quad x^1 x^1 - x^2 x^2 + x^3 x^3 = 0, \tag{5.2.65a}$$

$$\hat{SO}(2.1): \quad x^1 b_1{}^2 x^1 - x^2 b_2{}^2 x^2 + x^3 b_3{}^2 x^3 = 0, \tag{5.2.65b}$$

$$SO^d(2.1): \quad -x^1 x^1 + x^2 x^2 - x^3 x^3 = 0, \tag{5.2.65c}$$

$$\hat{SO}^d(2.1): \quad -x^1 b_1{}^2 x^1 + x^2 b_2{}^2 x^2 - x^3 b_3{}^2 x^3 = 0. \tag{5.2.65d}$$

The *isosphere of Class V* has not been investigated to date, and it is expected to permit the formulation of new notions of "spheres", such as spheres whose radius is a step function or a lattice.

It should be indicated that all *physical* applications known at this time are restricted to the isosphere of Class I, which unifies the sphere and all its ellipsoidical deformations and to the isodual isosphere of Class II, which unifies the isodual sphere and all its ellipsoidical deformations). This is due to the fact that there is no known physical event capable of altering, say, ellipsoids into hyperboloids, or viceversa.

Theorem 5.2.2 essentially states that *all quadrics (A)-(D) of Fig. 5.2.5 have the shape depicted only when expressed in the conventional Euclidean space, because when properly represented in isoeuclidean space they can all be reduced to perfect circles.*

This intriguing property should not be surprising for the reader now familiar with isotopic liftings. As it was the case for straight lines, *the isotopies of a sphere must remain a sphere as a necessary condition for the achievement of the isotopies themselves.* The unification of the sphere with all its infinitely possible ellipsoidical deformations then follows, with evidently broader unifications for higher classes.

One can now understand why distances which are very large in our perception of the universe in Euclidean space can become rather small in isospace. In fact, very large distances, say, in a hyperboloid are turned into relatively much shorter distances on the isosphere of Class III.

The reader should finally be aware that the unification of all quadrics into the isosphere is the geometric foundations of the unification of the compact SO(3) and non compact SO(2.1) symmetry into the isosymmetry $\hat{SO}(3)$ submitted by this author since the original proposal [5].

ISOTOPIC UNIFICATION OF QUADRICS

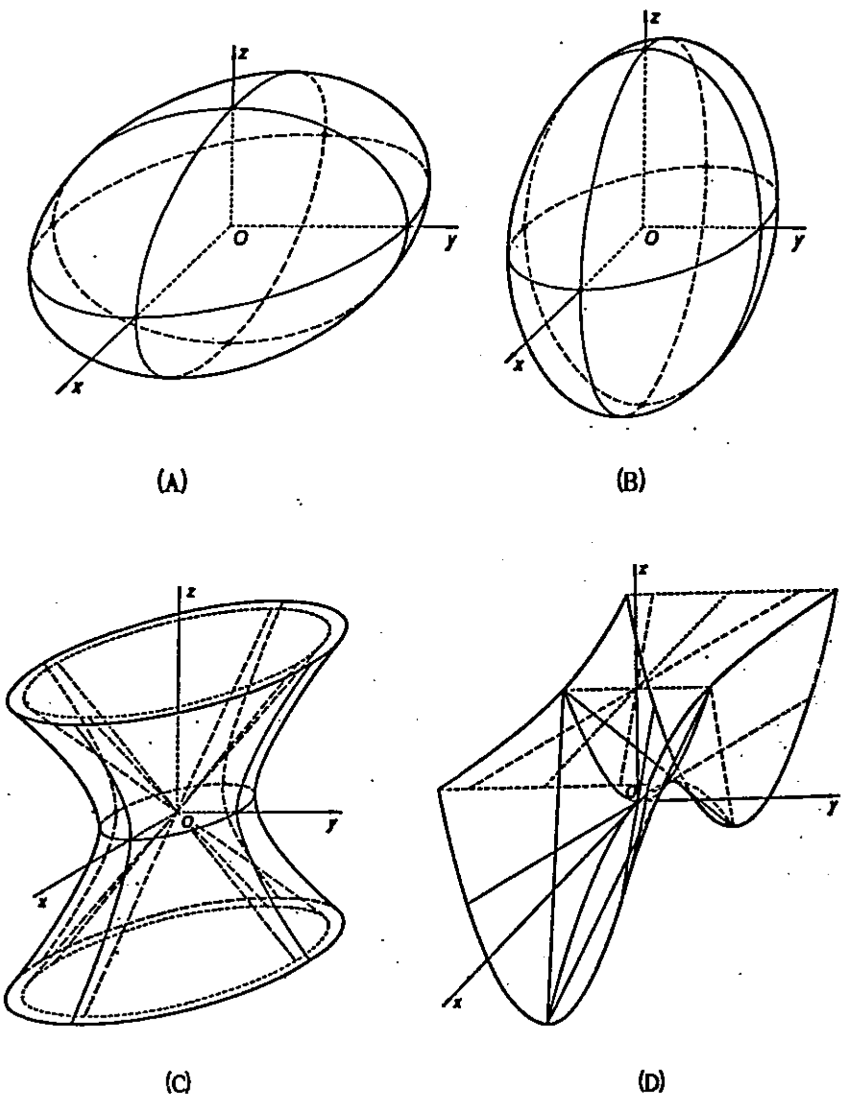


FIGURE 5.2.5: A schematic view of the unification into the isosphere of Class III of prolate ellipsoid (A), oblate ellipsoid (B), one sheet hyperboloid (C) and two sheets hyperboloid (D), plus the related cones and isodual images here omitted for brevity. All these quadrics are unified into one, single, unique geometric notion in isospace.

5.2.F: Connections with noneuclidean geometries and applications. Recall that the isotopies lift the conventional Euclidean metric δ = diag. (1, 1, 1) into the isometrics $\hat{\delta} = \hat{T} \times \delta$ with a well behaved, but otherwise unrestricted

functional dependence on time t, local coordinates r and their derivatives of arbitrary order, $\hat{\delta} = \hat{\delta}(t, r, \dot{r}, \ddot{r}, ...)$. The first noneuclidean property of the isoeuclidean geometry, apparently presented here for the first time, can be expressed as follows.

Lemma 5.2.1: *Isoeuclidean spaces are curved unless the isometric is independent on the local coordinates, but dependent on the remaining variables,* $\hat{\delta} = \hat{\delta}(t, \dot{r}, \ddot{r}, ...)$.

Proof. A given n-dimensional isoeuclidean space $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ admits the *non-null* Christoffel symbols (connection)

$$\Gamma_{h\ k}^{\ i} = \tfrac{1}{2}\hat{\delta}^{ij}\left(\frac{\partial\,\hat{\delta}_{kj}}{\partial\, r^h} + \frac{\partial\,\hat{\delta}_{jh}}{\partial\, r^k} - \frac{\partial\,\hat{\delta}_{hk}}{\partial\, r^j}\right), \tag{5.2.66}$$

which characterize the quantities

$$\hat{R}_{l\ h}^{\ j} = \frac{\partial\Gamma_{l\ h}^{\ j}}{\partial r^k} - \frac{\partial\Gamma_{l\ k}^{\ j}}{\partial r^h} + \Gamma_{q\ k}^{\ j}\,\Gamma_{l\ h}^{\ q} - \Gamma_{q\ h}^{\ j}\,\Gamma_{l\ k}^{\ q}, \tag{5.2.67}$$

representing non-null curvature, which is identically null when the isometric is independent from the local coordinates. **q.e.d.**

In short, *the isoeuclidean geometry provides a symbiotic unification of the Euclidean and Riemannian geometries.* The curved character of the isoeuclidean geometry has been computed in the projection of the isospace in the original space. Nevertheless, as we shall see later on in this chapter, the above curvature persists even in isospace.

This author must admit that the emergence of curvature on an isospace which is isoflat was basically unexpected and, for this reason, it was identified only here for the first time, twelve years following the identification of the isoeuclidean geometry [12]. As we shall see, the property is, by far, nontrivial, inasmuch as, when extended to the isotopies of our space-time, it permits a geometric unification of general and special relativities with ensuing operator form of gravity with curvature embedded in the unit of relativistic quantum theories.

The curved character of the isoeuclidean geometry when projected on conventional spaces over conventional fields can be studied via the same methods used for the Riemannian geometry [4]. Its study in isospace will be done in Sect. 5.6.

The advantages of the isoeuclidean representation of gravity can be indicated from these introductory lines as follows. Note that, in view of the arbitrariness in the functional dependence of the isometric, we can identify the

isoeuclidean metric with the space component of (3+1–dimensional Riemannian metrics,, $\hat{\delta}(r) \equiv g(r)$. For the case of Schwarzschild's exterior solution [9], this implies the following particular realization

$$\hat{\delta}(r) = \hat{T}(r) \times \delta \equiv g(r) = (1 - M/r)^{-1} \operatorname{diag.}(1, 1, 1), \qquad (5.2.68)$$

by reaching in this way the (space component of the) *gravitational isotopic element and isounits*

$$\hat{T}_{gr} = (1 - M/r)^{-1}, \quad \hat{1}_{gr} = (1 - M/r). \qquad (5.2.69)$$

It is evident that, at the limit of gravitational collapse all the way to a singularity, *the isounit becomes singular,* i.e., *gravitational singularities can be represented via the zeros of the (space) isounit,*

$$r = M \quad \rightarrow \quad \hat{1} = 1 - M/r = 0. \qquad (5.2.70)$$

This provides a first illustration of physical applications of isogeometries of Class IV. The above representation also permits *a novel conception of stars undergoing gravitational collapse all the way to a singularity as isospheres of Class IV,* that is, isosphere with singular radius.

Note that isointerpretation (5.2.70) is *external,* that is, conceived and realized *outside* the collapsing stars without any representation of internal effects. The representation of gravity on isoflat geometries then permits more realistic interior representations of gravitational collapse with interior nonlinear, nonlocal and nonlagrangian effects via the study of the zeros of general isounits, $\hat{1}_{gr}(r, \dot{r}, \ddot{r}, ...) = 0$.

All these possibilities, and other studied in Vol. II, are evidently precluded to the conventional Riemannian representation evidently because in the latter case the unit is the trivial constant value $I = \operatorname{diag.}(1, 1, 1)$.

A few comments are in order on the comparison of the isoeuclidean geometry and other *noneuclidean geometries* (see, e.g., ref.s [28,43] and quoted literature). As well known, Euclid's Fifth Axiom lead to a historical controversy that lasted for two millennia, until solved by Lobacevski in a rather unpredictable way, via the introduction of a new, non–Euclidean geometry today appropriately called *Lobacevski geometry* (see [loc. cit.]).

As it is also well known, Lobacevski geometry is abased on certain liftings of Euclidean expressions, although defined on the conventional unit. Thus, the Lobacevski and isoeuclidean geometries are structurally different.

Nevertheless, it is important to understand that *the Lobacevski geometry is a particular case of the projection of the isoeuclidean geometry in the Euclidean plane.* To see this point consider the following celebrated transformations

$$x' = \frac{x + a}{1 + a x}, \qquad y' = \frac{y(1 - a^2)^{\frac{1}{2}}}{1 + ax}, \qquad |a| < 1, \tag{5.2.71}$$

which have the peculiar property of carrying straight lines into straight lines and circles into circles (see ref. [28] for details) *while keeping the unit the same.*

Now, the isoeuclidean space $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ of class I in two dimensions can be equivalently reinterpreted as an ordinary Euclidean plane $\bar{E}(\bar{r},\delta,R)$ in the new coordinates

$$\bar{x} = b_1(x, y, ...)\, x\,, \quad \bar{y} = b_2(x, y, ...)\, y\,, \tag{5.2.72}$$

under which we have the identity

$$\bar{x}\,\bar{x} + \bar{y}\,\bar{y} = x\, b_1{}^2\, x + y\, b_2{}^2\, y. \tag{5.2.73}$$

It is then evident that Lobacevski transformations (5.2.68) are contained as a particular case of the much larger class of isotransformations (5.2.72).

The connection between Lobacevski and isoeuclidean geometries can therefore be expressed by saying that:

A) *the Lobacevski geometry identifies "one" particular lifting of the Euclidean geometry preserving straight lines and circles under the conventional value of the unit;* while

B) *the isoeuclidean geometry identifies "an infinite class" of liftings of the Euclidean geometry which preserve straight lines and circles under a joint lifting of the unit.*

Note finally that the Lobacevski geometry itself can be subjected to an isotopic lifting which has not been studied here for brevity.[37]

Numerous other noneuclidean geometries exist in the literature (besides the Minkowskian, symplectic, affine and Riemannian geometries studied later on in this chapter). One particularly intriguing geometry is the so-called *nondesarguesian geometry* studied by Shoeber [29], which has a significant connection with the studies of these volumes because it is also capable of representing variationally nonselfadjoint (that is, nonhamiltonian) systems.

This latter geometry too is different from the isoeuclidean one, again, because it is based on the conventional unit. However, the underlying mapping between the Euclidean and nondesarguesian geometry is also contained as a particular case of the infinite transformations (5.2.72) of the isoeuclidean geometry.

These comments are significant to focus the attention on an additional

[37] Note that the *isolobacevkii geometry* is no longer contained as a particular case of the isoeuclidean geometry because the original axioms of the two geometries are different.

reason for our selection of the isoeuclidean geometry over other possible choices, its "direct universality" for incorporating "all" infinitely possible maps of the Euclidean geometry (including singular maps for Class IV and discrete maps for Class V).

In summary, the isoeuclidean geometry appears to be unique because based on a unique set of isoaxioms, yet capable of unifying all possible noneuclidean geometries of the same dimension when projected in the conventional space.

Vol.s II and III contain numerous physical applications of the isoeuclidean geometry of Classes I and II. Its primary function is to provide a geometry directly applicable to interior dynamical problems, that is, applicable to the most general possible nonlinear, nonlocal, and nonhamiltonian systems studied in these volumes.

This physical objective is achieved via *the geometrization of physical media,* that is, via the characterization of the deviations in the geometric axioms of empty space caused by the presence of a physical medium. The geometrization is done via the restriction of the isogeometry to be of Class I, in which case the isometric is restricted to the positive-definite form

$$\hat{\delta} = \hat{T} \times \delta \equiv \hat{T} = \mathrm{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^2), \quad b_k > 0, \qquad (5.2.74)$$

where the b's, called *the characteristic functions of the medium considered,* have an unrestricted functional dependence of the type

$$b_k = b_k(t, r, \dot{r}, \ddot{r}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, \ldots) > 0, \; k = 1, 2, 3, \qquad (5.2.75)$$

including a dependence on basic physical characteristics of the medium to be geometrized, such as local density μ, local temperature τ, local index of refraction n, etc.

The above characterization is evidently not unique and can be done via other methods. However, *to be consistent with physical reality, the geometrization of physical media should be done with any appropriate method other than adding a potential to a Lagrangian or a Hamiltonian.* The is due to the intrinsically nonpotential-nonhamiltonian character of the effects to be represented. This basic condition is so compelling that the possible treatment of interior effects via a potential would imply trajectories not related to those of the physical reality.

The isoeuclidean geometry has been preferred over other possibilities because it verifies the above nonlagrangian-nonhamiltonian representation of interior effects, while preserving the same geometric axioms of empty space, thus permitting the geometric unity in the treatment of both exterior and interior problems achieved in these volumes.

The mechanism for the representation of interior problems is so simple to appear trivial. It is based on the now familiar lifting of the product of the

conventional Euclidean geometry.

Consider an extended free particle in empty space, which is evidently represented via the kinetic energy alone

$$L = \tfrac{1}{2} m\, v * v \in R, \quad v = dr/dt, \tag{5.2.76}$$

where r represents the trajectory of the center of mass.

Suppose now that the particle at a given value of time penetrates within a physical medium, thus experiencing nonpotential forces. *The transition from the exterior to the interior problem is merely expressed by the transition from the Euclidean geometry to its isoeuclidean covering of Class I.*

In turn, the transition is represented by writing the original Lagrangian in isospace, thus reaching the following *isolagrangian*

$$\hat{L} = \tfrac{1}{2} m\, \hat{v} \hat{*} \hat{v} \in \hat{R}. \tag{5.2.77}$$

The geometric aspect important for this section is that the two Lagrangians L and $\hat{L}$ coincide at the abstract level for all Class I isospaces. Yet the latter is indeed capable of representing nonpotential–nonhamiltonian forces via the basic isounit of the theory.

Numerous classical examples are now available (see ref.s [6,20]). the simplest one is the particle with linear velocity–damping along the x–axis

$$\ddot{x} + \gamma \dot{x} = 0, \quad m = 1, \quad \gamma > 0, \tag{5.2.78}$$

which is merely represented via the particular realization of the isotopic element and isounit

$$\hat{T} = e^{\gamma t}, \quad \hat{1} = e^{-\gamma t}, \quad \gamma > 0. \tag{5.2.79}$$

as the reader is encouraged to verify (see ref. [6], p. 101). The isorepresentation can be enlarged into the form

$$\hat{T} = \text{diag.}\,(b_1^2, b_2^2, b_3^2)\, e^{\gamma t}, \tag{5.2.80}$$

exhibiting a feature completely absent in Euclidean geometry, *a direct representation of the actual nonspherical shape of the particle considered* here assumed to be an ellipsoid with semiaxes b_k^2 and interpreted as the isosphere. The understanding is that the isoeuclidean geometry can also be realized via *nondiagonal isotopic elements,* as requested by the case at hand.

Note that the representation of shape is completely absent in Newton's equation of motion and it is a sole feature of the isoeuclidean geometry we shall study and apply in detail in Vol.s II and III. In fact, after computing the equation

of motion, the "shape factor" cancels out.

But perfectly rigid objects do not exist in the physical reality. The isoeuclidean geometry then permits a *direct representation of all infinitely possible deformations of the original shape,* which can be easily achieved via a dependence of the characteristics b–quantities in the local pressure, velocity, etc.

Note that $\hat{T} > 0$ and $\hat{1} > 0$ as verified for all known cases of particles in interior conditions (while for antiparticles we have $T = -e^{\gamma t} < 0$ resulting in the same equation of motion).

In summary, the isoeuclidean geometry has the following primary applications in physics: A) geometrization of physical media; B) representation of the resistive effects on the motion of extended particles; and C) representation of the actual, extended, nonspherical and deformable shape of particle via the notion of isospheres.

In Vol.s II and III we study examples and applications of these isogeometries in nuclear physics, particle physics and other fields. One application in the field of *theoretical conchology* is particularly significant to deserve an outline, not only because unexpected and intriguing, but also because it permits the illustration of the limitations of our geometric perception of Nature.

A mathematical representation of the growth of sea shells has been achieved by Illert [30]. The main result established via computer visualization is that *the shape of sea shells can be certainly represented with our three–dimensional Euclidean geometry, but their evolution in time is not because sea shells would generally "crack" if their growth occurs via the strict application of the Euclidean axioms.* The issue addressed here is therefore the identification of the appropriate geometry permitting a consistent representation of their growth.

As well known, sea shells grow by discrete increments $\Delta\xi$, thus requiring discrete methods. Their analytic representation in $E(r,\delta,R)$ has a "kinetic term" $K = \frac{1}{2}(\Delta\xi/\Delta t)*(\Delta\xi/\Delta t)$ and a "potential" term similar to that of the harmonic oscillator, $V = \frac{1}{2}\Delta\xi*\Delta\xi$. The emerging Lagrangian in $E(r,\delta,R)$ is therefore of the type

$$L = \tfrac{1}{2}(\Delta\xi/\Delta t)*(\Delta\xi/\Delta t) + \tfrac{1}{2}\,\Delta\xi*\Delta\xi\,. \qquad (5.2.81)$$

Illert's studies [loc. cit.] show that:

1) Euclidean models of type (5.2.81) are insufficient to represent the actual growth of sea shells, as illustrated by the disparity between reality and computer modelling.

2) The problem of growth of sea shells is analytically similar to *nonconservative* interior dynamical problems, evidently because growth is "nonconservative" by assumption; and

3) The growth of sea shells can be quantitatively represented via *noneuclidean* Lagrangians of the type

$$L = \tfrac{1}{2}\,e^{\Omega(\phi)}(\Delta\xi/\Delta t)*(\Delta\xi/\Delta t) + \tfrac{1}{2}\,e^{\Omega(\phi)}\Delta\xi*\Delta\xi\,, \qquad (5.2.82)$$

where $\Omega(\phi)$ is a function, varying from shell to shell, of the characteristic angle ϕ of growth of each shell (see ref. [30] for details).

ISOEUCLIDEAN EVOLUTION OF SEA SHELLS

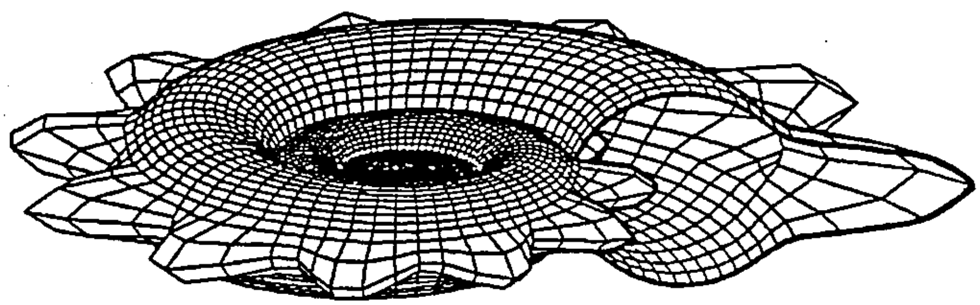


FIGURE 5.2.6: The computer visualization of a relatively "simple" sea shell from ref. [30], p. 64, the *Phanerotinus Spiralis.* The computer visualization clearly shows that the shells would crack during their growth if the Euclidean geometry is strictly implemented. However, the same computer modeling shows that growth is normal in isoeuclidean space with isorepresentation (5.2.44). In the transition to more complex sea shells, e.g., those with bifurcations, the need for noneuclidean geometries appear more compelling. In fact, a quantitative interpretation of growth at the bifurcations in Euclidean space would require *a discontinuous inversion of time* (see ref. [30], p. 98 and ff.). As we shall see in Vol. II, the isoeuclidean geometry of Class III permits instead a direct representation of the bifurcations without discontinuities [42].

It is then evident from the above results that *sea shells do not appear to evolve in Euclidean space.* In fact, they do admit an exact, quantitative representation as evolving in isoeuclidean space

$$\hat{L} = \tfrac{1}{2}(\Delta\xi/\Delta t)\,\hat{*}\,(\Delta\xi/\Delta t) + \tfrac{1}{2}\,\Delta\xi\,\hat{*}\,\Delta\xi\,, \quad \hat{1} = e^{-\Theta(\phi)}. \qquad (5.2.83)$$

We can therefore say that *a quantitative representation of sea shells requires the necessary alteration of the notion of scalar product while keeping the dimension unchanged,* which is precisely the basic mechanism of isotopies. This illustrates the statement made earlier to the effect that the strict imposition of the

Euclidean axioms, thus including the strict use of the *conventional* scalar product $V{*}V$, does not allow a representation of the growth of sea shells.

The isogeometries emerge as preferred over other noneuclidean geometries because, unlike the latter, they preserve the original axioms. As indicated in this section, a given line or a given shape cannot be told *a priori* to belong to the Euclidean or to the isoeuclidean geometry because this selection requires additional knowledge, such as the structure of the scalar product.

Sea shells *appear* to evolve in our Euclidean geometry because of the peculiar nature of the isogeometries of preserving the geometric axioms of our space, but they appear to evolve in a more complex geometry which is representable via the isotopy of the scalar product $V{*}V \rightarrow V\hat{*}V = V{*}\hat{T}{*}V$ *and* the joint isotopy of the unit $1 \rightarrow \hat{1} = \hat{T}^{-1}$. Note that the three–dimensional character of the geometry remains completely unaffected.

Deeper studies [30] have indicated that in actuality sea shells require the use of the isoeuclidean geometry of class more general than II. In fact, the interpretation of their growth at bifurcations clearly shows the need for an isogeometry capable of mastering the direction of time, thus requiring an isogeometry of Class III. Moreover, their structure is discrete, thus suggesting isounits of Kadeisvili's Class V which include all other classes as particular cases.

This is per sẽ intriguing inasmuch as we have indicated the *necessary* condition for *physical* events of restricting the isoeuclidean geometry, separately, to Class I for particles and to Class II for antiparticles. It therefore appears that *biological* structures belong to a a geometry structurally more general than that of the physical world.

The application of the isoeuclidean geometry to the growth of sea shells, even though evidently not unique, is instructive in suggesting an act of scientific humility: the expression of doubts prior to claiming final achievement of knowledge via a perception of Nature based on our manifestly limited three Eustachian tubes.In the final analysis, the complexity of the geometry of biological entities, such as a DNA molecule, is simply beyond the grasp of human comprehension at this time.

Our use of the geometries studied in this section is therefore the following:

1) Euclidean geometry, used for the characterization of *particles in vacuum;*

2) Isoeuclidean geometry of Class I, used for the characterization of *particles within physical media;*

3) isodual Euclidean geometry, used for the characterization of *antiparticles in vacuum;* and

4) Isodual Isoeuclidean geometry of Class II, used for the characterization of *antiparticles within physical media.*

5) Isogeometry of Class III, used for initial studies of biological structures;

6) Isoeuclidean geometries of Class IV, used for *gravitational*

singularities; and

7) Isoeuclidean geometry of Class V, suggested for representation of biological structures.

5.3: ISOMINKOWSKIAN GEOMETRY AND ITS ISODUAL

5.3.1: Basic properties. The *isominkowskian geometry* is the geometry of the isominkowski spaces of Class I over isoreal fields of the same class (Sect. I.3.5)

$$\hat{M}(\hat{x},\hat{\eta},\hat{R}): \quad \eta = \mathrm{diag.}\,(1,\,1,\,1,\,-1)\,, \quad \hat{\eta} = \hat{T}(x,\,\dot{x},\,\ddot{x},\,.....)\,\eta, \quad \hat{1} = T^{-1}, \qquad (5.3.1a)$$

$$\hat{x}^{\hat{2}} = \hat{x}^{\mu} \hat{\times} \hat{x}_{\mu} = [x^{\mu}\,\hat{\eta}_{\mu\nu}(x,\,\dot{x},\,\ddot{x},...)\,x^{\nu}\,]\,\hat{1} \in \hat{R}(\hat{n},\hat{+},\hat{\times})\,, \qquad (5.3.1b)$$

$$\hat{x} = \{\hat{x}^{\mu}\} = (r\hat{\times}\hat{1},\, x^4\hat{\times}\hat{1}), \qquad x^4 = c_o t\,, \qquad d\hat{s}^{\hat{2}} = (\,-\,dx^{\mu}\,\hat{\eta}_{\mu\nu}\,dx^{\nu}\,)\,\hat{1}, \qquad (5.3.1c)$$

$$\hat{x}_{\mu} = \hat{\eta}_{\mu\nu}\,\hat{x}^{\nu}\,, \quad \hat{N}^{\mu\nu} = \hat{1}^{\mu}{}_{\alpha}\,\eta^{\alpha\nu}, \quad \hat{\eta}_{\mu\alpha}\,\hat{\eta}^{\alpha\nu} = \delta_{\mu}{}^{\nu}\,, \qquad (5.3.1d)$$

where c_o is the *speed of light in vacuum* and s is called the *isotopic proper time* or *isotime* for short (t being the ordinary *isotopic time*), with isounits in the diagonal realization

$$\hat{1} = \mathrm{diag.}\,(\,b_1{}^{-2},\, b_2{}^{-2},\, b_3{}^{-2},\, b_4{}^{-2}\,) > 0, \qquad (5.3.2a)$$

$$\hat{x}^2 = (\,x^1\,b_1{}^2\,x^1 + x^2\,b_2{}^2\,x^2 + x^3\,b_2{}^2\,x^3 - x^4\,b_4{}^2\,x^4\,)\,\hat{1}\,. \qquad (5.3.2b)$$

or in nondiagonal form studied later on.

The isominkowskian geometry was first introduced by this author in paper [12] of 1993] and then studied in various publications (see monograph [20] of 1991 and quoted literature).

It is evident that *the space-component of the isominkowskian geometry is the isoeuclidean geometry in its entirety,* and no further comment is needed. Moreover, from the preceding section, it is evident that *the (3+1)-dimensional isominkowskian geometry of Class I is a particular case of the isoeuclidean geometry of Class III in 4-dimension.* In fact, this is the way it was originally derived in [12]. All main geometric lines of the isominkowskian geometry are therefore the same as those of the preceding section. We shall primarily consider below kinematical aspects important for the various applications of the new geometry.

The isominkowski geometry in Class I diagonal form is characterized by four functions b_{μ} which: A) are called *relativistic characteristic functions of the medium considered;* B) have a generally nonlinear and nonlocal dependence on space-time coordinates x, wavefunctions ψ, $\psi^{\dagger}$ their derivatives of arbitrary

order, $\dot{x}$, $\ddot{x}$, $\partial\psi$, $\partial\psi^\dagger$...., as well as on the physical characteristics of the medium considered, such as the local density μ, the local temperature τ, the local index of refraction n, etc.; and C) are assumed to be positive-definite for the Class I isogeometry

$$b_\mu = b_\mu(s, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, \ldots\ldots) > 0 , \quad \mu = 1, 2, 3, 4 . \quad (5.3.3)$$

The above general functional dependence is needed for the local study of interior dynamical problems; that is, the trajectory of an extended relativistic particle within a physical medium at one given interior point x.

If one studies global effects of physical media, such as the average speed of light throughout the medium, the characteristic functions can be averaged into the constants

$$b^\circ_\mu = < b_\mu(s, x, \dot{x}, \ddot{x}, \ldots) > , \qquad (5.3.4)$$

where < .... > represents an average appropriate for the problem at hand.

As we shall see in Ch. II.8, the above average provide a quantitative representation of interior effects, while permitting the recovering of conventional inertial systems for an outside observer.

A first property to keep in mind is that *the Minkowskian and isominkowskian geometries coincide, by construction, at the abstract, realization-free level.* This is due to the positive-definiteness of the b-quantities or, equivalently to the preservation under isotopies of the signature (+, +, +, –).

One of the implications for which the isogeometry was suggested in the first place [12], is the preservation of the axioms of the special relativity in the transition from the exterior to the interior problem, as studied in detail in Vol.s II and III. The subtle consequence is that *criticisms on the isominkowskian geometry may in the final analysis result to be criticisms on Einstein's axioms themselves.*

The primary physical application of the isominkowskian geometry is the relativistic geometrization of physical media (see Fig. 5.3.1 for comments).

By no means, is the isominkowskian geometry the only possible one for the geometrization here considered. In fact, the use of other geometries is conceivable, and their study is encouraged, because one of the beauties of mathematical and physical inquiries is their polyhedric character. However, other deformations of the Minkowskian geometry do not preserve the Einsteinian axioms.

Also, studies on the propagation of classical electromagnetic waves in physical media via operator approaches in first and second quantizations should be deferred after the achievement of a classical representation, because *conventional* operator treatments generally suppress the very characteristics to be represented, such as the inhomogeneity and anisotropy of the medium.

In essence, when first exposed to the propagation of light in our

inhomogeneous and anisotropic atmosphere, a natural mental attitude is the study of the propagation via old methods, e.g., via scattering of photons on the atoms of our atmosphere.

This is the approach which should be avoided on both theoretical and experimental grounds. Theoretically, the event depicted in Fig. 5.3.1 is *purely classical,* thus requiring a purely classical description, rather than the use of photons in second quantization. *After* the achievement of a geometric representation of the inhomogeneity and anisotropy of physical media at the classical level, studies based on first and second quantization should be considered.

But the strongest support against the preservation of old knowledge for the novel physical conditions of Fig. 5.3.1 comes from experimental data. In fact, as we shall see in Vol. III, physical media imply *shifts toward both, the red or the blue depending on their characteristics.* Assuming that adequate manipulations permit the interpretation of shift toward the red via scattering of photons on atoms, the same theory cannot evidently represent the opposite shift, precisely because lacking the characteristics to be represented.

A first intuitive understanding of the isominkowskian geometrization of physical media can be reached by writing the isoseparation in the equivalent form

$$x^2 = x^1 b_1{}^2 x^1 + x^2 b_2{}^2 x^2 + x^3 b_2{}^2 x^3 - x^4 b_4{}^2 x^4 =$$
$$= x^1 n_1{}^{-2} x^1 + x^2 n_2{}^{-2} x^2 + x^3 n_2{}^{-2} x^3 - x^4 n_4{}^{-2} x^4) \hat{1}, \qquad (5.3.5)$$

(where we have ignored the multiplicative factor $\hat{1}$ for simplicity), namely, by expressing the characteristic functions in the equivalent form $b_\mu = 1/n_\mu$. Now, the fourth term, $b_4 = 1 / n_4$, is already known, and represents the local index of refraction within a given medium, yielding the local speed of light

$$c = c_o b_4 = c_o / n_4 = c(x, \mu, \tau, ...) . \qquad (5.3.6)$$

One can therefore see the above distinction between the characteristic *functions* b_μ and the characteristic *constants* b°_μ. In fact, the quantity n_4 is the local index of refraction at one given point in space-time (characterizing the speed of light at one point of our atmosphere in Fig. 5.3.1), while n°_4 is the *average* index of refraction (characterizing the average speed of light throughout our entire atmosphere).

A first meaning of the isominkowskian geometry is therefore that of providing a *relativistic generalization of the familiar index of refraction* n_4 *to all space-time components* n_μ.

Proposition 5.3.1: *The isominkowskian geometry of class I with diagonal isounit uniquely follows from: 1) the use of a locally varying speed of light* $c =$

c_0/n_4; *2) the use of Lorentz transforms; and 3) the condition of preserving the original, abstract Minkowskian axioms.*

Proof. Assume the locally varying speed of light $c = c_0/n_4$. Then the conventional Minkowski separation is lifted to the following *Lorentz noninvariant* structure

$$x^2 = x^1 x^1 + x^2 x^2 + x^3 x^3 - x^4 n_4^{-2} x^4, \tag{5.3.7}$$

The use of the conventional Lorentz transforms then yields a structure precisely of the isominkowskian type

$$x^2 = x^1 n_1^{-2} x^1 + x^2 n_2^{-2} x^2 + x^3 n_2^{-2} x^3 - x^4 n_4^{-2} x^4, \tag{5.3.8}$$

which is now invariant under the Lorentz isotopic symmetry (Vol. II). However, the underlying space is no longer locally isomorphic to the Minkowski space when referred to the conventional unit over a conventional field R. The isominkowskian space and related geometry then uniquely follow from the imposition of the preservation of the original abstract axioms. **q.e.d.**

ISOMINKOWSKIAN GEOMETRIZATION OF PHYSICAL MEDIA

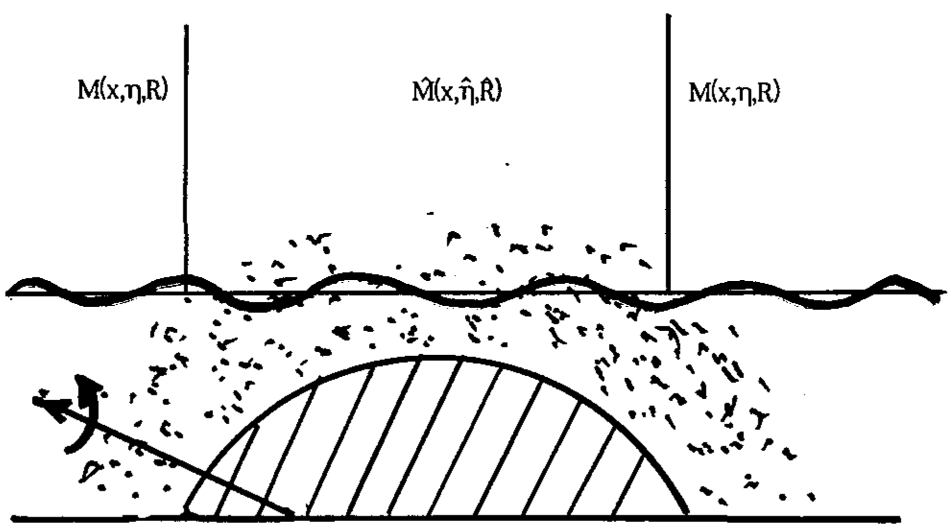


FIGURE 5.3.1. A schematic view of a primary physical application of the isominkowskian geometry: *the quantitative treatment in a form suitable for experimental verifications of the dynamical effects caused by the inhomogeneity and anisotropy of physical media in*

the propagation of electromagnetic waves and particles. Recall that the Minkowskian geometry is a geometrization of the homogeneity and isotropy of empty space. All predictions based on the Minkowskian geometry, such as Doppler's effects, dilation of time, etc., are therefore crucially dependent on the homogeneity and isotropy of empty space. Consider now an electromagnetic wave originating from a distant star which travels, first, in empty space (in which case the Minkowskian representation is exactly valid), then travels throughout our atmosphere, and finally returns to travel in empty space. Now, our atmosphere is manifestly inhomogeneous, and anisotropic, as discussed earlier in this volume. The physical issue requiring experimental verifications (which is studied in detail in Vol. III) is whether the inhomogeneity and anisotropy imply measurable deviations from the conventional Minkowskian predictions. Specifically, the experimental issue is whether Doppler's effect, time dilation, etc. have the same numerical values for events within inhomogeneous and anisotropic media or deviations are experimentally measurable. As we shall see in Vol. III a rather considerable body of experimental evidence supports the latter expectation, although in a predictable preliminary way. The mathematical issue considered here is therefore the achievement of a geometric representation, specifically, of the inhomogeneity and anisotropy of our atmosphere. The isominkowskian geometry appears to be particularly suited to: A) provide a direct geometric treatment of physical media, B) in a form suitable for experimental verifications, while C) preserving the basic Einsteinian axioms at the abstract level.

Stated in different terms, *the isominkowskian geometry emerges under any deviation from the constancy of the speed of light in vacuum even when not desired.*

At a deeper level, recall that only a small portion of physical media is transparent to light. A second meaning of the isominkowskian geometrization is therefore that of *extending the index of refraction to all physical media, whether transparent or not to light.* In the latter case the quantity $b_4 = 1/n_4$ acquires a purely geometric meaning similar, say, to the component g_{44} the Riemannian metric.

As we shall see in Vol. III, experimental evidence indicates quite clearly that *the space characteristic functions* b_k, $k = 1, 2, 3$, *have a velocity and other dependence, while the fourth characteristic quantity* b_4 *generally provides a geometrization of its density.*

The isominkowskian representation of the inhomogeneity and anisotropy of physical media is now evident. In fact, the former can be represented, e.g., via a dependence of the characteristic functions on the local density, while the latter can be represented, e.g., via a differentiation of the space–time quantities, $b_k \neq b_4$.

As a first example, a direct representation of water is given by the simplest possible isotopy, called *relativistic scalar isotopy* (see Ch. II.8 for details)

$$x^{\hat{2}} = \frac{1}{n^\circ} x^2, \quad n^\circ_\mu = n^\circ, \ \mu = 1, 2, 3, 4. \tag{5.3.9}$$

where n° is a known numerical quantity and x^2 is evidently the conventional Minkowskian invariant. In fact, water is a homogeneous and isotropic medium whose characteristics are then represented by isoinvariant (5.3.8).

A second example is our inhomogeneous and anisotropic atmosphere which requires the full isoinvariant (5.3.5) for its representations. The numerical values of the b°–constants will be computed in Vol. III from astrophysical data. Needless to say, the deviations of the b°–quantities from the value 1 are very small for our atmosphere, yet they produce measurable effects, as we shall see.

Intriguingly, isoinvariant (5.3.8) and related isospecial relativity permit a direct representation of relativistic kinematics in water, such as: the *decrease* of the speed of light according to law (5.3.7); the propagation of electrons faster than the local speed of light (Cherenkov's effect); the correct relativistic addition of speeds in water; and others (see Ch. II.8).

The extension of the results to inhomogeneous and anisotropic media is then consequential, and equally consequential are deviations from the Minkowskian prediction in vacuum.

A main characteristics of the isominkowskian geometry is *the alteration of the basic (dimensionless) units of space and time* from the conventional trivial form I = diag. (1, 1,1, 1) (the unit of the Lorentz group in its regular representation) to the expression

$$\hat{1} = \text{diag.}\,(\,b_1^{\,2}, b_2^{\,-2}, b_3^{\,-2}, b_4^{\,-2}\,)\,,\ \text{i.e.,}\ \ \hat{1}_\mu = b_\mu^{\,-2}\,, \mu = 1, 2, 3, 4\,, \quad (5.3.10a)$$

$$\hat{1}_{space} = \hat{1}_s = \text{diag.}\,(\,b_1^{\,2}, b_2^{\,-2}, b_3^{\,-2}\,)\,, \qquad \hat{1}_{tim} = \hat{1}_t = b_4^{\,-2}. \quad (5.3.10b)$$

In turn, the alteration of the unit has rather profound geometric implications. Those related to space have been discussed in the preceding section. Those including time are illustrated in Fig. 5.3.2 below.

We have considered until now isominkowskian geometries with a diagonal isotopic element. The reader should be aware of the existence of rather intriguing applications for isoMinkowskian geometry of Class I with *nondiagonal isotopic elements and isounits.* One of the most significant cases was proposed by Dirac [31] in two of his last (and little known) papers dealing with a generalization of his own celebrated equation. The ensuing "Dirac's generalization of Dirac's equation" has resulted to possess an essential isotopic structure, evidently without Dirac's awareness,[38] as we shall study in detail in Vol. II.

In this chapter we would like to identify only the rather intriguing isominkowskian geometry of Dirac's papers [31]. In essence, Dirac studied a deformation of the Minkowski space characterized by the nondiagonal element

[38] Dirac's papers [31] are of 1971–1972, while the isotopies were formulated in 1978 [5].

THE ISOBOX IN SPACE-TIME

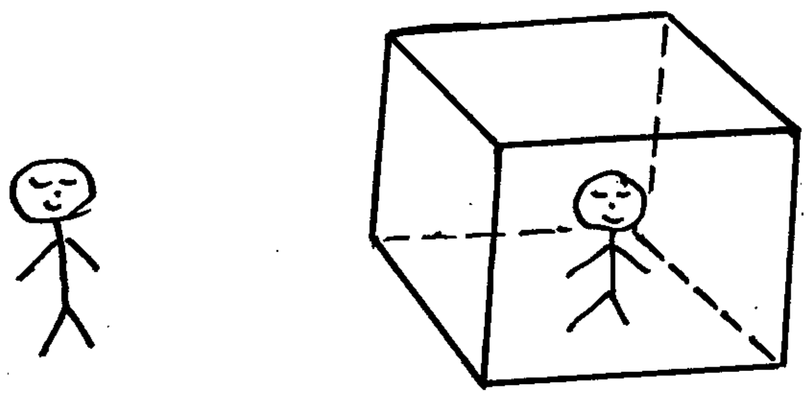


FIGURE 5.3.2: In Fig. 5.2.1 we have indicated how the transition from an observer in ordinary space to one in isospace alters the dimension and shape of a given object. In this figure we indicate the implications regarding the behaviour in time. In the outside of the isobox we have an observer in ordinary space–time with basic *units of space* I_s = diag. (1, 1, 1) and of *unit of time* $I_t = 1$. In the inside we have instead an observer in the isotopic space of Class I, with basic *isounits of space* $\hat{1}_s$ = diag. $(b_1^{\,2}, b_2^{\,-2}, b_3^{\,-2})$ and *isounit of time* $\hat{1}_t = b_4^{\,-2}$. Then the two observers have basically different forward time evolutions, in the same way as it occurs for space. The two observers not only see different shapes and sizes, but *observe the same isobox at different times.* When isominkowskian geometries of class III are admitted, this implies isounits of time $\hat{1}_t = f(x, \dot{x}, ...)$, where f is a well behaved function which can assume positive or negative values. It follows that, if the outside observer see, e.g., a cube of 10 m side at time t moving forward, the inside observer not only can see an arbitrary dimension and shape, but also the observation can occur at any given future time (for $\hat{1}_t > 0$) or past time (for $\hat{1}_t < 0$). One of the most important notions which we would like to convey with the isobox is that *if we see a given structure, such as a far away star or a near–by sea shell, this does not necessarily means that that particular structure must necessarily evolve with our time, because in reality it may belong to an arbitrary present or future time with respect to us.* The inclusion of nondiagonal isounits of Class I implies further departures from the conventional geometric perception. For instance, an inside observer with isounits (5,3,11) would lose completely any perception of the behavior in the x– and z–axes and see the isobox as a segment of arbitrary length in the y–axis at an arbitrary present, future or past time.

$$\hat{T} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{5.3.11}$$

with properties

$$\det \hat{T} = 1, \qquad \hat{1} = T^{-1} = T^{t}, \tag{5.3.12}$$

where t denotes transposed, which therefore qualify it as a fully acceptable isotopic element of Class I.

The isogeometry characterized by isotopic element (5.3.9) is intriguing indeed. Its most salient property is that *the isometric is nondegenerate*, det $\hat{\eta} = -1$, *but the isoinvariant is degenerate*,

$$\hat{x}^{\hat{2}} = x^{\mu}\,\hat{\eta}_{\mu\nu}\,x^{\nu} = \begin{pmatrix} x^1 & x^3 & x^3 & x^4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} =$$

$$= x^1 x^3 - x^2 x^4 - x^3 x^1 - x^4 x^2 = -2 x^2 x^4, \tag{5.3.13}$$

namely, the isoseparation is contracted under Dirac's isotopy from four to two dimensions. In turn, this contraction has truly remarkable implications, such as the lifting of the spin $s = \frac{1}{2}$ to spin $s = 0$, as originally derived by Dirac [31] and as confirmed by isotopic methods (seee Vol. II).

It is instructive for the interested reader to see that the same dimensional contraction occurs for other realizations, such as for $\eta = (+1, -1, -1, -1)$ and related ordering of the components $x = \{x^4, x^1, x^2, x^3\}$. As a result, the dimensional contraction $(1, 2, 3, 4) \rightarrow (2,4)$ is intrinsic in the isogeometry here considered, and so are its rather peculiar properties, such as the contraction of the three–dimensions (1, 2, 3) down to the line along the y–axis.

We shall have ample opportunities in Vol.s II and III to study the above isogeometry, the related "Dirac's generalization of Dirac's equation", and its novel physical implications.

The above isominkowskian geometries are used in these volumes for the characterization of *physical media of matter*, or a *particle* moving within such physical media. In order to characterize *antimatter*, it is not sufficient just to change the sign of the charge, because we need an *antiautomorphic map of the geometry itself*. As now familiar, that used in these volume is the *t of space and time*

$$\hat{1} = \text{diag.}\,(b_1^{2}, b_2^{-2}, b_3^{-2}, b_4^{-2}) \rightarrow \hat{1}^{d} = \text{diag.}\,(-b_1^{2}, -b_2^{-2}, -b_3^{-2}, -b_4^{-2}), \tag{5.3.14a}$$

$$\hat{1}_s = \text{diag.}\,(b_1^{2}, b_2^{-2}, b_3^{-2}) \rightarrow \hat{1}_s^{d} = \text{diag.}\,(-b_1^{2}, -b_2^{-2}, -b_3^{-2}), \tag{5.3.14b}$$

$$\hat{1}_t = b_4^{-2} \rightarrow \hat{1}_t^d = -b_4^{-2} \tag{5.3.14c}$$

The application of the above map top isospaces (5.3.1) then yields the *isodual isominkowski spaces*

$$\hat{M}^d(\hat{x}^d,\hat{\eta}^d,\hat{R}^d): \ \hat{\eta}^d = \hat{T}^d(x, \dot{x}, \ddot{x}, \mu, ...)\eta = -\hat{\eta}, \ \hat{1}^d = (\hat{T}^d)^{-1} = -\hat{1}, \tag{5.3.15a}$$

$$\hat{x}^{d\hat{2}d} = [x^\mu \hat{\eta}^d_{\mu\nu}(x, \dot{x}, \ddot{x},...) x^\nu]\hat{1}^d = [-x^\mu \hat{\eta}^d_{\mu\nu} x^\nu](-\hat{1}), \tag{5.3.15b}$$

$$ds^{d\hat{2}d} = (+dx^\mu \hat{\eta}^d_{\mu\nu} dx^\nu)\hat{1}^d \equiv d\hat{s}^{\hat{2}} \in \hat{R}^d(\hat{n}^d,+,\hat{x}^d), \tag{5.3.15c}$$

whose invariant can be written in the diagonal form

$$\hat{x}^{d2d} = (-x^1 b_1^2 x^1 - x^2 b_2^2 x^2 - x^3 b_2^2 x^3 + x^4 b_4^2 x^4)\hat{1}^d \in \hat{R}^d. \tag{5.3.16}$$

or in any nondiagonal realization preserving the negative-definite character of the isodual isounit.

The *isodual isominkowskian geometry* is the geometry of isospaces (5.3.13). It will be used for the description of *antiparticles in relativistic interior conditions.*

As a particular case for $\hat{1}^d \equiv I^d = \text{diag.}(-1, -1, -1, -1)$ we have the *isodual Minkowskian space*

$$M^d(x,\eta^d,R^d): \qquad \eta^d = -\eta, \qquad I^d = -I, \tag{5.3.17a}$$

$$x^{d2d} = [x^\mu \eta^d_{\mu\nu} x^\nu]I^d = [-x^\mu \eta^d_{\mu\nu} x^\nu](-I) \tag{5.3.17b}$$

$$ds^{2\,d} = (+dx^\mu \eta^d_{\mu\nu} dx^\nu)I^d \equiv ds^2 \in R^d(n^d, +, x^d), \tag{5.3.17c}$$

which is used for the characterization of antiparticles in vacuum.

Proposition 5.3.2: *The Minkowskian and isominkowskian separations are isoselfdual,*

$$x^2 = [x^\mu \eta_{\mu\nu} x^\nu]I = [x^\mu \eta^d_{\mu\nu} x^\nu]I^d \equiv x^2, \tag{5.3.18b}$$

$$\hat{x}^{\hat{2}} = [x^\mu \hat{\eta}_{\mu\nu}(x, \dot{x}, \ddot{x},...) x^\nu]\hat{1} = [x^\mu \hat{\eta}^d_{\mu\nu} x^\nu]\hat{1}^d \equiv \hat{x}^{\hat{2}}. \tag{5.3.18b}$$

The above properties may explain the reason why isodual spaces and isospaces were discovered only recently. It should be however stressed that, despite properties (5.3.18), *spaces (or isospaces) are mathematically and physically different than their isoduals.* In fact, mathematically, they are antiautomorphic to each other and, physically, all quantities which are positive-definite for the former become negative-definite for the latter.

As we shall see in Vol. II, the seemingly elementary identity (5.3.18a)

permits the formulation of a *fully causal motion backward in time.* In fact, motion backward in time, when referred to a negative-definite unit is fully equivalent to motion forward in time when referred to a positive-definite unit. This may illustrate the intriguing and far reaching implications of the isominkowskian geometry.

5.3.B The isolight cone and its isodual. One of the most significant and problematic insufficiencies in the use of the conventional Minkowskian geometry for the characterization of electromagnetic waves propagating within physical media is the *loss of the light cone.* In fact, a locally varying speed of light, as it occurs in a planetary atmospheres or astrophysical chromospheres with variable density, implies the necessary loss of the "cone" in favor of a more general surface in space-time in which the directrix is no longer a straight line.

The above loss is not a mere mathematical curiosity, because it carries rather deep *physical* implications of *numerical* character. As an example, gravitational horizons are today studied via the conventional light cone, as well known. But the exterior of a collapsing star is not, by far, "empty", being composed instead by huge chromospheres in which the the speed of light is not that in vacuum. Numerical results based on the conventional light light cone are then questionable.

One of the important implications of the isominkowskian geometry of Class I with diagonal isounit is the identification of a generalization of the light cone which permits more realistic calculations whenever the speed of light is no longer c_0. The latter was introduced for the first time by this author in ref. [20] under the name of *isolight cone.*

In line with all other isotopies, the isolight cone reproduces the exact cone in isospace to such an extend that even the maximal causal speed in isospace is the conventional speed *in vacuum* c_0. However, the *projection* of the isolight cone in our space-time yields the deformed cone we observe under a locally varying speed c.

The latter properties were proved for the first time by this author in ref. [22]. Their outline can be best done for clarity in the (z, t)-plane, in which the isolight cone can be written

$$\hat{x}^2 = (x^3 b_3{}^2 x^3 - x^4 b_4{}^2 x^4) \times \hat{1} = 0 , \hat{1} = \text{diag.} (b_3{}^{-2}, b_4{}^{-2}) , \qquad (5.3.19)$$

which clearly represents a *varying deformation of the light cone due to the locally varying speed* $c = c_0 b_4 = c_0/n_4(x, \mu, \omega, ...)$,.where n_4 is the locally varying index of refraction, μ the density of the medium, ω the frequency considered, etc.

The intriguing point is that deformation (5.3.19) appears only in the projection of the isominkowskian description in the original Minkowski space, because at the level of the isospace itself there we do have a perfect cone.

The proof is trivial for the isolight cone in water. In fact, isoinvariant

(5.3.19) for infinitesimal values Δz and Δt reads

$$\frac{\Delta z}{\Delta t} = \frac{b_4}{b_3} c_o \equiv c_o , \qquad (5.3.20)$$

(because $b_3 \equiv b_4$ in water due to its isotropic character).

The understanding of the isominkowskian geometry requires the knowledge that cone (5.3.15) is purely geometric because the speed of light in water is not c_o but $c = c_o/n_4$. The actual light cone is therefore that characterized by c and not c_o.

It is easy to prove that the above results also hold for arbitrary media, that is, for a locally varying speed of light within inhomogeneous and anisotropic media. In fact the general expression (5.3.20) for infinitesimal Δz and Δt becomes

$$\frac{\Delta z}{\Delta t} = \frac{b_4}{b_3} c_o \not\equiv c_o , \qquad (5.3.21)$$

because now $b_3 \not\equiv b_4$. The emergence of a *perfect cone in isospace* then results from the isotrigonometry of App. 5.C. In fact, by calling $\hat{v}$ the interior isoangle of the cone with the t–axis, we have

$$\Delta z = D \text{ isosin } \hat{v} = D\, b_3^{-1} \sin \hat{v} , \quad \Delta t = D \text{ isocos } \hat{v} = D\, b_4^{-1} \cos \hat{v} , \qquad (5.3.22a)$$

$$\frac{\Delta z}{\Delta t} = \text{isotang } \hat{v} = \frac{b_4}{b_3} \text{tang } \hat{v} = \frac{b_4}{b_3} c_o , \qquad (5.3.22b)$$

where D is the isohypothenuse.

The preservation of the maximal causal c_o is derivable from the property (see Vol. II, Ch. 8, for more details and alternative derivations)

$$\text{tang } \hat{v} = c_o . \qquad (5.3.23)$$

This implies that, even in physical media, the isolight cone remains a perfect cone and its characteristic angle remains characterized by c_o.

Proposition 5.3.3: *The isolight cone is isoselfdual,* i.e., invariant under isoduality,

$$\hat{x}^2 = (x^3 b_3^2 x^3 - x^4 b_4^2 x^4) \times \hat{1} \equiv (- x^3 b_3^2 x^3 + x^4 b_4^2 x^4) \times \hat{1}^d = \hat{x}^{d2d} = 0 \qquad (5.3.24)$$

This mathematically elementary property carries the implication (studied

in Vol. II) that *electromagnetic waves are isoselfdual and thus identically emitted by both matter and antimatter.* When we see a far away star, quasar or galaxy, we therefore have no known means at this writing for ascertain whether it is made of matter or antimatter.

The above results are confirmed via the use of the isohyperbolic functions of App. 5.C. The *hyperbolic angle* $\hat{v}$ of the isolight cone is given by

$$\hat{v} = v\, b_3\, b_4, \tag{5.3.225}$$

resulting in the *isohyperbolic functions*

$$\text{isosinh}\ \hat{v} = b_4^{-1} \sinh(v\, b_3\, b_4), \quad \text{isocosh}\ \hat{v} = b_3^{-1} \cosh(v\, b_3\, b_4), \tag{5.3.21}$$

with properties

$$b_3^2\, \text{isocosh}^2\, \hat{v} - b_4^2\, \text{isosinh}^2\, \hat{v} = 1. \tag{5.3.26a}$$

$$\text{isotangh}\ \hat{v} = b_4\, c_o / b_3, \quad \text{tangh}\ \hat{v} = c_o. \tag{5.3.26b}$$

ISOLIGHT CONE

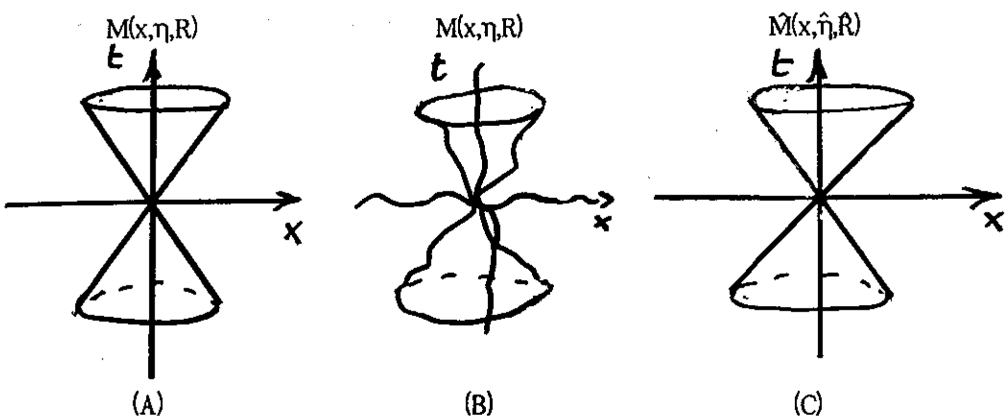


FIGURE 5.3.3. The three light cones of the isominkowskian geometry. Cone (A) is the conventional one in Minkowski space with $c_o = 1$ and $\alpha = 45°$. "Cone" (B) is the physical one in our space-time for a locally varying speed of light propagating within a generic medium. Cone (C) the *isolight cone*, that is, an isotopy of the original perfect cone. As such, it is also a perfect cone provided that it is computed in isospace. We learn in this way that the isolight cone essentially maps the locally varying "cone" (B) into the perfect cone (C). The axiom-preserving character of the isotopy is so strict to preserve the original numerical value c_o, i.e., the original angle $\alpha = 45°$. An understanding of these geometric occurrences is essential for the understanding of the isotopies of the special

relativity studied in Ch. II.8. In fact, as expected, the isospecial and the special relativities coincide at the abstract level to such an extent, as to admit the same light cone with the same speed of light c_0. Yet the physical predictions of the two relativities are profoundly different, as indicated by the inapplicability of the linear–local–canonical Lorentz transformations in favor of suitable nonlinear–nonlocal–noncanonical coverings, of the need to abandon the locally varying speed of light as the "universal invariant" for a geometrically more appropriate notion.

The isominkowskian geometry is studied in more detail in Vol. II after constructing its isosymmetry, the isotopic Poincaré symmetry. We shall then study the axiom–preserving lifting of the basic postulates of the special relativity which is inherent in the geometry, and review Aringazin's proof of its "direct universality" for all possible deformations of the Minkowski metric. Experimental verifications are studied in Vol. III. The gravitational content of the geometry is studied in App. 5.B.

5.3.C: Connections with nonminkowskian geometries. One of the most intriguing properties of the isominkowskian geometry is that it constitutes a symbiotic unification of the Minkowskian and Riemannian geometries in (3+1)–dimensional space–time, which is at the foundations of our isotopic unification of special and general relativities, as well as of the isotopic quantization of gravity.

In fact, as it was the case for the isoeuclidean geometry, it is easy to see that the isominkowskian space $\hat{M}(\hat{x},\hat{\eta},\hat{R})$ with isometric $\hat{\eta}(x, \dot{x}, \ddot{x}, ...)$ admits non–null Christoffel symbols and, therefore, non–null curvature. In particular, the latter quantities coincide with the conventional Riemannian quantities when the isominkowskian metric coincides with the Riemannian one, $\hat{\eta} = \hat{\eta}(x) \equiv g(x)$.

In summary, during our studies we shall use the following four isogeometries for the unified treatment of relativistic and gravitational aspects in classical and operator mechanics:

I) Minkowskian geometry, for the description of *particles in exterior relativistic conditions in vacuum;*

II) Isominkowskian geometry, for the description of *particles in interior relativistic conditions within physical media;*

III) Isodual Minkowskian geometry, for the description of antiparticles in *exterior relativistic conditions in vacuum;*

IV) Isodual isominkowski geometry, for the description of *antiparticles in interior relativistic conditions within physical media.*

5.4: ISOSYMPLECTIC GEOMETRY AND ITS ISODUAL

5.4A: Statement of the problem. In this section we study a generalization of the symplectic geometry[39] which is nonlinear (in all possible variables *and* their derivatives of arbitrary order), nonlocal-integral (also in all variables and their derivatives) and nonlagrangian-nonhamiltonian, yet preserving the original symplectic axioms.

The new geometry was introduced in memoir [15] under the name of *symplectic-isotopic geometry*, or *isosymplectic geometry* for short, and plays an important role in hadronic mechanics, e.g., for the isotopies of symplectic quantization, for interior gravitational problems, and other problems.

The objectives of the isosymplectic geometry are multifold. The first is to provide the geometric counterpart of the Lie-isotopic theory of the preceding chapter. This calls for the identification of generalized two-forms with an *integro-differential* structure which is the covariant counterpart of the contravariant Lie-Santilli isobrackets of Ch. I.4. The identification of the corresponding generalized analytic structures will be presented in Vol. II, including explicit examples of representations of nonlinear-nonlocal-noncanonical systems.

Other objectives are the resolution of the limitations and *physical* problematic aspects of the *Darboux Theorem* of the symplectic geometry. In the local formulation needed for physical applications, Darboux Theorem essentially states that (see, e.g., [4]) any given well behaved nonhamiltonian vector field on the 2-dimensional cotangent bundle (phase space) can always be reduced to a Hamiltonian form via a suitable transformation of the local coordinates $a = \{r, p\} \rightarrow a' = \{r'(r,p), p'(r,p)\}$.

The most transparent limitation of the above theorem is that it can be solely applied for *local-differential* vector fields, while the primary arena of interest of these volumes is that of *nonlocal-integral* systems. A second objective of this section is therefore the generalization of the Darboux theorem in such a way to accommodate integro-differential vector-fields.

Moreover, even when applicable, the Darboux theorem is afflicted by rather serious problematic aspects of *physical* character, the clear understanding being that the theorem is mathematically impeccable. In fact, the Darboux's transforms $a = \{r, p\} \rightarrow a' = \{r'(r,p), p'(r,p)\}$ are transparently noncanonical (as a necessary condition to map a nonhamiltonian into a Hamiltonian vector field) as well as *nonlinear*.

Therefore, Darboux's charts therefore map the frame of the observer actually used in experiments into highly nonlinear images which are not realizable in a laboratory (e.g., a new coordinate of the type $r' = r\exp(\alpha r p)$, where α

[39] An outline of the basic notions of the conventional symplectic geometry is presented in App. 5.A. For technical presentations, one may consult ref.s [3,4]. A comprehensive bibliography is contained in ref. [6].

is a suitable constant, simply cannot be realized in the real physical world). Moreover, the transformed frames are highly noninertial, as an evident occurrence caused by *nonlinear* transforms of *inertial* systems, thus implying the loss of Galilei's, Einstein's special and Einstein's general relativity.

In view of the latter, rather serious problematic aspects, one of the uncompromisable conditions of the studies presented in these volumes is that *the geometric representation of a given nonlinear, nonlocal-integral and nonhamiltonian vector field must first occur in the frame of the observer (direct representation).* Only when this objective is achieved, the use of the transformation theory may have a physical meaning.

The latter condition calls for an alternative formulation of Darboux's theorem which achieves "direct universality", that is, the representation of all vector fields of the class considered (universality), directly in the frame of the observer (direct universality). This objective is reached, apparently for the first time, in Sect. 5.4.F of this second edition.

In this section we shall merely review the main lines of the new isosymplectic geometry, and refer the reader to ref.s [15–20,25,26] for more details. Our entire treatment will be in a local chart so as to be ready for the applications of Vol.s II and III. The mathematical coordinate free treatment will be left to interested m,mathematicians.

All quantities considered are assumed to verify the needed continuity conditions, e.g., of being of Class $\mathcal{C}^{\infty}$, which shall be hereon omitted for brevity. Similarly, all neighborhoods of given points are assumed to be star-shaped, or have a similar topology also ignored hereon for brevity.

As a final point, we mention that the isotopies of the symplectic geometry presented in this section are not unique. In fact, another formulation of the isosymplectic geometry has already been presented in ref.,s [44,45] and will not be studied here for brevity.

5.4B: Isodifferential calculus and its isodual. Let M(R) be an n-dimensional manifold over the reals R(n.+,×) and let T*M(R) be its cotangent bundle. We shall denote with $T^*M_1(R)$ the manifold M(R) equipped with the *canonical one-form* [3,4]

$$\theta : \quad T^*M_1(R) \Rightarrow T^*(T^*M_1(R)), \quad \theta \in \Lambda_1(T^*M_1(R)). \qquad (5.4.1)$$

The *fundamental (canonical) symplectic form* is then given by

$$\omega = d\theta, \qquad (5.4.2)$$

which is nowhere degenerated, exact and closed (see App. 5.A). The manifold M(R), when equipped with the symplectic two-form ω becomes an (exact) *symplectic manifold* $T^*M_2(R)$ in canonical realization. The *symplectic geometry*

is the geometry of symplectic manifolds as characterized by exterior forms, Lie's derivative, etc.

The isotopies permit a chain of nonlinear, nonlocal and noncanonical generalizations of all that, one of isotopic type (Class I), another of isodual character (Class II), and the others of Classes III, IV and V.

In the original derivations we used the notation by Lovelock and Rund [4] in order to facilitate the identification of the differences between conventional and isotopic geometries, and we shall adopt the same approach here. Latin indices i, j, p, q, etc. will be used for a generic manifold, while Greek indices μ, ν, etc. will be used for specific physical applications.

The first visible implication of the isotopies of the symplectic geometry is that the basic differential calculus becomes inapplicable. This implies that the very notion of one-form θ or two-form ω are inapplicable and must be suitably generalized.

Consider an n-dimensional isomanifold $\hat{M}(\hat{R})$ (see ref. [26] for a technical definition) with local chart x over the isoreals $\hat{R}$, and let $T^*\hat{M}(\hat{R})$ be its *"isocotangent bundle"*, that is, the bundle of isoforms as more appropriately defined below. Introduce one of the infinitely possible, symmetric, nonsingular and real-valued isounits of Class I of the same dimension of M(R),

$$\hat{1} = \hat{1}(x, \dot{x}, \ddot{x}, ...) = (\hat{1}^i{}_j) = (\hat{1}_j{}^i) = (\hat{1}^j{}_i) = (\hat{1}_i{}^j) = T^{-1}. \qquad (5.4.3)$$

For mathematical consistency (e.g., to preserve isolinearity, see Sect. I.4.2), conventional linear transformations on T*M(R), $x' = Ax$, or $x'^i = A^i{}_j x^j$, must be generalized on $T^*\hat{M}(\hat{R})$ into the *isotransformations*

$$x' = A * x, \qquad \text{or} \qquad x' = A^i{}_r T^r{}_s x^s. \qquad (5.4.4)$$

In the conventional case, the differentials dx and dx' of the two coordinate systems are related by the familiar expression $dx' = Adx$, or $dx'^i = A^i{}_j dx^j$, with consequential known properties, e.g., for coordinates transformations [4,6].

However, the same differentials dx and dx' are inapplicable in the isocotangent bundle $T^*\hat{M}(\hat{R})$. The author therefore introduced the generalized notion of *isodifferentials* $\hat{d}x$ and $\hat{d}\bar{x}$ [15] which hold when interconnected by the isotopic laws

$$\hat{d}\bar{x} = A * \hat{d}x, \quad \text{or} \quad \hat{d}\bar{x}^i = A^i{}_r T^r{}_s \hat{d}x^s, \qquad (5.4.5)$$

with the particular realization, say, for the case of the isotransformations $x \Rightarrow \bar{x}(x)$

$$\hat{d}\bar{x} = \frac{\partial \bar{x}}{\partial x} * \hat{d}x, \quad \text{or} \quad \hat{d}\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^r} T^r{}_s \hat{d}x^s. \qquad (5.4.6)$$

Note that we have used the symbol $*$ for the isotopic product (in lieu of the symbol $\hat{\times}$ used earlier) which will be preserved in this section, including for the characterization of isofield herein denoted $\hat{R}(\hat{n},+,*)$.

As we shall see in the next chapter, two possibilities are significant for the characterization of the isodifferentials, $\hat{d}x = dx$ or $\hat{d}x = Tdx$ [40]. They are connected with the corresponding isointegrals $\hat{\int}\hat{d}x = \hat{x} \in \hat{R}$ or $\hat{\int}\hat{d}x = x \in R$. In this chapter we shall assume the former for simplicity and study the latter elsewhere.

Let $\phi(x)$ be a scalar function on $T^*\hat{M}(\hat{R})$. Then its isodifferential is given by

$$\hat{d}\phi = \frac{\partial\phi}{\partial x} * \hat{d}x, \quad \text{or} \quad \hat{d}\phi(x) = \frac{\partial\phi}{\partial x^r} T^r{}_s \, \hat{d}x^s. \tag{5.4.7}$$

where the partial derivative is the conventional one.

Similarly, a contravariant *isovector-field* $X = (X^i)$ on $T^*\hat{M}(\hat{R})$ is an ordinary vector-field although defined on an isospace. Then its isodifferential is given b

$$\hat{d}X = \frac{\partial X}{\partial x} * \hat{d}x, \quad \text{or} \quad \hat{d}X^i = \frac{\partial X^i}{\partial x^r} T^r{}_s \, \hat{d}x^s, \tag{5.4.8}$$

Thus, an isovector-field on $T^*\hat{M}(\hat{\mathfrak{R}})$ transforms according to the isotopic laws

$$\overline{X}(\overline{x}) = \frac{\partial\overline{X}}{\partial x} * X(x), \quad \text{or} \quad \overline{X}^i(\overline{x}) = \frac{\partial\overline{X}^i}{\partial x^r} T^r{}_s(x)\, X^s(x). \tag{5.4.9}$$

Note that, while for conventional transformations $dx' = Adx$ on $T^*M(x,R)$ we have $\partial x'/\partial x = A$, and thus we now have for isotransformations

$$\frac{\partial\overline{x}^i}{\partial x^j} = A^i{}_r T^r{}_j + A^i{}_r \frac{\partial T^r{}_s}{\partial x^j} x^s. \tag{5.4.10}$$

By using the above results and the usual chain rule for partial differentiation, one easily gets

$$\frac{\partial\overline{X}^j}{\partial\overline{x}^k} = \frac{\partial^2\overline{x}^j}{\partial x^s\,\partial x^i}\frac{\partial x^s}{\partial\overline{x}^k} T^i{}_r X^r + \frac{\partial\overline{x}^j}{\partial x^i}\frac{\partial x^s}{\partial\overline{x}^k} \text{Tir} \frac{\partial X^r}{\partial x^s} + \frac{\partial\overline{x}^j}{\partial x^i}\frac{\partial x^s}{\partial\overline{x}^k}\frac{\partial T^i{}_r}{\partial x^s} X^r. \tag{5.4.11}$$

Thus, in addition to the isotopy of the conventional two terms of this

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[40] This is the alternative formulation of the isodifferential calculus studied in ref.s [44,45].

expression (see ref. [4], Eq.s (3.5), p. 67), we obtain an additional third term. Note that the quantity $\partial\overline{X}_j/\partial\overline{x}^k$ is not a mixed tensor of rank (1.1), exactly as it happens in the conventional case.

From the preceding results one can then compute the isodifferential of a contravariant isovector-field

$$\hat{\partial}\,\overline{X}^j = \frac{\partial\overline{X}^j}{\partial\overline{x}^k}\,T^k{}_r\,\hat{\partial}\overline{x}^r =$$

$$= \frac{\partial T^i{}_r}{\partial x^s\,\partial x^i}\,T^i{}_r\,X^r\,\hat{\partial}x^s + \frac{\partial^2\overline{x}^j}{\partial x^i}\,T^i{}_r\,\frac{\partial\overline{x}^j}{\partial x^s}\,\hat{\partial}x^s + \frac{\partial X^r}{\partial x^i}\,\frac{\partial\overline{x}^j}{\partial x^s}\,X^r\,\hat{\partial}x^s\,. \quad (5.4.12)$$

A *contravariant isotensor* X^{ij} of rank two on $\hat{M}(\hat{R})$ is evidently characterized by the transformation laws

$$\overline{\overline{X}}^{(2)}(\overline{x}) = \frac{\partial\overline{x}}{\partial x} * \frac{\partial\overline{x}}{\partial x} * X^{(2)}(x), \quad \overline{X}^{ij}(\overline{x}) = \frac{\partial\overline{x}^i}{\partial x^r}\,T^r{}_p\,\frac{\partial\overline{x}^j}{\partial x^s}\,T^s{}_q\,X^{pq}(x), \quad (5.4.13)$$

Similar extensions to higher orders, as well as to contravariant isotensors of rank (0.s) and to generic tensors of rank (r.s) are left as an exercise for the interested reader.

All preceding expressions (5.4.4)–(5.4.13) have been written in both, the abstract form and their realization in local coordinates, to illustrate that the notion of isotransformations and isodifferentials do constitute isotopies, in the sense that all distinctions between conventional and isotopic notions cease to exist at the abstract, realization-free level.

For the identification of the *isodual isodifferential calculus* [15,19] recall that, under isodualities we have

$$\hat{\partial}\overline{x}^d = A^d *^d \hat{\partial}x^d = A^d * \hat{\partial}x\,. \quad (5.4.14)$$

The rest of the isocalculus can then be easily derived.

To have a guide in the use and meaning of isoduality, the reader should keep in mind that its primary classical function at this level is the characterization of the map from positive to negative energies. But energies are represented by the Hamiltonian H. A good guide for isodualities is therefore the map

$$\text{Energy } H > 0 \quad \stackrel{\text{isoduality}}{\Rightarrow} \quad \text{Energy } H^d = -H < 0\,. \quad (5.4.15)$$

As it has been the case for all preceding aspects studied so far, we have four distinguishable formulations:

A) the **conventional differential calculus** over the ordinary reals $R(n,+,\times)$ with basic transformation law $dx' = A\times dx$ which is and will remain the fundamental calculus for the exterior problem of ordinary matter in vacuum;

B) the **isodual differential calculus** over the isodual reals $R^d(n^d,+,\times^d)$ with law $dx' = A\times^d dx = A^d dx$ which is assumed the basic calculus for antimatter also in vacuum;

C) *the* **isodifferential calculus** over the isoreals $\hat{R}(\hat{n},+,*)$ with law $\hat{d}\overline{x} = A * \hat{d}x$ which is assumed as the basic calculus for the interior dynamical problem of matter; and

D) *the* **isodual isodifferential calculus** over the isodual isoreals $\hat{R}^d(\hat{n}^d,+,*^d)$ which is assumed as the basic calculus for the interior problem of antimatter.

The reader should keep in mind that all the above formulations can be unified by the abstract isotopic treatment of Class III, although in these volumes we shall study the individual formulations for clarity.

5.4C: Isoforms and their isoduals. The isotopies of the symplectic geometry of Class I were constructed [15] via the use of the isodifferential calculus, which permits the introduction of the following *one-isoform*

$$\hat{\Phi}_1 = A * \hat{d}x = A_i\, T^i_{\ j}\, \hat{d}x^j. \tag{5.4.16}$$

and the study of the algebraic operations on them. The *isocotangent bundle* $T^*\hat{M}_1(\hat{R})$ is then the bundle of all possible one-isoforms. The sum of two one-isoforms $\Phi_1{}^1 = A*\hat{d}x$ and $\hat{\Phi}_1{}^2 = B*\hat{d}x$ is the conventional expression

$$\hat{\Phi}_1{}^1 + \hat{\Phi}_1{}^2 = (A+B) * \hat{d}x. \tag{5.4.17}$$

The isoproduct of one-isoform $\hat{\Phi}_1 = A*\hat{d}x$ with an isonumber $\hat{n} \in \hat{R}$ is the conventional product,

$$\hat{n} * \hat{\Phi}_1 \equiv n\, \hat{\Phi}_1. \tag{5.4.18}$$

For the product of two or more one-isoforms $\hat{\Phi}_1{}^k = A^k*\hat{d}x$, $k = 1, 2, 3, \ldots$we introduce the *isoexterior*, or *isowedge product* denoted with the symbol $\hat{\wedge}$, which verifies the same axioms of the conventional exterior product, that is, the distributive laws

$$(\hat{\Phi}_1{}^1 + \hat{\Phi}_1{}^2)\,\hat{\wedge}\,\hat{\Phi}_1{}^3 = \hat{\Phi}_1{}^1\,\hat{\wedge}\,\hat{\Phi}_1{}^3 + \hat{\Phi}_1{}^2\,\hat{\wedge}\,\hat{\Phi}_1{}^3, \tag{5.4.19a}$$

$$\hat{\Phi}_1{}^1\,\hat{\wedge}\,(\hat{\Phi}_1{}^2 + \Phi_1{}^3) = \hat{\Phi}_1{}^1\,\hat{\wedge}\,\hat{\Phi}_1{}^2 + \hat{\Phi}_1{}^1\,\hat{\wedge}\,\hat{\Phi}_1{}^3, \tag{5.4.19b}$$

and the antisymmetry law

$$\hat{\Phi}_1{}^1 \wedge \hat{\Phi}_1{}^2 = - \Phi_1{}^2 \hat{\wedge} \hat{\Phi}_1{}^1. \qquad (5.4.20)$$

although it is defined on an isomanifold.

The isoproduct of two one-isoforms $\hat{\Phi}_1{}^1 = A{*}\hat{d}x$ and $\hat{\Phi}_1{}^2 = B{*}\hat{d}x$ is the *two-isoform*

$$\hat{\Phi}_2 = \hat{\Phi}_1{}^1 \hat{\wedge} \hat{\Phi}_1{}^2 = A_i\, T^i{}_r\, B_j\, T^j{}_s\, \hat{d}x^r \hat{\wedge} \hat{d}x^s =$$

$$= \tfrac{1}{2}(A_i\, T^i{}_r\, B_j\, T^j{}_s - A_i\, T^i{}_s\, B_j\, T^j{}_r)\, \hat{d}x^r \hat{\wedge} \hat{d}x^s =$$

$$= \tfrac{1}{2} A_i\, B_j\, (T^i{}_r\, T^j{}_s - T^i{}_s\, T^j{}_r)\, \hat{d}x^r \hat{\wedge} \hat{d}x^s, \qquad (5.4.21)$$

which characterizes the isocotangent bundle $T^*\hat{M}_2(x,\hat{R})$. Note the clear deviations from the conventional exterior calculus (compare with ref. [4], p. 132).

The isoexterior product of three one-isoforms yields the *three-isoform*

$$\hat{\Phi}_3 = \hat{\Phi}_1{}^1 \wedge \hat{\Phi}_1{}^2 \hat{\wedge} \hat{\Phi}_1{}^3 =$$

$$= A^1{}_{i_1}\, A^2{}_{i_2}\, A^3{}_{i_3}\, \delta^{j_1 j_2 j_3}{}_{k_1 k_2 k_3}\, T^{i_1}{}_{j_1}\, T^{i_2}{}_{j_2}\, T^{i_3}{}_{j_3}\, \hat{d}x^{k_1} \wedge \hat{d}x^{k_2} \hat{\wedge} \hat{d}x^{k_3}, \qquad (5.4.22)$$

where [4,6]

$$\delta^{i_1 i_2}{}_{j_1 j_2} = \det\begin{pmatrix} \delta^{i_1}{}_{j_1} & \delta^{i_1}{}_{j_2} \\ \delta^{i_2}{}_{j_1} & \delta^{i_2}{}_{j_2} \end{pmatrix}, \quad \delta^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \det\begin{pmatrix} \delta^{i_1}{}_{j_1} & \delta^{i_1}{}_{j_2} & \delta^{i_1}{}_{j_3} \\ \delta^{i_2}{}_{j_1} & \delta^{i_2}{}_{j_2} & \delta^{i_2}{}_{j_3} \\ \delta^{i_3}{}_{j_1} & \delta^{i_3}{}_{j_2} & \delta^{i_3}{}_{j_3} \end{pmatrix}, \qquad (5.4.23)$$

with a consequential extension to *p–isoforms*

$$\hat{\Phi}_p = A_{i_1 i_2 \ldots i_p}\, T^{i_1}{}_{j_1}\, T^{i_2}{}_{j_2} \times\times\times\, T^{i_p}{}_{j_p}\, \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} \hat{\wedge} \ldots \hat{\wedge} \hat{d}x^{j_p}, \qquad (5.4.24)$$

characterizing the corresponding isocotangent bundle $T^*\hat{M}_p(\hat{R})$.

Given n one-isoforms $\hat{\Phi}_1{}^k = A^k{*}\hat{d}x$, k = 1, 2,... n, they are said to be *isolinearly dependent* when $\hat{\Phi}_1{}^1 \hat{\wedge} \ldots\ldots \hat{\wedge} \hat{\Phi}_1{}^n \equiv 0$. Note that given n one-isoforms linearly dependent on $M(x,\Re)$, they can be isolinearly independent, evidently because of the functional dependence of the isotopic product.

In an n–dimensional isomanifold $\hat{M}(\hat{R})$ there exist a maximum of n linearly independent one–isoforms as in the conventional case, with isobasis $\hat{d}x^1, \ldots, \hat{d}x^n$. The isobasis of $T^*\hat{M}_2(\hat{R})$ are then given by the ordered set $\hat{d}x^i \hat{\wedge} \hat{d}x^j$, $i < j$. A similar situation occurs for p–isoforms and related isomanifolds $T^*\hat{M}_p(\hat{R})$.

As an incidental note we point out without treatment the *Grassmann-isotopic algebra* $\hat{G}$, or *isograssmann algebra*, which is given by the direct sum

[15]

$$\hat{G} = \sum_{k=0,1,2,\ldots,n} T^* \hat{M}_k(\hat{R}). \qquad (5.4.25)$$

The necessary and sufficient conditions for a two–isoform to be identically null are

$$\delta^{i_1 i_2}{}_{j_1 j_2} A^1{}_{k_1} A^2{}_{k_2} T^{k_1}{}_{i_1} T^{k_2}{}_{i_2} = A^1{}_{k_1} A^2{}_{k_2} (T^{k_1}{}_{i_1} T^{k_2}{}_{i_2} - T^{k_1}{}_{i_2} T^{k_2}{}_{i_1}) \equiv 0. \qquad (5.4.26)$$

A similar situation occurs for p–isoforms.

The reader should keep in mind the nontriviality of the above liftings. As an example, the linear, local and canonical one– and two–forms are lifted into the respective structures

$$\hat{\Phi}_1 = A_i\, T^i{}_j(x, \dot{x}, \ddot{x}, \mu. \tau, n, \ldots)\, \hat{d}x^j\,, \qquad (5.4.27a)$$

$$\hat{\Phi}_{\hat{2}} = A_i\, T^i{}_r(x, \dot{x}, \ddot{x}, \mu, \tau, n, \ldots B_j\, T^j{}_s(x, \dot{x}, \ddot{x}, \mu, \tau, n, \ldots)\, \hat{d}x^r \hat{\wedge} \hat{d}x^s \qquad (5.4.27b)$$

which are nonlinear (in the local coordinates x as well as their derivatives), nonlocal–integral (in all variables) and noncanonical when projected in the original manifold M(R). What is remarkable is that the above forms are *isolinear, isolocal and isocanonical* (Sect. 4.2) on $\hat{M}(\hat{R})$, that is, they coincide with the corresponding conventional forms at the abstract level despite the indicated differences. Perhaps, this abstract unity is the reason why the isosymplectic geometry has been discovered only recently.

It is evident that all the above quantities admit a new image under isoduality. To begin, the basic manifold M(R) is now mapped into the isodual manifold $M^d(R^d)$ with the isobasis $dx^{1d}, \ldots, dx^{nd}$ over R^d. We therefore have the *isodual one–forms*

$$\theta^d = A^d \times^d d\, x^d\,, \qquad (5.4.28)$$

and the isodual operations in them. We then have th *isodual one–isoforms*

$$\hat{\theta}^d = A \hat{*}^d \hat{d}x = - \hat{\theta}\,, \qquad (5.4.29)$$

and isodual isooperations on them.

5.4D: Isotopies and isodualities of the Poincare´ lemma. The Poincaré lemma (see. e.g., ref.s [3,4,6) has a particular mathematical and physical meaning inasmuch as it establishes that the symplectic geometry is the geometry

underlying Lie's theory. For the case of two–forms on a 2n–dimensional manifold

$$d\omega = d\,[\,\omega_{ij}\, dx^i \wedge dx^j\,] = d(d\theta) \equiv 0, \qquad (5.4.30)$$

the Poincaré lemma provides the integrability conditions for the brackets characterized by the contravariant tensor $\omega^{ij} = |\,(\,\omega_{rs}\,)^{-1}\,|^{ij}$

$$[\,A\,,B\,] = \frac{\partial A}{\partial x^i}\;\omega^{ij}\;\frac{\partial A}{\partial x^j}, \qquad (5.4.31)$$

to be Lie. Thus, the rather complex integrability conditions for brackets (5.4.31) to be Lie (see, e.g., the detailed study in ref. [6]) are reduced to the simple and elegant geometric property $d\omega = d(d\theta) \equiv 0$.

A central objective of memoir [15] was to show that a similar situation occurs under isotopy, namely; that the isosymplectic geometry is the correct integro–differential geometry underlying the Lie–isotopic algebras. This property was established by showing that the Poincaré lemma is a true geometric axiom because it persists under isotopies.

To review this result, let us first study the isodifferential calculus of p–isoforms. Let $\hat\Phi_1 = A{*}\hat d x$ be a one–isoform. The *isoexterior derivative* $\hat d\hat\Phi_1$ of $\hat\Phi_1$ (also called *isoexterior differential*) is the two–isoform [loc. cit.]

$$\hat\Phi_2 = \hat d\hat\Phi_1 = \frac{\partial\,(A_{i_1}\, T^{i_1}{}_{j_1})}{\partial x^{i_2}}\, T^{i_2}{}_{j_2}\, \hat d x^{j_1} \hat\wedge \hat d x^{j_2} = \qquad (5.4.32)$$

$$= (\frac{\partial A_{i_1}}{\partial x^{i_2}}\, T^{i_1}{}_{j_1}\, T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_2}}\, T^{i_2}{}_{j_2}\,)\, \hat d x^{j_1} \hat\wedge \hat d x^{j_2} =$$

$$= \tfrac{1}{2}\delta^{j_1 j_2}{}_{k_1 k_2}\, (\frac{\partial A_{i_1}}{\partial x^{i_2}}\, T^{i_1}{}_{j_1}\, T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j}}{\partial x^{i_2}}\, T^{i_2}{}_{j_2}\,)\, \hat d x^{k_1} \hat\wedge \hat d x^{k_2}$$

from which one can see that $\hat d\hat\Phi_1$ is no longer the curl of the vector field A_{i_1}, but the more general *isocurl* encountered in Sect. 5.2.

The isoexterior derivative of a two–isoform (5.2.32) is given by the three–isoform

$$\hat\Phi_3 = \hat d\hat\Phi_2 = (\frac{\partial A_{i_1 i_2}}{\partial x^{i_3}}\, T^{i_1}{}_{j_1}\, T^{i_2}{}_{j_2}\, T^{i_3}{}_{j_3} + A_{i_1 i_2} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_3}}\, T^{i_2}{}_{j_2}\, T^{i_3}{}_{j_3} +$$

$$+ A_{i_1 i_2}\, T^{i_1}{}_{j_1} \frac{\partial T^{i_2}{}_{j_2}}{\partial x^{i_3}}\, T^{i_3}{}_{j_3}\,)\, \hat d x^{j_1} \hat\wedge \hat d x^{j_2} \wedge \hat d x^{j_3}. \qquad (5.4.33)$$

∂x^{i_3}

It is easy to see that the isoexterior derivative of the isoexterior product of a p–isoform Φ_p and a q–isoform $\hat{\Phi}_q$ is given by

$$\hat{\partial}(\hat{\Phi}_p \hat{\wedge} \hat{\Phi}_q) = (\hat{\partial}\hat{\Phi}_p) \hat{\wedge} \hat{\Phi}_q + (-1)^p \hat{\Phi}_p \hat{\wedge} (\hat{\partial}\hat{\Phi}_q). \qquad (5.4.34)$$

A p–isoform $\hat{\Phi}_p$ is then said to be *isoexact* when there exists a (p–1)–isoform $\hat{\Phi}_{p-1}$ such that $\hat{\Phi}_p = \hat{\partial}\Phi_{p-1}$, and *isoclosed* when $\hat{\partial}\hat{\Phi}_p \equiv 0$. We are thus equipped to formulate the following important

Lemma 5.4.1 - Isotopic Poincare´ lemma [15,19]: *The Poincaré Lemma admits an infinite number of isotopic liftings of Class I, i.e., given an exact p–form* $\Phi_p = d\Phi_{p-1}$, *there exists an infinite number of isotopies*

$$\Phi_{p-1} \Rightarrow \hat{\Phi}_{p-1}, \qquad \Phi_p = d(\Phi_{p-1}) \Rightarrow \hat{\Phi}p = \hat{\partial}(\hat{\Phi}_{p-1}), \qquad (5.4.35)$$

for each of which the isoexterior derivative of the isoexact p–isoforms are identically null,

$$\hat{\partial}(\hat{\partial}\hat{\Phi}_{p-1}) \equiv 0. \qquad (5.4.36)$$

The proof is an instructive exercise for the reader interested in acquiring a knowledge of the isotopic techniques. We merely note that $d\hat{\Phi}_1 \equiv 0$, iff

$$\tfrac{1}{2}\delta^{j_1 j_2}{}_{k_1 k_2} \left(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j}}{dx^{i_2}} T^{i_2}{}_{j_2} \right) \equiv 0, \qquad (5.4.37)$$

namely, the isoclosure of a one–isoform does not imply that the conventional curl of the vector A is null, but that the isocurl is null.

Similarly, given a exact two–isoform $\Phi_2 = \hat{\partial}\hat{\Phi}_1$, the property $\hat{\partial}\hat{\Phi}_2 \equiv 0$ holds iff

$$\delta^{j_1 j_2 j_3}{}_{k_1 k_2 k_3} \left(\frac{\partial^2 A_{i_1}}{\partial x^{i_2} \partial x^{i_3}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + \frac{\partial A_{i_1}}{\partial x^{i_2}} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_3}} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + \right.$$

$$\left. + \frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} \frac{\partial T^{i_2}{}_{j_2}}{\partial x^{i_3}} T^{i_3}{}_{j_3} \right) \equiv 0. \qquad (5.4.38)$$

Thus, the abstract axioms $d\Phi_2 = \hat{\partial}(\hat{\partial}\hat{\Phi}_1) \equiv 0$, $d\Phi_3 = \hat{\partial}(d\Phi_2) \equiv 0$, etc., admit the conventional linear–local–canonical realization based on an ordinary manifold, as well as an infinite number of additional, nonlinear–nonlocal–

noncanonical realizations for each given original form, via covering isomanifolds. The latter realizations are geometrically equivalent among themselves, but physically inequivalent owing to the generally different isotopic elements or isounits.

The *isodualities of the Poincaré lemma* can now be easily formulated. We first have the isodual Poincaré lemma which is characterized by the isodual calculus, and then the isodual isopoincaré lemma based on the isodual isodifferential calculus.

5.4E- Isosymplectic geometry and its isodual. Let us review the interplay between exact symplectic two-forms and Lie-isotopic algebras. Recall (see ref. [6] for details) that the most general possible, *local-differential* and conventional two-form on an even, 2n-dimensional manifold $T^*M_2(R)$ with covariant geometric tensor $\Omega_{i_1 i_2}$

$$\Phi_2 = \tfrac{1}{2}\Omega_{i_1 i_2}(x)\, dx^{i_1} \wedge dx^{i_2}, \tag{5.4.39}$$

characterizes, in its corresponding contravariant version, the brackets among functions A(x) and B(x) on $T^*M_2(R)$

$$[A \hat{,} B] = \frac{\partial A}{\partial x^{i_1}} \Omega^{i_1 i_2} \frac{\partial B}{\partial x^{i_2}}, \qquad \Omega^{i_1 i_2} = (|\Omega_{j_1 j_2}|^{-1})^{i_1 i_2}. \tag{5.4.40}$$

Now, the integrability conditions for two-form (5.4.39) to be an exact symplectic two-form are given by [loc. cit]

$$\Omega_{i_1 i_2} + \Omega_{i_2 i_1} \equiv 0, \qquad \frac{\partial \Omega_{i_1 i_2}}{\partial x^{i_3}} + \frac{\partial \Omega_{i_2 i_3}}{\partial x^{i_1}} + \frac{\partial \Omega_{i_3 i_1}}{\partial x^{i_2}} \equiv 0, \tag{5.4.41}$$

which general solution in terms of 2n functions $R_i(x)$

$$\Phi_2 = d\,[\,R_i(x)\, dx^i\,], \qquad \Omega_{j_1 j_2} = \frac{\partial R_{j_1}}{\partial x^{j_2}} - \frac{\partial R_{j_2}}{\partial x^{j_1}}, \tag{5.4.42}$$

characterizing the *Birkhoffian generalization of Hamiltonian mechanics* [5,6]. The above conditions are equivalent to the integrability conditions

$$\Omega^{i_1 i_2} + \Omega^{i_2 i_1} \equiv 0, \tag{5.4.43a}$$

$$\Omega^{i_1 k} \frac{\partial \Omega^{i_2 i_3}}{\partial x^k} + \Omega^{i_2 k} \frac{\partial \Omega^{i_3 i_1}}{\partial x^k} + \Omega^{i_3 k} \frac{\partial \Omega^{i_1 i_2}}{\partial x^k} \equiv 0, \tag{5.4.43b}$$

for generalized brackets (5.4.40) to be Lie–isotopic, i.e., to verify the Lie algebra axioms in their most general possible, classical, regular realization on $T^*M_2(R)$

$$[A \hat{,} B] + [B \hat{,} A] \equiv 0, \quad [[A \hat{,} B] \hat{,} C] + [[B \hat{,} C] \hat{,} A] + [[C \hat{,} A] \hat{,} B] \equiv 0. \qquad (5.4.44)$$

Thus, the exact character of the general two–form $\Phi_2 = d\Phi_1$ implies its closure $d\Phi_2 \equiv 0$ (Poincaré Lemma), which, in turn, guarantees that the underlying brackets are Lie–isotopic, with the canonical case being a trivial particular case (see the analytic, algebraic, and geometric proofs in ref. [6], Sect. 4.1.5).

Lemma 5.4.1 establishes that the above general but local–differential interplay between algebra and geometry persists under the most general possible nonlocal–integral isotopies. We therefore have the following:

Definition 5.4.1 [15,19]: *The "general exact isosymplectic manifolds" of Class I are 2n–dimensional isomanifolds $T^*\hat{M}_2(x,\hat{R})$ over the isofields $\hat{R}(\hat{n},+,*)$ with isounits $\hat{1}_2$ equipped with an isoexact and nowhere degenerate two–isoform*

$$\Phi_2 = \tfrac{1}{2}\,\Omega_{j_1 j_2}(x,\dot{x},\ddot{x},...)\,\hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} = \hat{d}\Phi_1 = \frac{\partial (A_{i_1} T^{i_1}{}_{j_1})}{\partial x^{i_2}} T^{i_2}{}_{j_2}\, \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} =$$

$$= \Big(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \Big)\, \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} =$$

$$= \tfrac{1}{2}\delta^{j_1 j_2}{}_{k_1 k_2} \Big(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \Big)\, \hat{d}x^{k_1} \hat{\wedge} \hat{d}x^{k_2}, = \qquad (5.4.45)$$

which is such to admit the factorization

$$\Phi_{2_2} = \Omega_{i_1 k}(x) \times T_2{}^{k}{}_{i_2}(x, \dot{x}, \ddot{x}, ...)\, \hat{d}x^{i_1} \hat{d}x^{i_2}, \qquad \hat{T}_2 > 0, \qquad (5.4.46)$$

where T_2 is the isotopic element of the underlying isofield, i.e., it is such that $\hat{1}_2 = T_2^{-1}$ and

$$\Omega_{i_1 i_2} = \frac{\partial A_{i_2}}{\partial x^{i_1}} - \frac{\partial A_{i_1}}{\partial x^{i_2}}, \qquad (5.4.47)$$

is Birkhoff's tensor [6], i.e., the most general possible local, exact symplectic tensor. The corresponding Lie–Santilli isotheory is then characterized by the brackets

∂A ∂B

$$[A \hat{,} B] \;=\; \frac{}{\partial x^{i_1}} \hat{1}_2{}^{i_1}{}_k(x, \dot{x}, \ddot{x}, ...)\, \Omega^{ki_2}(x) \frac{}{\partial x^{i_2}}, \tag{5.4.48a}$$

$$\hat{1}_2 \;=\; \hat{T}_2{}^{-1}, \;(\hat{\Omega}^{i_1 i_2}) \;=\; (\|\Omega_{k_1 k_2}\|^{-1}), \tag{5.4.48b}$$

where $\hat{1}_2 = \hat{T}_2^{-1}$ *is the isounit of the universal enveloping isoassociative algebra.*

The "isocanonical isosymplectic isomanifolds" are the same manifolds as above, except that the two-isoform is reducible to the isocanonical form

$$\hat{\omega}_{2_2} \;=\; \omega_{i_1 k} \times T_2{}^k{}_{i_2}(x, \dot{x}, \ddot{x}, ...)\, \hat{d}x^{i_1}\, \hat{d}x^{i_2}, \qquad \hat{T}_2 > 0, \tag{5.4.49}$$

where ω *is the conventional symplectic tensor. The corresponding Lie–Santilli isotheory is then characterized by the brackets*

$$[A \hat{,} B] \;=\; \frac{\partial A}{\partial x^{i_1}} \hat{1}_2{}^{i_1}{}_k(\, x, \dot{x}, \ddot{x}, ...)\, \omega^{ki_2} \frac{\partial B}{\partial x^{i_2}}, \tag{5.4.50a}$$

$$\hat{1}_2 \;=\; \hat{T}_2{}^{-1}, \;(\hat{\omega}^{i_1 i_2}) \;=\; (\|\omega_{k_1 k_2}\|^{-1}), \tag{5.4.50b}$$

The "isosymplectic geometry" is the geometry of the general and isocanonical isosymplectic manifolds.

The "exact isodual isosymplectic manifolds" are defined by the isodual exact two-isoforms

$$\Phi^d{}_2 = \Omega_{i_1 k}(x) \times T^d{}_2{}^k{}_{i_2}(s, x, \dot{x}, \ddot{x}, ...)\, \hat{d}x^{i_1} \hat{\wedge} \hat{d}x^{i_2}, \quad \hat{T}^d{}_2 < 0, \tag{5.4.51}$$

which are now defined on the "isodual isocotangent bundle" $T^*\hat{M}^d(\hat{R}^d)$ *over the isodual isofield* $\hat{R}^d(\hat{n}^d,+,*^d)$ *with isodual isounit* $\hat{1}^d{}_2 = (T^d{}_2)^{-1} = -\hat{1}_2$.

Note the complete lack of restriction in the functional dependence of the isotopic element T_2 which is at the foundation of the "direct universality" of the isogeometry for all possible nonlinear, nonlocal and noncanonical systems (the "direct universality" for the nonlinear and noncanonical but local systems was proved in ref. [6]).

Note also that factorization (5.4.46) is possible for all two-isoforms (5.4.45). In the above definition one can either pre-assign an isounit $\hat{1}_2$ and then select the two-isoforms (5.4.46) verifying the condition $\hat{1}_2 = T_2^{-1}$, or pre-assign the two-isoform (5.4.46) and then select the isounit $\hat{1}_2$ accordingly. In this way, *all two-isoforms whose antisymmetric tensor* Ω_{ij} *is symplectic can always be interpreted as characterizing an isosymplectic manifold.* As a matter of fact, this is an illustration of the existence of the infinite variety of isotopies $\hat{R}(\hat{n},+,*)$ of the field of real numbers $R(n,+,\times)$.

The isosymplectic geometry focuses the attention on a subtle aspect which

is absent in the conventional formulation of the geometry [3,4]: *the relationship between the two-forms and the underlying unit.* For the conventional Hamiltonian (or Birkhoffian) case, the underlying unit is the unit of the enveloping associative algebra of the related Lie algebra. As such it is the 2n-dimensional unit

$$I \;=\; (I^i_{\;j}) \;=\; (I_i^{\;j}) \;=\; \mathrm{Diag.}\,(1, 1, \ldots, 1) \quad (2n\ \mathrm{dim})\,. \tag{5.4.52}$$

which is trivially symmetric. The most general possible symplectic two-form is then characterized in a local chart by *two* tensors, the totally antisymmetric symplectic tensor Ω_{ij} and the totally symmetric one $I^{-1} = \mathrm{diag.}\,(1, 1, \ldots, 1) \equiv I$

$$\Omega \;=\; d\Theta = \Omega_{i_1 k}(x)\, I^k_{\;i_2}\, dx^{i_1} \wedge dx^{i_2}\,, \tag{5.4.53}$$

In the transition to an arbitrary isounit $\hat{1}_2$ the symplectic tensor Ω_{ij} is preserved, but the totally symmetric tensor $I = (I^i_{\;j})$ is lifted for mathematical consistency into the isotopic form $T_2 = (T_2{}^i_{\;j}) = \hat{1}_2^{-1}$ or, equivalently, the totally symmetric tensor in the factorization (5.4.46) must be interpreted as the isotopic element of the related enveloping associative algebra.

The geometrical and physical implications of the above isotopies and isodualities are intriguing, and it is hoped that they will received a much needed attention by geometers. As an example, it has been assumed until now in differential geometry that the only possible degeneracy is that in the symplectic tensor, e.g.,

$$\mathrm{Det}\,\Omega(x) \;\equiv\; 0. \tag{5.4.54}$$

in which case one evidently lose the symplectic character of the geometry and the possibility to characterize a corresponding Lie algebra owing to the impossibility to perform the transition to the contravariant tensor Ω^{ij}.

The isotopies imply the existence of a second "hidden" degeneracy, that of the isotopic element

$$\mathrm{Det}\,T_2(x) \;\equiv\; 0. \tag{5.4.55}$$

which the symplectic tensor is nondegenerate, det $\Omega \neq 0$, which characterizes the *isosymplectic geometry of Class IV.* This latter form of isogeometry represents gravitational collapse into a singularity at x and, as such, need suitable study.

Note that the primitive notion here is that of *isonumbers with a singular unit.* The degeneracy of the geometry is only consequential.

The generalized analytic equations characterized by the isosymplectic geometry will be identified in Vol. II, jointly with explicit examples.

6.EF: Direct universality of the isosymplectic geometry. In monograph [6] this author proved the "direct universality" of the (conventional) symplectic geometry with an exact, but otherwise general noncanonical-Birkhoffian two form for the characterization of all possible local-differential, analytic and regular Newtonian systems (universality) in a star-shaped neighborhood of the *fixed* variables of the experimenter (direct universality).

The underlying action resulted to be of the following general Pfaffian first-order type

$$A = \int_{t_1}^{t_2} dt\, [\, A_k(x)\, v^k \; - \; B(t, x)\,] =$$

$$= \int_{t_1}^{t_2} dt\, [\, P_\alpha(r, p)\, \dot{r}^\alpha \; + \; Q_\beta(r, p)\, \dot{p}^\beta - \; B(t, r, p)\,] \qquad (5.4.56)$$

$$A = \{ A_k \} = \{ P_\alpha, Q_\beta \} \;\; x = \{x^k \; \{ \; r^\alpha, p^\beta\} , \;\; k = 1, 2, ..., 2N, \;\; \alpha, \beta = 1, 2, ..., N,$$

with corresponding Birkhoff's equations

$$\Omega_{ik}(x)\, dx^k/dt \; = \; \partial\, B(t, x) \,/\, \partial x^i \, , \qquad (5.4.57)$$

where B is a function (generally different than the Hamiltonian) called the *Birkhoffian.*

Theorem 4.5.1, p. 54, ref. [6] essentially proved that, given a vector field $\Xi^k(t, x)$ which is local-differential, analytic and regular but otherwise arbitrarily nonhamiltonian in T^*Mx, R), there always exists a star-shaped region of the variables t, x in which Ξ^k is a *Birkhoffian vector field,* i.e., there always exist a function B(t, x) and an exact symplectic tensor $\Omega_{ij}(x)$ such that all the following identities are verified (see also App. 5.A for more details),

$$\Omega_{ik}(x)\, \Xi^k t, x) \; = \; \partial\, B(t, x) \,/\, \partial x^i \, , \;\; i \; = \; 1, 2, ..., 2N \, . \qquad (5.4.58)$$

Note that the direct universality was achieved in the fixed local chart t, x of the experimenter, but the canonical form ω had to be abandoned in favor of the Birkhoffian form Ω.

The studies on isotopies subsequent to ref. [6] permit a significant revision and enlargement of the above results. To begin, it is easy to see that the use of the isotopies permit the restoration of the *canonical* symplectic form ω. In fact, we have the following

Proposition 5.4.1: *Under sufficient continuity and regularity conditions, all possible Birkhoffian tensors always admit the decomposition into the antisymmetric canonical tensor and a symmetric tensor,*

$$\Omega_{ij}(x) \; = \; \omega_{ik}\, \hat{1}^k{}_j(x) \, , \qquad (5.4.59)$$

as a result, all possible general isosymplectic two-forms (5.4.46) always admit their reformulation as isocanonical two-isoforms (5.4.49) in different isounits, but in the same local coordinates.

In different terms, the isotopies permit the reformulation of Theorem 5.4.1 of ref. [6] in terms of the *canonical* symplectic tensor, but a change of the basic unit of the theory. We also have the following additional properties of self-evident proof.

Proposition 5.4.2: *Under sufficient continuity and regularity conditions, all possible, general, first-order, Pfaffian actions always admit an identical reformulation in terms of the "isocanonical action" in the same local coordinates,*

$$A = \int_{t_1}^{t_2} dt\,[\, A_k(x)\, v^k - B(t, x)\,] =$$

$$= \int_{t_1}^{t_2} dt\,[\, P_\alpha(r, p)\, \dot{r}^\alpha + Q_\beta(r, p)\, \dot{p}^\beta - B(t, r, p)\,] = \tag{5.4.60a}$$

$$= \int_{t_1}^{t_2} dt\,[\, p_\alpha * \dot{r}^\alpha - H(t, r, p)\,] = \int_{t_1}^{t_2} dt\,[\, p_\alpha\, \hat{1}_1{}^\alpha{}_\beta(r, p)\, \dot{r}^\beta - H(t, r, p)\,] = \hat{A},$$

$$\hat{1}_1 = (\hat{1}_1{}^i{}_j) = \{\hat{1}_1{}^\alpha{}_\beta\, \delta_{\alpha\beta},\ \hat{1}_1{}^\alpha{}_\beta\, \delta_{\alpha\beta}\} =$$

$$= \tfrac{1}{2}\{\, P_\alpha/p_\alpha + Q_\alpha \dot{p}^\alpha / p_\alpha \dot{r}^\alpha,\ P_\alpha/p_\alpha + Q_\alpha \dot{p}^\alpha / p_\alpha \dot{r}^\alpha \}\ \text{(no sum)}. \tag{5.4.60b}$$

In different terms, general, first-order one-forms of the conventional symplectic geometry can always be identically rewritten as a isocanonical one-isoforms in the same local variables, but different units.

Proposition 5.4.3: *The fundamental analytic equations underlying the isosymplectic geometry are the "isohamilton equations"* [15]

$$\omega_{ik}\, \hat{1}_2{}^k{}_j(x)\, dx^j / dt = \partial H(t, x) / \partial x^i, \tag{5.4.61}$$

where the isounit $\hat{1}_2$ *is derivable from the expression* $\hat{1}_1$ *of the one-isoform via the techniques of Lemma 5.4.1.*

The proof of Theorem 5.4.1 of ref. [6] and the preceding propositions then imply the proof of the following important property here presented apparently for the first time.

Theorem 5.4.1 (Direct Universality of the Isosymplectic Geometry in Isocanonical Form): *All possible analytic and regular nonlinear, nonlocal-*

integral and nonhamiltonian vector fields $\Xi^k(t, x)$ *on an isocotangent bundle* $T^*\hat{M}(x,\hat{R})$ *always admit a star-shaped neighbour of their variables in which they are "isohamiltonian", i.e., there always exist a Hamiltonian* $H(t, x)$, *and an isotopic unit* $\hat{1}_2(x)$ *for which all the following identity hold*

$$\omega_{ik}\, \hat{1}^k{}_j(x)\, \Xi^j(t, x) = \partial Hpt, x) / \partial x^i. \tag{5.4.62}$$

Note that nonlocal–integral systems, while prohibited in Theorem 5.4.1 of ref. [6] as well as for the conventional symplectic geometry at large, are admitted under isotopies because they are admitted in the isounit.

A simpler formulation of the symplectic geometry via the use of a different realization of the isodifferential calculus, with a more effective Theorem of Direct Universality, has been formulated in the recent papers [44,45] which are not reviewed at this time for brevity.

5.2G. Isodual representation of negative energies. As well known in particle physics, antiparticles are characterized by negative–energy solutions of field equations. In this section (which is evidently purely classical), we can only study the geometric characterization of the negative–energies via isodualities.

First, we simply note that a conventional Hamiltonian representing the kinetic energy over $R(n,+,\times)$, naturally becomes negative–definite when mapped into the isodual field $R^d(n^d,+,\times^d)$

$$H_R = \dot{r} \times \dot{r}\ \ 2m > 0 \ \Rightarrow\ H^d = H_{R^d} = \dot{r} \times^d \dot{r} / 2m = -\dot{r} \times \dot{r} / 2m = -H < 0 \tag{5.4.63}$$

and evidently the same holds for the isoduality of the Lagrangian

$$L_{\Re} = \tfrac{1}{2}\, m\, \dot{r} \times \dot{r} > 0 \ \Rightarrow\ L^d = L_{\Re^d} = \tfrac{1}{2}\, m\, \dot{r} \times^d \dot{r} = -\tfrac{1}{2}\, m\, \dot{r} \times \dot{r} = -L < 0\,. \tag{5.4.64}$$

Jointly, the equation of motion reads

$$m^d \times^d \ddot{r}^d \equiv -m\, \ddot{r} = 0, \tag{5.4.65}$$

with a similar result for arbitrary equations of motion (see Ch. II.1).

More generally, the *isodual Legendre transform* is given by

$$L = p \times \dot{r} - H \setminus \ \Rightarrow\ L^d = p \times^d \dot{r} - H^d = -p \times r + H\,. \tag{5.4.66}$$

The construction of the *isodual isolegendre transform* is an instructive exercise for the interested reader.

All these features of antiparticles in vacuum are directly represented by the isodual symplectic geometry. In fact, the integrand of the conventional action is precisely the one–form of the symplectic geometry

$$A = \int_{[\,,\,]} dt\, L(t, r, \dot r) = \int_{[\,,\,]} R_k(t, r) \times dr_k = \int_{[\,,\,]} \Theta . \qquad (5.4.67)$$

The property identified earlier of the change of sign of a one-form under isoduality then constitutes precisely the desired geometrization of the negative-energy solutions of field equations.

5.5: ISOAFFINE GEOMETRY

5.5A: Isoaffine spaces and their isoduals. As an intermediate step prior to the isotopies of Riemann, we shall now review the isotopies of the affine geometry introduced in in memoir [16] under the name of *affine-isotopic geometry*, or *isoaffine geometry* for short, then studied in ref.s [17–21], reviewed in ref. [26] and studied at the mathematical level in the recent papers [46,47]. This author is aware of no additional studies in the new geometry at this writing (fall 1995).

The central technical objective is the achievement of a generalization of basic notions such as connection, curvature, etc., which is of *nonlocal-integral* type, as well as dependent on the *velocities and accelerations* in a nonlinear and nonlocal way, while jointly *preserving the original axioms of the geometry.*

The literature in the conventional affine geometry is predictably vast, although Scrödinger's presentation [2] remains valid to this day. In this section we shall continue to follow the treatise by Lovelock and Rund [4] of which we preserve the notation unchanged for clarity in the comparison of the results.

Let M(x,R) be an n-dimensional *affine space* here referred as a differentiable manifold with local coordinates $x = (x^i)$, i = 1, 2, ...,n, over the reals R(n,+,×). We shall denote: the conventional scalars on M(x,R) with $\phi(x)$; contravariant and covariant vectors with $X^j(x)$ and $X_j(x)$, respectively; and mixed tensors of rank (r.s) with the notation $X^{(r.s)} = X^{j_1 j_2 \dots j_r}{}_{k_1 k_2 \dots k_s}(x)$. Unless otherwise stated, all tensors considered on M(x,$\mathfrak{R}$) will be assumed to be local-differential and to verify all needed continuity conditions.

Definition 5.5.1 [16,19]: *The infinitely possible isotopic liftings* $\hat M(x,\hat R)$ *of Class I of an n-dimensional affine space* M(x,R) *over the reals* R(n,+,×), *called "isoaffine spaces", are characterized by the same local coordinates* x *and the same local-differential tensors* $X^{(r.s)}$ *of* M(x,R) *but now defined over the isoreals* $\hat R(\hat n,+,*)$ *for all infinitely possible n-dimensional isounits* $\hat 1$ *of Class I. "The "isoaffine*

geometry" of Class I is then the geometry of vectorfields on $\hat{M}(x\hat{R})$. *The "isodual isoaffine spaces" of Class II* $\hat{M}^d(x, \hat{R}^d)$ *are the original spaces* $\hat{M}(x,\hat{R})$ *defined over the isodual isoreals* $\hat{R}^d(\hat{n}^d,+,\hat{*}^d)$. *The 'isodual isoaffine geometry" is then the geometry of isodual vectorfields on* $\hat{M}^d(x,\hat{R}^d)$.

Recall in the conventional case that, given two contravariant vectors x_1 and x_2 on M(x,R), their difference $\Delta x = x_1 - x_2$ is a contravariant vector iff the transformation is *linear* (as well as local) [4]. Similarly, Δx is a contravariant vector on $\hat{M}(x,\hat{R})$ iff the transformation is *isolinear* (as well as isolocal).

The first difference between affine and isoaffine spaces can be seen by noting that coordinate differences which are not contravariant in the conventional geometry can be turned into a contravariant form via a suitable selection of the isotopic element.

The *left and right modular isotransforms* on $\hat{M}(x,R)$ are defined by

$$\bar{x}^t = x^t * A^t = x^t T A^t, \qquad \bar{x} = A * X = A T x, \tag{5.5.1}$$

where t denotes conventional transpose. The *inverse, right-modular isotransformations* are given by

$$x = A^{\hat{-1}} * \bar{x} = A^{\hat{-1}} \bar{T} x, \tag{5.5.2}$$

where $A^{\hat{-1}}$ is the *isoinverse,* i.e., it verifies the isotopic rules $A^{\hat{-1}}*A = A*A^{\hat{-1}} = \hat{1}$ and $\bar{T} = \bar{T}(\bar{x}, \dot{\bar{x}}, \ldots) \equiv T(x, \dot{x}, \ldots)$.

Note the preservation of the isotopic element for the left and inverse isotransformations which is ensured by the assumed Hermiticity of the element T for Classes I and II herein considered, and it is at the very foundations of the Lie–isotopic theory. Isoaffine spaces $\hat{M}(x,\hat{R})$ are then *isomodules* for the isorepresentations of Lie–isotopic algebras, while the isodual spaces $\hat{M}^d(x, \hat{R}^d)$ are correct isodual isomodules for the isorepresentations of the isodual algebras.

The fundamental difference between the conventional affine geometry and its isotopic covering is given by the fact that the former uses the conventional differential calculus, while the latter is based on the *differential calculus.*

In turn, as indicated in the preceding section, the latter admits three known realizations: the first via the use of the conventional differential, $\hat{d}x \equiv dx$, and the generalization of the product $Adx \rightarrow A*dx$ which is the form studied in this chapter; the second via the use of the conventional products $A\hat{d}x$ and the generalization instead of the isodifferential itself, $\hat{d}x = Tdx$, which is studied in the recent papers [44,45]; and the third via the combination of both preceding degrees of freedom.

At the level of one–isoforms, the above two realizations of the isodifferential calculus are identical because $\Phi_1 = A*dx = ATdx \equiv A\hat{d}x$. However, the reader should be aware that at higher levels rather important differences

emerge. The isotopies presented in this volume are sufficient for a first study of the program, and we are forced to refer the interested reader to papers [44,45] for brevity.

We finally note that the local coordinates should be everywhere isocoordinates $\hat{x} = x \times \hat{1}$. However, in this case all products should be isotopic, thus eliminating the isounit in the coordinates. For simplicity we shall therefore use conventional coordinates x.

5.5B: Isocovariant differentials and their isoduals. Recall that the conventional differentials dx′ and dx interconnected by the linear and local transformations $dx' = A \times dx$ cannot be defined under isotopies and must be lifted into the *isodifferentials* $\hat{d}\bar{x}$ and $\hat{d}x$ interconnected by the isotopic rules (5.5.5).

The *isodifferential of an scalar* $\phi(x)$ on $\hat{M}(x,\hat{R})$ is then given by law (5.5.7); the *isodifferential of a contravariant isovector* $X = (X^i(x))$ on $\hat{M}(x,\hat{R})$ is given by rule (5.5.8); the *isotransformation laws* of the contravariant isovector is rule (5.5.9); and the *isotransformations of a contravariant isotensor* X^{ij} of rank two on $\hat{M}(x,\hat{R})$ is given by Eq.s (5.5.13).

By using these results, the *isodifferential of a contravariant isovectorfield* on $\hat{M}(x,\hat{R})$ is given by

$$\hat{d}\,\bar{X}^j = \frac{\partial \bar{X}^j}{\partial \bar{x}^k} T^k{}_r\, \hat{d}\bar{x}^r = \tag{5.5.3}$$

$$= \frac{\partial^2 \bar{x}^j}{\partial x^s\, \partial x^i} T^i{}_r X^r\, \hat{d}x^s + \frac{\partial \bar{x}^j}{\partial x^i} T^i{}_r \frac{\partial X^r}{\partial x^s}\, \hat{d}x^s + \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial T^i{}_r}{\partial x^s} X^r\, \hat{d}x^s$$

By using the above quantities, one can introduce the *isocovariant (or isoabsolute) differential* [10]

$$\hat{D}X^j = \hat{d}X^j + \hat{P}^j(x, X, \hat{d}x), \tag{5.5.4}$$

under the condition that it preserves the original axioms (see ref. [4], p. 68), i.e.,

1) $\hat{D}(X^j + Y^j) = \hat{D}X^j + \hat{D}Y^j$, which can hold iff $\hat{P}^j$ is isolinear in X^r;

2) $\hat{D}X^j$ is isolinear in $\hat{d}x^s$; and

3) $\hat{D}X^j$ transforms as a contravariant isovector.

By again using Lovelock–Rund's symbols with a "hat" to denote isotopy, we can write

$$\hat{D}X^j = \hat{d}X^j + \hat{\Gamma}_h{}^j{}_k\, T^h{}_r X^r\, T^k{}_s \hat{d}x^s, \tag{5.5.5}$$

where the $\hat{\Gamma}$'s are called the component of an *isoaffine connection.*

By lifting the conventional procedure, one can readily see that the necessary and sufficient conditions for the n^3 quantities $\hat{\Gamma}_m{}^s{}_n$ to be the

coefficient of an isoaffine connection are given by

$$\hat{\Gamma}_{m\,p}^{\ j}\, T^m{}_r \frac{\partial \bar{x}^r}{\partial x^s} T^s{}_t\, X^t\, T^p{}_q \frac{\partial \bar{x}^q}{\partial x^w} T^w{}_z\, \hat{\partial} x^z =$$

$$= \frac{\partial \bar{x}^j}{\partial x^r} T^r{}_s \hat{\Gamma}_m{}^s{}_n\, T^m{}_p\, X^p\, T^n{}_q\, \hat{\partial} x^q - \frac{\partial^2 \bar{x}^j}{\partial x^s\, \partial x^i} T^i{}_r\, X^r\, \hat{\partial} x^s +$$

$$+ \frac{\partial x^j}{\partial x^i} T^h{}_r \frac{\partial X^r}{\partial x^s} (T^s{}_t \hat{\partial} x^t - \hat{\partial} x^s) - \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial T^i{}_r}{\partial x^s} X^r\, \hat{\partial} x^s. \qquad (5.5.6)$$

As in the conventional case, the $\hat{\Gamma}$'s do not constitute a tensor of rank (1.2). The extra terms in conditions (5.5.6), therefore, do not affect the consistency of the isoaffine geometry, but constitute the *desired generalization.*

The extension of the above notions to the *isocontravariant derivatives* is evidently given by

$$\hat{D}X_j = \hat{\partial} X_j - \hat{\Gamma}_j{}^s{}_n\, T_s{}^r\, X_r\, T^n{}_p\, \hat{\partial} x^p . \qquad (5.5.7)$$

As a result, *the isocovariant derivative of a scalar coincides with the isodifferential*, as in the conventional case, i.e., $\hat{D}\phi = \hat{D}(X^i X_i) = \hat{\partial}\phi$.

The isoaffine connection is *symmetric* if $\hat{\Gamma}_m{}^s{}_n = \hat{\Gamma}_n{}^s{}_m$. Also, the isotopic image of a symmetric connection is symmetric in isospace. However, the following property can be easily proved (but carries important consequences).

Proposition 5.5.1 [16,19]: *The isotopic image* $\hat{\Gamma}_h{}^j{}_k$ *of a symmetric affine connection* $\Gamma_h{}^j{}_k$ *is not necessarily symmetry when projected in* M(x,R).

The isotopic liftings of all remaining properties of covariant derivatives, as well as the extension to the isocovariant differential of tensors, will be left for brevity to the interested reader.

It is important to verify that the isocovariant (isoabsolute) differential preserves the basic axioms of the conventional differential because this is a necessary condition for consistency of the isotopies. In fact, we have the following axioms which coincide at the abstract level with the conventional ones (ref. [4], p. 74),

Axiom 1: *The isocovariant differential of a constant is identically null; that of a scalar coincides with the isodifferential; and that of a tensor of rank (r.s) is a tensor of the same rank.*

Axiom 2: *The isocovariant differential of the sum of two tensors of the same*

rank is the sum of the isoabsolute differentials of the individual tensors. And
Axiom 3: *The isocovariant differential of the product of two tensors of the same rank verifies the conventional chain rule of differentiation.*

By following again the conventional formulation, and as a natural generalization of the isocovariant differential, we introduced the *isocovariant derivative* of a contravariant vector field X^p [loc. cit.]

$$X^j{}_{\uparrow k} = \frac{\partial X^j}{\partial x^k} + \hat{\Gamma}_h{}^j{}_k \, T^h{}_r \, X^r, \tag{5.5.8}$$

under which the isocovariant differential can be written

$$\hat{D}X^j = X^j{}_{\uparrow k} \, T^k{}_s \, \hat{d}x^s. \tag{5.5.9}$$

It is an instructive exercise for the interested reader to prove that the isocovariant derivatives (5.5.9) constitute the components of a (1.1) isotensor. It is also easy to verify that the isocovariant derivatives preserve the axioms of the conventional covariant derivatives (ref. [4], p. 77):

Axiom 1': *The isocovariant derivative of a constant is identically null; that of a scalar is equal to the conventional partial derivative; and that of an isotensor of rank (r. s) is an isotensor of rank (r. s+1).*
Axiom 2': *The isocovariant derivative of the sum of two tensors of the same rank is the sum of the isocovariant derivatives of the individual tensors. And*
Axiom 3': *The isocovariant derivative of the product of two isotensors of the same rank is that of the usual chain rule of partial derivatives.*

It is easy to see that all the preceding notions admit a consistent and significant image under isoduality. Intriguingly, the isodifferential of a vectorfield does not change under isoduality $\hat{d}^dX^d = \hat{d}X$. Similarly, we have the following isodual isocovariant differentials (see the 2–nd edition of Vol. I, ref. [47])

$$\hat{D}^dX^{dj} = \hat{d}^dX^{dj} + \Gamma^d{}_h{}^j{}_k \, T^{dh}{}_r X^{d^r} \, T^{dk}{}_s \hat{d}x^s = -\hat{D}X^j, \tag{5.5.10a}$$

$$\hat{D}^dX_j = \hat{d}^dX_j - \Gamma^d{}_j{}^s{}_n \, T^d{}_s{}^r \, X_r \, T^{dn}{}_p \, \hat{d}x^p = \hat{D}X_j. \tag{5.5.10b}$$

We therefore have the following important

Proposition 5.5.2 [loc. cit.]: *The isoaffine connection changes sign under isoduality,*

$$\hat{\Gamma}^d{}_i{}^s{}_n = -\hat{\Gamma}_i{}^s{}_n \tag{5.5.11}$$

The preservation of all basic axioms, although in their isodual form, is then consequential.

Axioms 1, 2, 3 and 1′, 2′, 3′ imply the most important result of this section, which can be expressed via the following

Theorem 5.5.1 [16,19]: *All infinitely possible nonlinear, nonlocal and noncanonical isoaffine geometries of Class I coincide with the conventional affine geometry at the abstract, coordinate-free level, while all the infinitely possible isodual isoaffine geometries of Class II coincide with the isodual affine geometry at the abstract level.*

5.5C: Isocurvature, isotorsion and their isoduals. We now pass to the study of a central notion of the isoaffine geometry, the generalized curvature, called *isocurvature,* and generalized torsion, called *isotorsion,* which are inherent in the isoaffine geometry prior to any introduction of an isometric (to be done in the next section).

For this purpose, let us study the lack of commutativity of the isocovariant derivatives on isoaffine spaces $\hat{M}(x,\hat{R})$ with respect to an arbitrary, not necessarily symmetric, isoconnection $\hat{\Gamma}_h{}^j{}_k$. Via a simple isotopy of the corresponding equations (see ref. [4], pp. 82–83), and by noting that

$$X^j{}_{\hat{\uparrow} h \hat{\uparrow} k} = \frac{\partial}{\partial x^k}(X^j{}_{\hat{\uparrow} h}) + \hat{\Gamma}2_p{}^j{}_k\, T^p{}_q\, (X^q{}_{\hat{\uparrow} h}) - \hat{\Gamma}2_h{}^p{}_k\, T_p{}^q (X^j{}_{\hat{\uparrow} q}), \tag{5.5.12}$$

one gets the expression

$$\begin{aligned} X^j{}_{\hat{\uparrow} h \hat{\uparrow} k} - X^j{}_{\hat{\uparrow} k \hat{\uparrow} h} &= \Big(\frac{\partial \hat{\Gamma}2_l{}^j{}_h}{\partial x^k} - \frac{\partial \hat{\Gamma}2_l{}^j{}_k}{\partial x^h} + \\ &+ (\hat{\Gamma}2_m{}^l{}_k\, T^m{}_r\, \hat{\Gamma}2_l{}^r{}_h - \hat{\Gamma}2_m{}^l{}_h\, T^m{}_r\, \hat{\Gamma}2_l{}^r{}_k)\, T^l{}_s\, X^s - (\hat{\Gamma}2_h{}^l{}_k - \hat{\Gamma}2_k{}^l{}_h) T_l{}^r X^j{}_{\hat{\uparrow} r} - \\ &- (\hat{\Gamma}2_l{}^j{}_h \frac{\partial T^l{}_r}{\partial x^k} - \hat{\Gamma}2_l{}^j{}_k \frac{\partial T^l{}_r}{\partial x^h}) X^r, \end{aligned} \tag{5.5.13}$$

Definition 5.5.2 [16,19]: *The "isocurvature" of a vector field* X^r *on an* n*–dimensional isoaffine space* $\hat{M}(x,\hat{R})$ *is given by the isotensor of rank (1.3)*

$$\hat{R}_l{}^j{}_{hk} = \frac{\partial \hat{\Gamma}_l{}^j{}_h}{\partial x^k} - \frac{\partial \hat{\Gamma}_l{}^j{}_k}{\partial x^h} + \hat{\Gamma}_m{}^j{}_k\, T^m{}_r\, \hat{\Gamma}2_l{}^r{}_h - \hat{\Gamma}_m{}^j{}_h\, T^m{}_r\, \hat{\Gamma}2_l{}^r{}_k +$$

$$+ \hat{\Gamma}_{r\,h}^{\;j} \frac{\partial T^r{}_s}{\partial x^k} \hat{T}^s{}_l \quad - \quad \hat{\Gamma}_{r\,k}^{\;j} \frac{\partial T^r{}_s}{\partial x^h} \hat{T}^s{}_l \,; \qquad (5.5.14)$$

while the "isotorsion" is given by the isotensor

$$\hat{\tau}_{h\,k}^{\;l} = \hat{\Gamma}_h{}^l{}_k - \hat{\Gamma}_k{}^l{}_h \,; \qquad (5.5.15)$$

The "isodual isocurvature" and "isodual isotorsion" are the opposite of the corresponding isotopic quantities.

Expression (5.5.12) can then be written

$$X^j{}_{\hat{|}h\hat{|}k} - X^j{}_{\hat{|}k\hat{|}h} = \hat{R}_l{}^j{}_{hk} T^l{}_s X^s - \hat{\tau}_{h\,k}^{\;l} T^s{}_l X^j{}_{\hat{|}s}. \qquad (5.5.16)$$

Comparison with the corresponding conventional expression (Eq.s (6.9), p. 83, ref. [4]) is instructive to understand the *modification of the curvature as well as of the torsion caused by the isotopic geometrization of interior physical media.* As we shall see, this modification is the desired feature to avoid excessive approximations of physical reality, such as the admission of the perpetual motion within a physical environment which is inherent in all rotationally invariant (torsionless) theories.

The extension of the results to a (0.2)–rank tensor is tedious but trivial, yielding the expression

$$X^{jl}{}_{\hat{|}h\hat{|}k} - X^{jl}{}_{\hat{|}k\hat{|}h} = \hat{R}_r{}^j{}_{hk} T^r{}_s X^{sl} + \hat{R}_r{}^l{}_{hk} T^r{}_s X^{js} - \hat{\tau}_h{}^r{}_k T_r{}^s X^{jl}{}_{\hat{|}s}, \qquad (5.5.17)$$

Similarly, for contravariant isovectors and isotensors one obtains the expressions

$$X_{j\hat{|}h\hat{|}k} - X_{j\hat{|}k\hat{|}h} = -\hat{R}_j{}^r{}_{hk} T_r{}^s X_s - \hat{\tau}_h{}^r{}_k T_r{}^s X_{j\hat{|}s}\,, \qquad (5.5.18a)$$

$$X_{jl\hat{|}h\hat{|}k} - X_{jl\hat{|}k\hat{|}h} = -\hat{R}_j{}^r{}_{hk} T_r{}^s X_{sl} - \hat{R}_l{}^r{}_{hk} T_r{}^s X_{js} - \hat{\tau}_h{}^r{}_k T^s{}_k X_{jl\hat{|}s}\,. \qquad (5.5.18b)$$

called the *isoricci identities.*

The following first property is an easy derivation of definition (5.5.14).

Property 1:

$$\hat{R}_l{}^j{}_{hk} = -\hat{R}_l{}^j{}_{kh}. \qquad (5.5.19)$$

The second property requires some algebra, which can be derived via a simple isotopy of the conventional derivation (ref. [26], pp. 91–92).

Property 2:

$$\hat{R}_l{}^j{}_{hk} + \hat{R}_h{}^j{}_{lh} + \hat{R}_k{}^j{}_{lh} = \hat{\tau}_l{}^j{}_{h\uparrow k} + \hat{\tau}_h{}^j{}_{kl\uparrow l} + \hat{\tau}_k{}^j{}_{l\uparrow h} +$$
$$+ \hat{\tau}_l{}^j{}_r T^r{}_s \hat{\tau}_h{}^s{}_k + \hat{\tau}_h{}^j{}_r T^r{}_s \hat{\tau}_k{}^s{}_l +$$
$$+ \hat{\tau}_k{}^j{}_r T^r{}_s \hat{\tau}_l{}^s{}_h + \hat{\tau}^j{}_{rh} \frac{\partial T^r{}_s}{\partial x^k} \hat{1}^s{}_l + \hat{\tau}^j{}_{rk} \frac{\partial T^r{}_s}{\partial x^l} \hat{1}^s{}_h + \hat{\tau}^j{}_{rl} \frac{\partial T^r{}_s}{\partial x^h} \hat{1}^s{}_k. \qquad (5.5.20)$$

where, again, the reader should note the isotopies of the conventional terms, plus two new terms.

Note that, for a symmetric isoconnection, the isotorsion is null and the above property reduces to the familiar form

$$\hat{R}_l{}^j{}_{hk} + \hat{R}_h{}^j{}_{kl} + K_k{}^j{}_{lh} = 0. \qquad (5.5.21)$$

The third property identified in ref.s [16,19] also requires some tedious but simple algebra given by an isotopy of the conventional derivation (ref. [4], pp. 92–93), which results in

Property 3:

$$(\hat{R}_j{}^l{}_{h\uparrow p} + \hat{R}_j{}^l{}_{kp\uparrow h} + \hat{R}_j{}^l{}_{ph\uparrow k}) Y_l =$$
$$= (\hat{S}_h{}^r{}_k T_r{}^s K_j{}^l{}_{sp} + \hat{S}_k{}^r{}_p T_r{}^s \hat{R}_j{}^l{}_{sh} + \hat{S}_k{}^r{}_h T_r{}^s \hat{R}_j{}^l{}_{sk}) Y_l +$$
$$+ (\hat{R}_j{}^r{}_{hk} T_r{}^l{}_{\uparrow p} + \hat{R}_j{}^r{}_{kp} T_r{}^l{}_{\uparrow h} + \hat{R}_j{}^r{}_{ph} T_r{}^l{}_{|k}) Y_l +$$
$$+ (\hat{S}_h{}^r{}_k T_r{}^l{}_{\uparrow p} + \hat{S}_k{}^r{}_p T_r{}^l{}_{|h} + \hat{S}_p{}^r{}_h T_r{}^l{}_{\uparrow k}) T Y_{j\uparrow l}, \qquad (5.5.22)$$

called the *isobianchi identity* , and which can be written in a number of equivalent forms here left to the interested reader (see an alternative expression in the next section).

Again, as it was the case for property (5.5.20), the isobianchi identity for the case of a symmetric isoconnection reduces to

$$\hat{R}_j{}^l{}_{hk\uparrow p} + \hat{R}_j{}^l{}_{kp\uparrow k} + \hat{R}_j{}^l{}_{ph\uparrow k} \equiv 0. \qquad (5.5.23)$$

The corresponding properties for isodual quantities can be easily derived. This completes the identification of the primary properties of an isocurvature tensor prior to the introduction of the isometric.

5.6: ISORIEMANNIAN GEOMETRY AND ITS ISODUAL

5.6A: Statement of the problem. The *isoriemannian geometry* of Class V is the most general possible geometry on a curved manifold possessing:

A) a *nonlinear, nonlocal and nonlagrangian structure* in the local coordinates and their derivatives of arbitrary order;

B) "directly universality" for all possible interior gravitational problems; and

C) admitting the conventional Riemannian geometry and exterior gravitation as a particular case when the isounits $\hat{1}$ recovers the conventional unit I = diag. (1, 1, 1, 1) (physically, when motion returns to be in vacuum).

In this section we shall solely study some of the mathematical properties of the new geometry for the specific case of Class I, with only basic elements for the case of Class II. All physical applications are deferred to Ch. II.8, 9, while experimental verifications are be studied in Vol. III.

THE DICHOTOMY OF EXTERIOR AND INTERIOR GEOMETRIES IN GRAVITATIONAL PROBLEMS

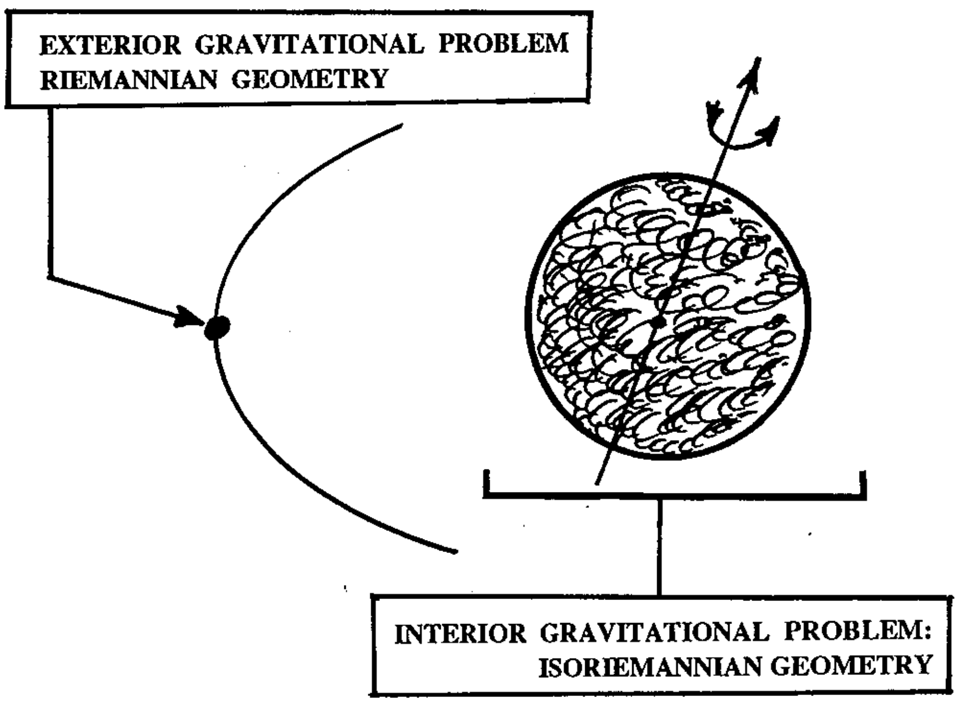

FIGURE 5.6.1: A schematic view of the dual geometric treatment of gravitation characterized by: A) the conventional Riemannian geometry on spaces $\Re(x,g,R)$ assumed as exact for the exterior motion of dimensionless test body in vacuum; B) the covering isoriemannian geometry on isospaces $\hat{\Re}(x,\hat{g},\hat{R})$ for the interior structural problem; C) under the general condition that the latter recovers the former identically in vacuum, e.g., for

null density μ, $\hat{\Re}(x,\hat{g},\hat{R})|_{\mu=0} \equiv \Re(x,g,R)$. As we shall see in Ch. I.7, despite the considerable enlargement of the scientific horizon, the use of the isoriemannian geometry alone is still insufficient for the interior problem because it is time-reversible, thus particularly suited for the "global" treatment of the structure as a whole with conserved total quantities. The complementary approach of Ch. I.7 will then be the irreversible treatment of one interior test body, while considering the rest of the system as external.

The new geometry was proposed by this author in memoir [16], developed in more details in ref. [18,19] and applied to the generalization of Einstein's gravitation for the interior problem in ref.s [20,21]. The only additional contributions on the new *geometry* on record at this time (fall 1995) are Lopez's [24] application to the exterior problem, Kadeisvili's review [25], and the studies on isoriemannian manifolds and related topology (see next section) by Tsagas and Sourlas [46,47]. The most recent formulation of the isoriemannian geometry is that by this author in the recent papers [44,45] via the use of a new form of isodifferential calculus.

Additional contributions in the field are those by Gasperini [32–34] who was the first to study the isotopies of Einstein's gravitation. However, Gasperini formulated his studies on a *conventional* Riemannian geometry, while the primary emphasis of this section is on the *generalization* of the Riemannian geometry itself. A review of Gasperini studies is available in monograph [35], and it is therefore omitted here for brevity.

Also, Gasperini formulated his locally isopoincaré studies everywhere in space–time, thus reaching predictable restrictions from available exterior experiments. On the contrary, in the studies herein considered, all generalized geometrical and physical theories are specifically formulated for the *interior* problem only under Condition C) of recovering identically the conventional formulations in the exterior problem.

In this way all available experiments in gravitation have no bearing on the interior isotopic treatment by construction. As we shall see in Vol. II, the test of the isoriemannian geometry for the interior problem requires *novel* experiments, that is, experiments which cannot be even formulated, let alone quantitatively treated with the conventional Riemannian geometry.

The central objectives of the isoriemannian geometry are the achievement of an axiom–preserving generalization of the Riemannian geometry with an isometric $\hat{g}$ which, besides being sufficiently smooth, bounded, real valued and symmetric, possesses the most general possible dependence on all needed quantities

$$\hat{g}(x, \dot{x}, \ddot{x}, \mu, \tau, n,...) \equiv \hat{g}_{ji}(s, x, \dot{x}, \ddot{x}, \mu, \tau, n, ...) = \hat{g}^t, \quad \det \hat{g} \neq 0 , \qquad (5.6.1)$$

as a pre–requisite to achieve the desired "direct universality" for the interior gravitational problem.

As we shall see, the above objectives are permitted by the *isodifferential*

calculus. The latter, however, has at least *three* known realization, as discussed in the preceding sections. It then follows that *the isoriemannian geometry has at least three known realizations,* the first based on the conventional differential $\hat{d}x = dx$ and the isotopic product $A*dx = A\hat{T}dx$, the second based on the conventional product Adx and the isotopic lifting of the differential $\hat{d}x = \hat{1}dx$, and the third based on the combination of the preceding two degrees of freedom.

In this chapter we shall solely study the first form of the isodifferential calculus, and defer the interested reader to papers [44,45] for brevity. In fact, the selected form is amply sufficient for the scope of these monographs.

5.6B: Isoriemannian spaces and their isoduals. To begin, let us perform the transition from the n-dimensional isoaffine spaces $\hat{M}(x,\hat{R})$ of the preceding section, to the corresponding isospaces $\hat{M}(x,\hat{g},\hat{R})$ equipped with the symmetric isotensor (5.6.1) on $\hat{M}(x,\hat{R})$, called *isometric*.

Similarly, we perform the transition from the isodual isoaffine spaces $\hat{M}^d(x,\hat{R}^d)$ to the corresponding spaces $\hat{M}^d(x,\hat{g}^d,\hat{R}^d)$ equipped with the isodual symmetric isotensor $\hat{g}^d = (\hat{g}^d{}_{ij})$.

Definition 5.6.1 [16,19]: *The "isotopic liftings" of Class I $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ of a conventional Riemannian space $\mathfrak{R}(x,g,R)$ in n-dimension, called "isoriemannian spaces", are the isoaffine spaces $\hat{M}(x,\hat{R})$ in the same dimension equipped with an isometric*

$$\hat{g} = (\hat{g}_{ij}) = \hat{g}(x, \dot{x}, \mu, \tau, n, ...) = T(x, \dot{x}, \mu, \tau, n,...)\, g(x) = \hat{g}^t, \ \det \hat{g} \neq 0, \quad 5.6.2)$$

i, j = 1, 2, ..., n, *where* T *is the* n×n*-dimensional isotopic element of the underlying isofield* $\hat{R}(\hat{n},+,*)$, $\hat{n} = n\hat{1}$, $\hat{1} = T^{-1}$, *which characterizes a symmetric isoaffine connection, called "isochrisoffell symbols of the first kind"*

$$\hat{\Gamma}^1{}_{hlk} = \tfrac{1}{2}\left(\frac{\partial \hat{g}_{kl}}{\partial x^h} + \frac{\partial \hat{g}_{lh}}{\partial x^k} - \frac{\partial \hat{g}_{hk}}{\partial x^l}\right) = \hat{\Gamma}^1{}_{klh} \qquad (5.6.3)$$

as well as the "isochristoffel symbols of the second kind"

$$\hat{\Gamma}^2{}_h{}^i{}_k = \hat{g}^{ij}\,\hat{\Gamma}^1{}_{hjk} = \hat{\Gamma}^2{}_k{}^i{}_h \qquad (5.6.4)$$

where the capability for an isometric of raising and lowering the indices is understood (as in any affine space), and $\hat{g}^{ij} = |(\hat{g}_{rs})^{-1}|^{ij}$. *The "isoriemannian geometry" is the geometry of isospaces* $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$.

The "isodual isoriemannian isospaces" are then given by the isodual map of isospaces $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$

$$\hat{\mathfrak{R}}^d(x^d, \hat{g}^d, \hat{R}^d), \quad \hat{g}^d = T^d g = -\hat{g}, \quad \hat{R}^d \approx \hat{R}\hat{1}^d, \hat{1}^d = (T^d)^{-1} = -\hat{1}, \quad (5.6.5)$$

with "isodual isochristoffel symbols"

$$\Gamma^{d\ l}{}_{hlk} = \tfrac{1}{2}\left(\frac{\partial \hat{g}_{kl}}{\partial x^h} + \frac{\partial \hat{g}_{lh}}{\partial x^k} - \frac{\partial \hat{g}_{hk}}{\partial x^l}\right)^d = \hat{\Gamma}^l{}_{hlk} \qquad (5.6.6a)$$

$$\Gamma^{d\,2}{}^{\ i}_{h\ k} = \Gamma^{2}{}^{\ i}_{h\ k} \qquad (5.6.6b)$$

where one should keep in mind that the isodual map must be applied, for consistency, to all quantities as well as their operations such a quotient.

In essence, the above definition is centered on the requirement that the alteration (also called *"mutation"* [loc. cit.] $g(x) \Rightarrow T(s, x, \dot{x}, \ddot{x}, \mu, \tau, n, \ldots)\, g(x) = \hat{g}$ of the original Riemannian metric g is characterized by the isotopic element T of the base field and, thus of the base multiplication. The joint liftings $g \Rightarrow \hat{g} = Tg$ and $R(n,+,\times) \Rightarrow R(\hat{n},+,*)$, $\hat{n} = n\hat{1}$, $\hat{1} = T^{-1}$, leave the functional dependence of the isometric totally unrestricted, thus verifying the fundamental pre-requisite for "direct universality".

The above new structures imply that the transformation theory of the conventional Riemannian space must be lifted into the isotopic form of the preceding sections. In turn, this ensures that the isoriemannian geometry is isolinear, isolocal and isolagrangian (Sect. 4.2) on $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$, although generally nonlinear, nonlocal and nonlagrangian when projected on $\mathfrak{R}(x,g,R)$.

On physical grounds, the isotopies $\mathfrak{R}(x,g,R) \Rightarrow \hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ imply that we have performed the transition from the exterior to the interior gravitational problem. Throughout our analysis the reader should keep in mind that the isotopic elements T (or isounit $\hat{1}$) assume their conventional unit value $I = \mathrm{diag.}\,(1, 1, 1, 1)$ everywhere in the exterior of the minimal surface $S°$ encompassing all matter of the interior problem, i.e., for null density μ, in which case $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})_{\mu=0} \equiv \mathfrak{R}(x,g,R)$.

Note that *each given Riemannian geometry can be subjected to an infinite number of isotopic liftings* which are expected to represent the infinite number of possible, different, interior physical media for each given total gravitational mass. This is the reason for the use the plural in "isotopies".

As indicated in Definition 5.6.1, the introduction of a metric on an affine space implies the capability of raising and lowering the indices. The same property evidently persists under isotopies. Given a contravariant isovector X^i on $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$, one can define its covariant form via the familiar rule

$$X_i = \hat{g}_{ij} X^j. \qquad (5.6.7)$$

Similar conventional rules apply for the lowering of the indices of all other quantities.

It is easy to see that the inverse g^{-1} is a bona-fide contravariant isotensor

of rank (2.0). Given a covariant isovector X_i on $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$, its contravariant form is then defined by

$$X^i = \hat{g}^{ij} X_j. \tag{5.6.8}$$

Rules (5.6.7) and (5.6.8) can then be used to raise or lower the indices of an arbitrary isotensor of rank (r. s).

The first important property of the isoriemann geometry can be derived by writing from Eq.s (5.6.3)

$$\frac{\partial \hat{g}_{hl}}{\partial x^k} = \hat{\Gamma}^l{}_{hlk} + \hat{\Gamma}^l{}_{lhk}, \quad \hat{g}_{hl\uparrow k} = \frac{\partial g_{hl}}{\partial x^k} - \hat{\Gamma}^l{}_{hlk} - \hat{\Gamma}^l{}_{lhk}, \tag{5.6.9}$$

for which,

$$\hat{g}_{hl\uparrow k} \equiv 0, \qquad \hat{g}^{hl}{}_{\uparrow k} \equiv 0, \tag{5.6.10}$$

with similar results for the isodual isometrics. We reach in this way the following

Lemma 5.6.1 – Isoricci lemma [16,19]: *All isotopic liftings of Class I and II of the Riemannian geometry preserve the vanishing character of the covariant derivative of the isometrics.*

In different terms, the familiar property of the Riemannian geometry $g_{ij\,|\,k} = 0$ is a true geometric axiom because it is invariant under all infinitely possible isotopies. As shown below, this property is not shared by all gravitational quantities, such as Einstein's tensor.

The isotransformation law of the isometric $\hat{g}$ is given by expression of type (5.4.13). By repeating the conventional procedure (ref. [4], pp. 78–70) under isotopy, one obtains the following expression for the *isochristoffel symbol of the first kind*

$$\hat{\Gamma}^l{}_{hlk} = \tfrac{1}{2}\left(\frac{\partial \hat{g}_{kl}}{\partial x^h} + \frac{\partial \hat{g}_{lh}}{\partial x^k} - \frac{\partial \hat{g}_{hk}}{\partial x^l}\right) =$$

$$= \hat{g}_{jp} T^j{}_r \frac{\partial^2 x^r}{\partial x^h \partial x^k} T^p{}_s \frac{\partial x^s}{\partial x^l} + \frac{\partial \bar{g}_{jp}}{\partial \bar{x}^m} T^j{}_r T^p{}_s \left(\frac{\partial \bar{x}^r}{\partial x^h}\frac{\partial \bar{x}^s}{\partial x^k}\frac{\partial \bar{x}^m}{\partial x^l} + \right.$$

$$\left. + \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial \bar{x}^m}{\partial x^h} - \frac{\partial \bar{x}^r}{\partial x^l}\frac{\partial \bar{x}^s}{\partial x^h}\frac{\partial \bar{x}^m}{\partial x^k}\right) + \tfrac{1}{2}\hat{g}_{jp} T^p{}_s \left[\frac{\partial T^j{}_r}{\partial x^l}\left(\frac{\partial \bar{x}^r}{\partial x^h}\frac{\partial \bar{x}^s}{\partial x^k} + \frac{\partial \bar{x}^s}{\partial x^h}\frac{\partial \bar{x}^r}{\partial x^k}\right) + \right.$$

$$\left. + \frac{\partial T^j{}_r}{\partial x^h}\left(\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial \bar{x}^s}{\partial x^l} + \frac{\partial \bar{x}^s}{\partial x^k}\frac{\partial x^r}{\partial x^l}\right) - \frac{\partial T^j{}_r}{\partial x^k}\left(\frac{\partial \bar{x}^r}{\partial x^l}\frac{\partial \bar{x}^s}{\partial x^h} + \frac{\partial \bar{x}^s}{\partial x^l}\frac{\partial \bar{x}^r}{\partial x^h}\right)\right], \tag{5.6.11}$$

with a number of alternative formulations and simplifications, e.g., for diagonal isotopic elements T, which are left to the interested reader for brevity.

5.6C: Basic identities. In order to proceed with our review, we need the following

Definition 5.6.2 [loc. cit.]: *Given an n-dimensional isoriemannian space $\hat{\Re}(x,\hat{g},\hat{R})$ of Class I, the "isocurvature tensor" is given by*

$$\hat{R}_{l\,h}^{\ j} = \frac{\partial \hat{\Gamma}^{2\ j}_{\ l\,h}}{\partial x^k} - \frac{\partial \hat{\Gamma}^{2\ j}_{\ l\,k}}{\partial x^h} + \hat{\Gamma}^{2\ j}_{\ m\,k}\, T^m_{\ r}\, \hat{\Gamma}^{2\ r}_{\ l\,h} - \hat{\Gamma}^{2\ j}_{\ m\,h}\, T^m_{\ r}\, \hat{\Gamma}^{2\ r}_{\ l\,k} +$$

$$+ \hat{\Gamma}^{2\ j}_{\ r\,h} \frac{\partial T^r_{\ s}}{\partial x^k} \hat{1}^s_{\ l} - \hat{\Gamma}^{2\ j}_{\ r\,k} \frac{\partial T^r_{\ s}}{\partial x^h} \hat{1}^s_{\ l}, \qquad (5.6.12)$$

and can be rewritten

$$\hat{R}_{l\,hk}^{\ j} = \tfrac{1}{2}\,\hat{g}^{jp}\left(\frac{\partial \hat{g}_{ph}}{\partial x^k\,\partial x^l} - \frac{\partial \hat{g}_{pk}}{\partial x^h\,\partial x^l} + \frac{\partial \hat{g}_{lk}}{\partial x^h\,\partial x^j} - \frac{\partial \hat{g}_{lh}}{\partial x^k\,\partial x^j}\right) +$$

$$+ \hat{g}^{jp}\left(\hat{\Gamma}^{1}_{\ prh}\, T^r_{\ s}\, \hat{\Gamma}^{2\ s}_{\ l\,h} - \hat{\Gamma}^{1}_{\ prk}\, T^r_{\ s}\, \hat{\Gamma}^{2\ s}_{\ l\,k}\right) +$$

$$+ \hat{\Gamma}^{2\ j}_{\ r\,h} \frac{\partial T^r_{\ s}}{\partial x^k} \hat{1}^s_{\ l} - \hat{\Gamma}^{2\ j}_{\ r\,k} \frac{\partial T^r_{\ s}}{\partial x^h} \hat{1}^s_{\ l}; \qquad (5.6.13)$$

the "isoricci tensor" is given by

$$\hat{R}_{lh} = \hat{R}_{l\,hj}^{\ j} = g^{ij}\,\hat{R}_{lihj}; \qquad (5.6.14)$$

the "isoeinstein tensor" is given by

$$\hat{G}_i^{\ j} = \hat{R}_i^{\ j} - \tfrac{1}{2}\,\delta_i^{\ j}\,\hat{R}; \qquad (5.6.15)$$

and the "completed isoeinstein tensor" is given by

$$\hat{S}_i^{\ j} = \hat{R}_i^{\ j} - \tfrac{1}{2}\,\delta_i^{\ j}\hat{R} - \tfrac{1}{2}\,\delta_i^{\ j}\hat{\Theta}; \qquad (5.6.16)$$

where $\hat{R}$ is the "isocurvature isoscalar"

$$\hat{R} = \hat{R}_i^{\ i} = \hat{g}^{ij}\,\hat{R}_{..}, \qquad (5.6.17)$$

and $\hat{\Theta}$ *is the "isotopic isoscalar"*

$$\hat{\Theta} = \hat{g}^{jh}\hat{g}^{lk}(\hat{\Gamma}^{l}{}_{rjk}\,T^{r}{}_{s}\,\hat{\Gamma}^{2s}{}_{l\,h} - \hat{\Gamma}^{l}{}_{rjh}\,T^{r}{}_{s}\,\hat{\Gamma}^{2s}{}_{l\,k}) = $$
$$= \hat{\Gamma}^{l}{}_{rjk}\,T^{r}{}_{s}\,\hat{\Gamma}^{2s}{}_{l\,h}\,(\hat{g}^{jh}\hat{g}^{lk} - \hat{g}^{jk}\hat{g}^{lh}). \qquad (5.6.18)$$

Isodual quantities are defined accordingly.

We are now equipped to review the isotopies of the various properties[41] of the Riemannian geometry [10,12]. From definition (5.4.12) we readily obtain

Property 1: Antisymmetry of the last two indices of the isocurvature tensor

$$\hat{R}_{l}{}^{j}{}_{hk} = - \hat{R}_{l}{}^{j}{}_{kh}. \qquad (5.6.19)$$

The specialization of properties (3.22) to the case at hand easily implies the following

Property 2: Vanishing of the totally antisymmetric part of the isocurvature tensor

$$\hat{R}_{l}{}^{j}{}_{hk} + \hat{R}_{h}{}^{j}{}_{kl} + \hat{R}_{k}{}^{j}{}_{lh} = 0, \qquad (5.6.20)$$

or, equivalently,

$$\hat{R}_{lmhk} + \hat{R}_{hmkl} + \hat{R}_{kmlh} = 0. \qquad (5.6.21)$$

The use of property (5.6.19) and Lemma 5.6.1 then yields

Property 3: Antisymmetry in the first two indices of the isocurvature tensor

$$\hat{R}_{jlhk} = \hat{R}_{ljhk}, \qquad (5.6.22)$$

or, equivalently,

$$\hat{R}_{ljhk} = \hat{R}_{hklj}. \qquad (5.6.23)$$

From Definition (5.6.12) and the use of Lemma 5.6.1, after tedious but

[41] The reader should be aware that the properties below are different for different realizations of the isodifferential calculus and of the isoriemannian geometry, as shown in ref.s [44,45].

simple calculations, we obtain the following:

Property 3: Isobianchi identity

$$\hat{R}_{l\;hk\hat{|}p}^{\;j} + \hat{R}_{l\;ph\hat{|}k}^{\;j} + \hat{R}_{l\;kp\hat{|}h}^{\;j} = \hat{S}_{l\;hkp}^{\;j}, \tag{5.6.24}$$

where

$$\hat{S}_{l\;hkp}^{\;j} = \hat{\Gamma}2^{\;j}_{r\;h}\,(T^r{}_{s|k}\,\hat{\Gamma}2_l{}^s{}_p - T^r{}_{s\hat{|}p}\,\hat{\Gamma}2_l{}^s{}_k) +$$

$$+ \hat{\Gamma}2^{\;j}_{r\;p}\,(T^r{}_{s|h}\,\hat{\Gamma}2_l{}^s{}_k - T^r{}_{s|k}\,\hat{\Gamma}2_l{}^s{}_h) + \hat{\Gamma}2^{\;j}_{r\;k}\,(T^r{}_{s\hat{|}p}\,\hat{\Gamma}2_l{}^s{}_h - T^r{}_{s|h}\,\hat{\Gamma}_l{}^s{}_p) +$$

$$+ \hat{\Gamma}2^{\;j}_{r\;h}\,(\hat{Q}_k{}^r{}_{l\hat{|}p} - \hat{Q}_p{}^r{}_{l\hat{|}k}) + \hat{\Gamma}2^{\;j}_{r\;p}\,(\hat{Q}_h{}^r{}_{l\hat{|}k} - \hat{Q}_k{}^r{}_{l\hat{|}h}) + \hat{\Gamma}2^{\;j}_{r\;k}\,(\hat{Q}_p{}^r{}_{l\hat{|}h} - \hat{Q}_h{}^r{}_{r\hat{|}p}) \tag{5.4.25a}$$

$$\hat{Q}_k{}^r{}_{l\hat{|}p} = \left(\frac{\partial T^r{}_s}{\partial x^k}\,\hat{1}^s{}_l\right)_{\hat{|}p}. \tag{5.6. 25b}$$

For isotopic liftings independent from the local coordinates (but dependent on the velocities and other variables, as it is often the case for the characteristic functions of interior physical media, isodifferential property (5.6.25) assumes the simpler form

$$\hat{R}_l{}^j{}_{hk\hat{|}p} + \hat{R}_l{}^j{}_{ph\hat{|}k} + \hat{R}_l{}^j{}_{kp|h} \equiv 0. \tag{5.6.26}$$

The isobianchi identity can also be equivalently written in the general case

$$\hat{R}_{ljhk\hat{|}p} + \hat{R}_{ljph\hat{|}k} + \hat{R}_{ljkp|h} = \hat{S}_{ljhkp}, \tag{5.6.27}$$

where the $\hat{S}$–term is that defined by Eq.s (5.6.26), with the reduced form for the isotopies not dependent on the local coordinates (or constant)

$$\hat{R}_{ljhk\hat{|}p} + \hat{R}_{ljph\hat{|}k} + \hat{R}_{ljkp\hat{|}h} \equiv 0. \tag{5.6.28}$$

We now consider the isotopic liftings of *Freud identity* which was originally identified by Freud [36] in 1939, reviewed in details by Pauli []37, and then forgotten for a long time by virtually all textbooks in gravitation. The identity was "rediscovered" by Yilmaz [38] who brought it to the attention of this author. The identity was then subjected to a mathematical study by Rund [39] (in perhaps his last paper). In memoir [19] published jointly with Rund's article [39], this author followed Rund's treatment, and reached the following property:

Property 5: The isofreud identity

$$\hat{\mathrm{U}}^{k}{}_{j} + \hat{\mathrm{G}}^{k}{}_{j} = \frac{\partial \hat{\mathrm{V}}^{kl}{}_{j}}{\partial x^{l}}, \tag{5.6.29}$$

where

$$\hat{\mathrm{V}}^{kl}{}_{j} = \tfrac{1}{2}\hat{\Delta}^{\frac{1}{2}}\{\hat{g}^{rs}(\delta^{k}{}_{j}\hat{\Gamma}^{2}{}^{l}{}_{rs} - \delta^{l}{}_{j}\hat{\Gamma}^{2}{}_{r}{}^{k}{}_{s}) + $$
$$+(\delta^{l}{}_{j}\hat{g}^{kr} - \delta^{k}{}_{j}\hat{g}^{lr})\hat{\Gamma}^{2}{}^{s}{}_{rs} + \hat{g}^{lr}\hat{\Gamma}^{2}{}^{k}{}_{jr} - \hat{g}^{kr}\hat{\Gamma}^{2}{}^{l}{}_{jr}\}, \tag{5.6.30a}$$

$$\hat{\mathrm{U}}^{k}{}_{j} = \tfrac{1}{2}\hat{\Delta}^{\frac{1}{2}}\Big(\frac{\partial \hat{\mathrm{G}}}{\partial \hat{g}^{lm}{}_{\uparrow k}}\hat{g}^{lm}{}_{\uparrow j} - \delta^{k}{}_{j}\hat{\mathrm{G}}\Big), \tag{5.6.30b}$$

$$\hat{\mathrm{G}} = \hat{g}^{jk}(\hat{\Gamma}^{2}{}^{p}{}_{js}\,T^{q}{}_{p}\hat{\Gamma}^{2}{}^{s}{}_{kq} - \hat{\Gamma}^{2}{}^{p}{}_{jk}\,T^{q}{}_{p}\,\hat{\Gamma}^{2}{}^{s}{}_{qs}), \tag{5.6.30c}$$

$$\hat{\mathrm{G}}^{k}{}_{j} = \hat{\Delta}^{\frac{1}{2}}\hat{\mathrm{G}}^{k}{}_{j}, \quad \hat{\Delta}^{\frac{1}{2}} = \sqrt{\hat{g}}. \tag{5.6.30d}$$

Rund's [39] reached the important result that *the Freud identity holds for all symmetric and nonsingular metrics on a (conventional) Riemannian space of dimension higher than one.* The same property evidently persist under isotopies. Thus, *Property 5 is automatically satisfied for all symmetric and nonsingular isometrics on isoriemannian spaces of dimension higher than one.* Despite this inherent compatibility of the identity with the geometry, the Freud identity and its isotopic image have important consequences in gravitation, e.g., for the vexing problem of the *source* of the gravitational field in vacuum.

In fact, Yilmaz's [38] points out that the conventional Freud identity on a Riemannian space raises the fundamental question, apparently still open to debates at this writing, whether a sourceless gravitational theory in vacuum does or does not verify all axioms of the Riemannian geometry.

We are now in a position to identify some of the first consequences of the isoriemannian geometry. First, it is an instructive exercise for the reader interested in acquiring a technical knowledge of the isotopies of the Riemannian geometry to prove the following important property:

Lemma 5.6.2 [19]: *Einstein's tensor* $G^{i}{}_{j} = R^{i}{}_{j} - \frac{1}{2}\delta^{i}{}_{j}R$ *does not preserve under isotopies the vanishing value of its covariant divergence (contracted Bianchi identity)*

$$G^{i}{}_{j|i} = R^{i}{}_{j|i} - \tfrac{1}{2}\delta^{i}{}_{j}R_{|i} \equiv 0, \tag{5.6.31}$$

that is, the isoeinsteinian tensor (5.6.15) is such that

$$\hat{G}^{k}{}_{i|k} = \hat{R}^{k}{}_{i\uparrow k} - \tfrac{1}{2}\delta^{k}{}_{i}\hat{R}_{\uparrow k} \neq 0, \tag{5.6.32}$$

Therefore, Einstein's tensor does not possess an axiomatically complete structure.

This unexpected occurrence has rather deep connections with the Freud identity, and implications for the identification of the correct theory of exterior gravitation in vacuum because it raises again the fundamental question, this time from an independent viewpoint, of the geometric consistency of a sourceless theory in vacuum.

It is interesting to note that *the Freud identity is a true geometric axiom of the Riemannian geometry in the sense that it persists under isotopies, while the contracted Bianchi identity is not, evidently because not preserved by isotopies.*

These occurrences shift the emphasis, from the historically predominant use of the contracted Bianchi identity, to the geometrically more rigorous Freud identity with predictable important implications for the entire theory of gravitation, both external and internal.

. The following property can also be proved via tedious but simple calculations from isodifferential property (5.6.25).

Lemma 5.6.3 [16,19]: *The completed isoeinstein tensor does possess an identically null isocovariant isodivergence, i.e.,*

$$\hat{S}^{i}_{j\uparrow j} = (\hat{R}^{i}_{j} - \tfrac{1}{2}\delta^{i}_{j}\hat{R} - \tfrac{1}{2}\delta^{i}_{j}\hat{\Theta})_{\uparrow i} \equiv 0. \qquad (5.6.33)$$

called the "completed and contracted isobianchi identity".

5.6.D: The fundamental theorem for interior isogravitation. As now familiar, we have initially considered conventional gravitational theories on $\mathfrak{R}(x,g,R)$ which have *null torsion,* and have reached an infinite family of isotopies all of which also have a *null isotorsion* on $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ because of the axiom-preserving character of the isotopies. In fact, the original symmetric connection $\Gamma^{2}{}_{h}{}^{s}{}_{k}$ has been lifted into an infinite family of isoconnections which are also symmetric

$$\tau_{h}{}^{s}{}_{k} = \Gamma^{2}{}_{h}{}^{s}{}_{k} - \Gamma^{2}{}_{k}{}^{s}{}_{h} \equiv 0 \;\Rightarrow\; \hat{\tau}_{h}{}^{s}{}_{k} = \hat{\Gamma}^{2}{}_{h}{}^{s}{}_{k} - \hat{\Gamma}^{2}{}_{k}{}^{s}{}_{h} \equiv 0. \qquad (5.6.34)$$

However, *the null value of torsion occurs at the level of isospace* $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ *which is not the physical space of the experimenter,* the latter remaining the conventional space-time in vacuum (see for details ref. [20], Ch. V).

The physical issue whether or not the isotopies of Einstein's gravitation for interior conditions have the non-null torsion required to avoid perpetual motion approximations, must therefore be inspected in the physical space and not in the

geometrical isospace.

This can be done by projecting the isocovariant derivative of an isovector on $\hat{\Re}(x,\hat{g},\hat{R})$ in the ordinary space $\Re(x,g,R)$, i.e.,

$$X^i{}_{\hat{|}k} = \frac{\partial X^i}{\partial x^k} + \hat{\Gamma}^2{}_h{}^i{}_k\, T^h{}_r\, X^r = \frac{\partial X^i}{\partial x^k} + \hat{\Gamma}^2{}_r{}^i{}_k\, X^r, \tag{5.6.35a}$$

$$\hat{\Gamma}^2{}_r{}^i{}_k = \hat{\Gamma}^2{}_h{}^i{}_k\, T^h{}_r. \tag{5.6.35b}$$

It is then evident that, starting with a symmetric isoconnection $\hat{\Gamma}_h{}^i{}_k$ on $\hat{\Re}(x,\hat{g},\hat{R})$, the corresponding connection $\hat{\Gamma}_r{}^i{}_k$ on $\Re(x,g,R)$ is no longer necessarily symmetric, and we have the following

Theorem 5.6.1 [18,19]: *The isotopic liftings $\Gamma^2{}_h{}^i{}_k \Rightarrow \hat{\Gamma}^2{}_h{}^i{}_k$ of a symmetric connection $\Gamma^2{}_h{}^i{}_k$ on a Riemannian space $\Re(x,g,R)$ into an infinite family of isotopic connections $\hat{\Gamma}^2{}_h{}^i{}_k$ on Isoriemannian spaces $\hat{\Re}(x,\hat{g},\hat{R})$ of the same dimension, imply that the isospace always possesses a null isotorsion, but, when the isotopies are projected into the original space, a non–null torsion generally occurs.*

The above property was first reached by Gasperini in ref. [32–34] in the language of conventional differential forms on a conventional Riemannian space. The geometrization of the property into a symmetric isotorsion was achieved by the author in ref. [18].

Theorem 5.6.1 is physically fundamental inasmuch as it ensures the needed structural differences for a realistic, quantitative representation of interior trajectories. We are referring to a representation of the *differences* in the trajectory of a test body from motion in vacuum with stable orbit (and thus null torsion) to motion within a physical medium with an unstable trajectory (and, therefore, non–null torsion, but null isotorsion).

Theorem 5.6.1 is also fundamental for our achievement of a geometric unit between the exterior and interior problem which will be more evident later on in this section. In fact, the instability of the interior trajectories is achieved via the same geometric axiom (null torsion) of the exterior problem, although realized in its most general possible isotopic form.

Finally, Theorem 5.6.1 necessarily requires *two different, but compatible theories: one for the exterior gravitational problem with null torsion, and one for the interior gravitational problem with null isotorsion but non–null torsion*.

The most important result of the analysis of this section can be expressed via a repetition under isotopies of ref. [26], p. 313 and the Theorem of p. 321, with the addition of the isofreud identity plus the completed Einstein's tensor (5.6.16), lead to the following:

Theorem 5.6.2 –Fundamental theorem for interior gravitation [18,19]: *In a (3+1)-dimensional isoriemannian isospace of Class I, $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$, the most general possible isolagrangian equations*

$$\hat{E}^{ij} = 0, \tag{5.6.36}$$

verifying the properties:
1) symmetric condition

$$\hat{E}^{ij} = \hat{E}^{ji}, \tag{5.6.37}$$

2) the contracted isobianchi identity

$$\hat{E}^{il}{}_{\hat{\uparrow} l} = 0, \tag{5.6.38}$$

and 3) the isofreud identity

$$\hat{U}^{k}{}_{j} + \hat{G}^{k}{}_{j} = \frac{\partial \hat{V}^{kl}}{\partial x^{l}}, \tag{5.6.39}$$

are characterized by the isolagrangian principle [42]

$$\begin{aligned} \delta\hat{A} &= \delta\int \hat{L}(\hat{g}_{ij}, \hat{g}_{ij,k}, \hat{g}_{ij,kl}, \tau_{ij}, t_{ij})\, dx = \\ &= \delta\int \hat{\Delta}^{\frac{1}{2}}[\lambda(\hat{R} + \hat{\Theta}) + 2\Lambda + \rho(\hat{\tau} + \hat{t})]\, dx = 0, \end{aligned} \tag{5.6.40}$$

where λ, Λ, and ρ are constants, $\hat{t}$ is the isotopic generalization of stress-energy tensor, $\hat{\tau}$ is an isotopic source tensor, $\hat{R}$ the isotopic curvature scalar and $\hat{\Theta}$ the isotopic scalar. For the case $\lambda = \rho = 1$, $\Lambda = 0$ and appropriate units, the isolagrange equations are given by

$$\hat{E}^{ij} = \hat{R}^{ij} - \tfrac{1}{2}\hat{g}^{ij}\hat{R} - \tfrac{1}{2}\hat{g}^{ij}\hat{\Theta} - \hat{\tau}^{ij} - \hat{t}^{ij} = 0, \tag{5.6.41}$$

and can be written in terms of the completed isoeinstein tensor

$$\hat{S}^{ij} = \hat{R}^{ij} - \tfrac{1}{2}\hat{g}^{ij}\hat{R} + \tfrac{1}{2}\hat{\Theta} = \hat{\tau}^{ij} + \hat{t}^{ij}, \tag{5.6.42}$$

or, equivalently, in terms of the isoeinstein tensor

---

[42] We are now in a position to clarify the meaning of "non-first-order-Lagrangians" in interior gravitation. As now well known, the Lagrangians emerging under isotopies, when projected in the original space, are of arbitrary order higher than the first, $L = L(s, x, \dot{x}, \ddot{x}, ...)$. However, the isolagrange equations remain of the second-order, evidently because they only depend on the second-order derivative of the isometric with respect to the local coordinates.

$$\hat{G}^{ij} = \hat{R}^{ij} - \tfrac{1}{2}\hat{g}^{ij}\hat{R} + \tfrac{1}{2}\hat{\theta} = \hat{T}^{ij} + \hat{t}^{ij}, \quad \hat{T}^{ij} = \hat{\tau}^{ij} - \tfrac{1}{2}\hat{\Theta}, \qquad (5.6.43a)$$

The reformulation of the above theorem in terms of isointegrals (Sect. 6.7) is an intriguing exercise for the interested reader.

The physical implications of the above theorem will be studied in Vol. II. Here were merely note the *dual* revision of conventional equations, one caused by the isoscalar $\hat{\Theta}$ and the other by the Freud identify which implies the identification

$$\hat{M}^{ij}{}_{\text{Einstein}} = \hat{\tau}^{ij} + \hat{t}^{ij}. \qquad (5.6.44)$$

As we shall see in Vol. II, this turns the exterior "description" of the gravitational field in vacuum into an *interior theory on the origin of the gravitational field* with numerous, rather intriguing and far reaching implications.

5.6E: Description of antimatter via the isodual isoriemannian geometry. We close this section with a brief study of the image of the isoriemannian geometry under isoduality, including the isodual definition of operations (such as fraction and derivatives) which can be expressed via the following

Theorem 5.6.3 [20,21]: *The interior problem of antimatter verifies Theorem 5.6.2 under isoduality characterized by the following maps:*

Basic unit	$\hat{1} \rightarrow \hat{1}^d = -\hat{1}$,
Isotopic element	$T \rightarrow T^d = -T$,
Isometric	$\hat{g} = Tg \rightarrow \hat{g}^d = -\hat{g}$,
Isoconnection coefficients	$\hat{\Gamma}_{klh} \rightarrow \hat{\Gamma}^{d\ l}{}_{klh} = -\hat{\Gamma}^l{}_{klh}$,
Isocurvature tensor	$\hat{R}_{lijk} \rightarrow \hat{R}^d{}_{lijk} = -\hat{R}_{lijk}$,
Isoricci tensor	$\hat{R}_{\mu\nu} \rightarrow \hat{R}^d{}_{\mu\nu} = -\hat{R}_{\mu\nu}$,
Isoricci scalar	$\hat{R} \rightarrow \hat{R}^d = \hat{R}$,
Isoeinstein tensor	$\hat{G}_{\mu\nu} \rightarrow \hat{G}^d{}_{\mu\nu} = -\hat{G}_{\mu\nu}$,
Isotopic scalar	$\hat{\Theta} \rightarrow \hat{\Theta}^d = \hat{\Theta}$,
Compl. isoeinstein tensor	$\hat{S}_{\mu\nu} \rightarrow \hat{S}^d{}_{\mu\nu} = -\hat{S}_{\mu\nu}$,
Electromagnetic potentials	$\hat{A}_\mu \rightarrow \hat{A}^d{}_\mu = -\hat{A}_\mu$,
Electromagnetic field	${}_{\mu\nu} \rightarrow \hat{F}^d{}_{\mu\nu} = -\hat{F}_{\mu\nu}$,
Elm energy–mom. tensor	$\hat{T}_{\mu\nu} \rightarrow \hat{T}^d{}_{\mu\nu} = -\hat{T}_{\mu\nu}$,
Stress–energy tensor	$\hat{t}_{\mu\nu} \rightarrow \hat{T}^d{}_{\mu\nu} = -\hat{t}_{\mu\nu}$.

The proof of the above properties is simple but instructive. In particular, it can show the necessity of the use of the isodual spaces to reach negative energies. In fact, in conventional Minkowski $M(x,\eta,R)$ and Riemannian spaces $\Re(x,g,R)$ the

electromagnetic potentials and fields do change sign for antiparticles, but the energy–momentum tensor remains the same. The latter changes sign only when computed in isodual Minkowski spaces $M^d(x,\eta^d,R^d)$ and isodual Riemannian spaces $\Re^d(x,g^d,R^d)$. These basic properties then persist when passing to the covering isospaces $\hat{M}(x,\hat{\eta},\hat{R})$, $\hat{\Re}(x,\hat{g},\hat{R})$ and their isoduals $\hat{M}^d(x,\hat{\eta}^d,\hat{R}^d)$, $\hat{\Re}^d(x,\hat{g}^d,\hat{R}^d)$.

The proof of Theorem 5.6.3 also shows that antimatter represented via the isodual isoriemannian geometry evolves "backward in time", as anticipated in Sect. 5.1, with intriguing epistemological conceptual and geometrical possibilities for advances, e.g., a theoretical conception of antigravity [43] studied in Vol. II.

5.7: ISOTOPIES AND ISODUALITIES OF PARALLEL TRANSPORT AND GEODESIC MOTION

Recall from Sect. I.5.1 that one of the primary objectives of the isotopies of the Riemannian geometry is the *geodesic* representation of the *free* fall of extended objects within physical media, such as a leaf in free fall in atmosphere.

A geometrically consistent generalization of the Riemannian geometry and of Einstein's gravitation cannot be reached without consistent isotopic coverings of conventional *parallel transport* and *geodesic motion* [4].

These generalized notions were introduced for the first time in memoir [16], expanded in ref.s [18,19], applied to interior gravitation in ref. [20,21] under the names of *isoparallel transport* and *isogeodesic motion* and reviewed in [25]. The most recent formulation is available in papers [44, 45] which is not reviewed here for brevity.

The new notions represent the maximal geometric achievements of the isotopies. They can be stated in figurative terms by saying that "physical media disappear under their isogeometrization". In fact, as we shall see, the trajectories of the isoparallel transport and the isogeodesics coincide with the original trajectories in vacuum when represented in isospaces.

Their knowledge is particularly important for hadronic mechanics. Recall that the sections of the perfect sphere, i.e., the circles, are geodesics of the rotational symmetry O(3). Isogeodesics are then important to understand that the sections of the ellispoidically deformed charge distributions of hadrons, the ellipses, are *bona fide* geodesics of the isorotational symmetry $\hat{O}(3)$ in isospace.

Since the times of Galileo Galilei and his experiments at the Pisa tower (1609), we know that *the free fall of a body in Earth's gravitational field is geodesic only in the absence of the resistive forces due to our atmosphere.* It is therefore well know that the trajectory of a test particle within a physical medium is not geodesic, owing to the resistive forces. Our isogeodesic then permits an ultimate geometric unity of motion in vacuum and within physical media which is the true foundation of the isorelativities of Vol. II.

Moreover, it is also well known since Lagrange's and Hamilton's times (see the historical notes of ref. [20]) that the forces between the body and the medium are of nonpotential type and, thus, of a type outside the representational capabilities of the conventional, local-differential, Riemannian geometry. A fully similar situation occurs for parallel transport, thus implying the inapplicability of the geometry itself for interior conditions.

Isoparallel transport and isogeodesic motions are crucial for a technical understanding of the isotopic relativities and of their underlying form-invariant description of physical laws via isosymmetries, because they complete the abstract geometric unity between interior and exterior problems found at the preceding levels in vector spaces, algebras, groups, etc.. In fact, parallel transport and geodesic motion are reached in interior conditions via the same abstract axioms of the corresponding quantities in vacuum, only realized in their most general possible way.

To begin, let $\Re(x,g,R)$ be a conventional n-dimensional Riemannian space. Under sufficient smoothness and regularity conditions hereon assumed, a vector field X^i on $\Re(x,g,R)$ is said to be *parallel along a curve* C if it satisfies the differential equation along C [4]

$$DX^i = X^i{}_{|s}\, dx^s = \Big(\frac{\partial X^i}{\partial x^s} + \Gamma 2^{\,i}_{r\,s}\, X^r \Big)\, dx^s = 0, \qquad (5.7.1)$$

where $\Gamma 2^{\,i}_{r\,s}$ is a symmetric connection. Then, by recalling the notions of isodifferential of Sect. 5.4, we have the following

Definition 5.7.1 [16,19]: *An isovector field* X^i *on an n-dimensional isoriemannian space of Class I* $\hat{\Re}(x,\hat{g},\hat{R})$ *is said to be transported via "isoparallel displacement" along a curve* C *on* $\hat{\Re}(x,\hat{g},\hat{R})$, *iff it verified the isotopic equations along* C

$$\hat{D}X^i = X^i{}_{\hat{|} r}\, T^r{}_s(x, \dot{x},..)\, \hat{d}x^s = \Big[\frac{\partial X^i}{\partial x^s} + \hat{\Gamma} 2^{\,i}_{r\,s}\, T^r{}_t(x, \dot{x},..)\, X^t \Big]\, T^s{}_p(x, \dot{x},..)\, \Big]\, \hat{d}x^p = 0 \qquad (5.7.2)$$

where $\hat{\Gamma}\, 2^{\,i}_{r\,s}$ *is the symmetric isoconnection and* $T = (T^r{}_s)$ *is the isotopic element of the underlying isofield* $\hat{R}(\hat{n},+,*)$.

The identity of axioms (5.7.1) and (5.7.2] at the abstract level is evident, again, because of the loss of all distinction between the right, modular, associative product, say Xx, and its isotopic generalization $X{*}x = XT(s, x, \dot{x}, \ddot{x}, ...)x$.

To understand the physical differences between the above two definitions, let us consider the independent (invariant) parameter s, such that the isovector field $\dot{x} = \hat{d}x/\hat{d}s$ is tangent to C, and let $X^i = X^i(s)$. Consider the curve C at a point P(1) for $s = s_1$ and let $X^i(1)$ be the corresponding value of the isovector field

X^i at P(1).

Consider now the transition from P(1) to P(2), i.e., from s_1 to $s_1 + \hat{d}s$. The corresponding transported value of the isovector field $X^i(2) = X^i(1) + \hat{d}X^i$ is said to occur under an *isoparallel displacement* on $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ in accordance with Definition 5.7.1, iff

$$\hat{d}X^i = \frac{\partial X^i}{\partial x^r} T^r{}_s \hat{d}x^s = -\Gamma^2{}^i{}_{rs} T^r{}_p X^p T^s{}_q \hat{d}x^q. \qquad (5.7.3)$$

The iteration of the process up to a finite displacement is equivalent to the solution of the integro-differential equation

$$\frac{\hat{d}X^i}{\hat{d}s} = \frac{\partial X^i}{\partial x^r} T^r{}_s \frac{\hat{d}x^s}{\hat{d}s} = -\Gamma^2{}^i{}_{rs} T^r{}_p X^p T^s{}_q \frac{\hat{d}x^q}{\hat{d}s}, \qquad (5.7.4)$$

By integrating the above expression in the finite interval (s_1, s_2), one reaches the following property (expressed in terms of isointegrals of Sect. 6.7)

Lemma 5.7.1 [loc. cit.]: *The isoparallel transport of an isovectorfield* $X^i(s)$ *on an n-dimensional isoriemannian space* $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ *of Class I from the point* s_1 *to a point* s_2 *on a curve* C *verifies the isotopic laws*

$$\hat{X}^i(2) = \hat{X}^i(1) - \hat{\int}_1^2 \Gamma^2{}^i{}_{rs}(x, \dot{x},..) T^r{}_p(x, \dot{x},..) X^p(x) T^s{}_q(x, \dot{x},..) \dot{x}^q \hat{d}s, \quad (5.7.5)$$

where

$$\hat{X}^i(2) - \hat{X}^i(1) = \hat{\int}_1^2 \hat{d}X^i = \int_1^2 \frac{\partial X^i}{\partial x^p} T^p{}_q \frac{\hat{d}x^q}{\hat{d}s} \hat{d}s. \qquad (5.7.6)$$

The physical implications are pointed out by the fact that the isotransported isovector does not start at the value $X^i(1)$, but at the *modified value* $\hat{X}^i(1)$ characterized by Eq.s (5.7.5). Additional evident modifications are characterized by the isotopic connection $\hat{\Gamma}^2{}^i{}_{rs}$ and the two isotopic elements T of the r.h.s. of Eq.s (57.5).

These departures from the conventional case can be better understood in a *flat isospace*, via the following evident

Corollary 5.7.1A [Loc. cit.]: *In a flat isospace, such as the isominkowski space* $\hat{M}(x,\hat{\eta},\hat{R})$ *in (3.1)-space-time dimensions, or the isoeuclidean space* $\hat{E}(r,\hat{\delta},\hat{R})$ *in 3-dimension, the conventional notion of parallelism no longer holds, in favor of the following flat isoparallelism*

$$\hat X^i(2) - \hat X^i(1) = \int_1^2 \hat\partial X^i = \int_1^2 \frac{\partial X^i}{\partial x^p} T^p{}_q \frac{\hat\partial x^q}{\hat\partial s} \hat\partial s. \qquad \hat\Gamma^2{}^i_{rs} \equiv 0, \qquad (5.7.7)$$

ISOPARALLEL TRANSPORT

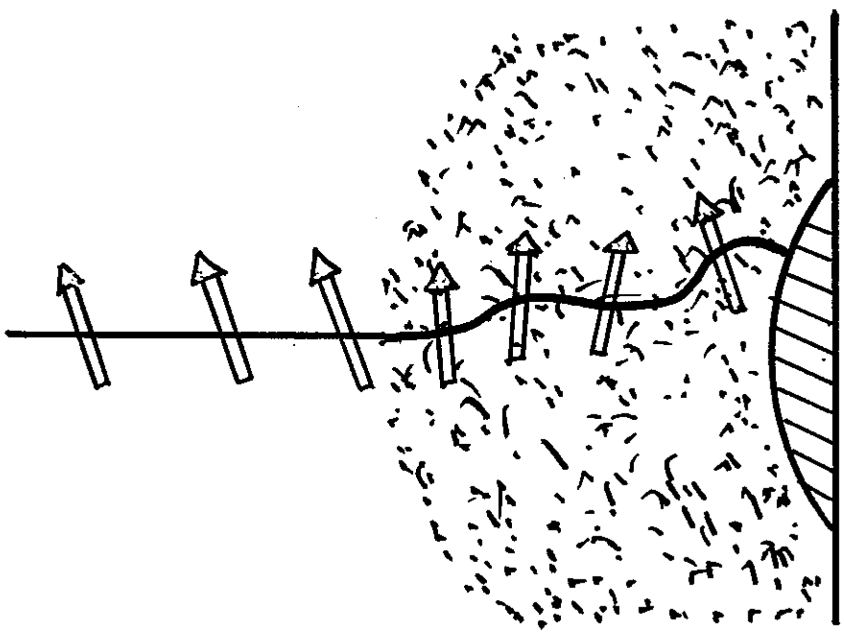

FIGURE 5.7.1: A schematic view of the isotopic representation of parallel transport in isoriemannian space. Consider, say, a rocket under parallel transport in empty space, e.g., due to free fall toward Earth. When penetrating within physical media, the same object is, first, twisted depending on its shape, and then moves along an anomalous trajectory. The isoriemannian geometry permits the geometrization of the latter motion via the *isoparallel transport.* Its understanding requires the knowledge that the anomalous trajectory depicted in the figure occurs in our space, while in isospace the object continues with exactly the same original trajectory.

Consider, as an illustration, a straight line C in conventional Euclidean space $R_t \times E(r,\delta,R)$, with only two space–components. Then a vector $\overline{R}(1)$ at $s = t_1$ is transported in a parallel way to $\overline{R}(2)$ at $s = t_2$ by keeping unchanged the characteristic angles with the reference axis, i.e.,

$$\hat R(2) - \hat R^k(1) = \int_1^2 \left(\frac{\partial R^k(r)}{\partial x^1} \hat\partial x^1 + \frac{\partial R^k(r)}{\partial x^2} \hat\partial x^2. \qquad (5.7.8)$$

Under isotopy, the situation is no longer that trivial. In fact, assume the simple diagonal isotopy

$$T = \mathrm{diag.}\,(b_1{}^2(\hat{r}),\, b_2{}^2(\hat{r})) > 0. \qquad (57.9)$$

Then Eq.s (5.7.5) yield into the form

$$\hat{R}^k(2) - \hat{R}^k(1) = \int_1^2 \Big(\frac{\partial R^k(r)}{\partial r^1} b_1{}^2(\hat{r})\, \hat{d}r^1 + \frac{\partial R^k(r)}{\partial r^2} b_2{}^2(\hat{r})\, \hat{d}r^2 \qquad (5.7.10)$$

The *irreducibility* of the notion of isoparallel transport to the conventional notion can be illustrated even in the case of null curvature. In fact, consider for simplicity the isominkowski-space $M(x,\hat{\eta},\hat{R})$ with local coordinates $x = (x^\mu)$, $\mu = 1, 2, 3, 4$, with constant diagonal isotopy

$$\hat{\eta} = T\eta, \quad T = \mathrm{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^2, b_4{}^2) > 0. \qquad (5.7.11)$$

and introduce the redefinitions $\hat{x}^\mu = b_\mu{}^2 x^\mu$ (no sum), $X^\mu(x(\hat{x})) = \hat{X}^\mu(\hat{x})$.

Then Eq.s (5.7.5) become

$$\int_1^2 [\frac{\partial X^\mu(x)}{\partial x^\alpha} b_\alpha{}^2 \hat{d}x^\alpha \equiv \int_1^2 \frac{\partial \hat{X}^\mu(\hat{x})}{\partial \hat{x}^\alpha} b_\alpha{}^2\, \hat{d}\hat{x}^\alpha, \qquad (5.7.12)$$

namely, the isotopy persists even under the simplest possible constant isotopy (5.7.11), thus confirming the achievement of a novel geometrical notion.

By submitting the conventional treatment (ref. [4], Sect. 3.7) to isotopies, one can identify the *integrability conditions for the existence of isoparallelism* result in the condition

$$\frac{\partial X^i}{\partial x^s\, \partial x^t} = - \frac{\partial \hat{\Gamma}^{2\,i}_{\ r s}}{\partial x^t} T^r_{\ p} X^p + \hat{\Gamma}^{\ i}_{r s} T^r_{\ q} \Gamma^{2\ q}_{\ m t} T^m_{\ n} X^n +$$

$$+ \hat{\Gamma}^{2\,i}_{\ r s} \frac{\partial T^r_{\ p}}{\partial x^t} X^p = \frac{\partial^2 X^i}{\partial x^t\, \partial x^s} = \frac{\partial \hat{\Gamma}^{2\,i}_{\ r t}}{\partial x^s} T^r_{\ p} X^p +$$

$$+ \hat{\Gamma}^{2\,i}_{\ r t} T^r_{\ p} \hat{\Gamma}^{2\ p}_{\ m s} T^m_{\ n} X^n + \hat{\Gamma}^{2\,i}_{\ r t} \frac{\partial T^r_{\ p}}{\partial x^s} X^p \qquad (5.7.13)$$

from which the following property holds.

Lemma 5.7.2 [loc. cit.]: *Necessary and sufficient conditions for the existence of*

an isoparallel transport of an isovector X^i *on an n–dimensional isoriemannian isospace* $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ *are that all the following equations hold*

$$\hat{R}_{l\ hk}^{\ i}\, T^l_{\ s}\, X^s = 0, \qquad (5.7.14)$$

where $\hat{R}_{r\ pq}^{\ i}$ *is the isocurvature, Eq.s (5.6.12).*

ISOGEODESIC MOTION

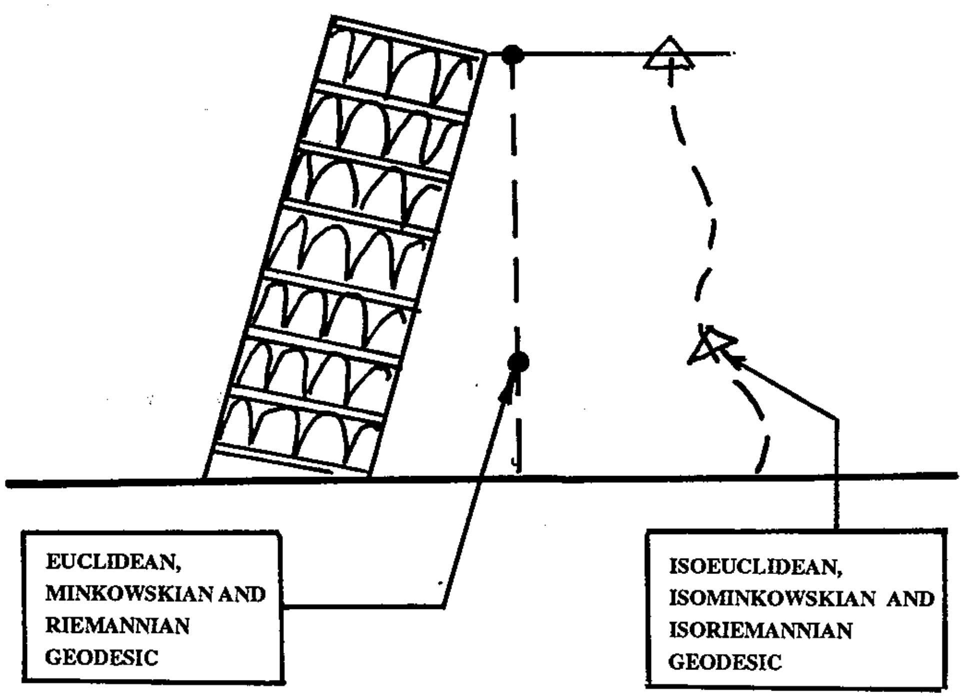

FIGURE 5.7.1: The birth of the notion of geodesic motion can be seen in Galilei's historical conception of uniform motion in vacuum, i.e., via the celebrated Galilei's boosts

$$r'^k = r^k + t^\circ v^{\circ k}, \qquad p'_k = p_k + m\, v^\circ_k, \qquad (1)$$

which can be formulated in terms of the contemporary modular action

$$T(v^\circ)\, r^k = r^k + t^\circ v^{\circ k}, \quad T(v^\circ)\, p_k = p_k + m\, v^\circ_k. \qquad (2)$$

As well known, Galilei established the above law by ignoring the friction due to the air. Our studies essentially aim at the achievement of a geodesic characterization of the motion of free objects within physical media in such a way as to preserve the original axioms of the free motion in vacuum.

Stated in different terms, the understanding of the content of this chapter can be reduced to the understanding that the irregular trajectory of this figure describing the free fall of an objective under the resistive force due to the atmosphere does indeed verify the same geodesic axioms of Galilei's free fall in the absence of the atmosphere. In fact, in isoeuclidean space it is a straight isoline (Sect. 5.2), exactly as the trajectory in the absence of air, and a similar occurrence holds for curved spaces.

The fundamental tool is provided by the isospaces. In fact, we represent the transition from motion in vacuum to motion within a physical medium via the transition from conventional Euclidean, Minkowskian or Riemannian space to the corresponding isoeuclidean, isominkowskian and isoriemannian spaces, respectively. By recalling that the conventional spaces provide a geometrization of the vacuum (empty space), one can then confirm the isogeometrization of interior physical media of Sect.s 5.2 and 5.3.

This yields the most general possible, nonlinear, nonlocal and noncanonical generalization of laws (1) in $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$

$$r'^k = r^k + t^\circ v^\circ B^{k-2}(t,r,p,...), \qquad p'_k = p_k + m v^\circ B_k{}^{-2}(t,r,p,...), \tag{3}$$

and represented via the isotopic group action (see ref. [20] for a detailed classical treatment and Vol. II for the operator counterpart)

$$\hat{T}(v^\circ) * r^k = r^k + t^\circ v^{\circ k} B^{k-2}, \quad \hat{T}(v^\circ) * p_k = p_k + m v^\circ{}_k B_k{}^{-2}. \tag{4}$$

where the B's are certain nonlinear–nonlocal functions computable from the knowledge of the isounit.

The arbitrariness of the isounits, that is, of the B–function then illustrate the "direct universality" of the *isogalilean relativity* for the form–invariant description of interior trajectories. The preservation of the original Galilean axioms can also be seen by nothing that isoboosts (4) form an isogroup (Sect. 4.5), e.g., the composition of two successive Galilean boosts

$$T(v^\circ)\, T(v^{\circ\prime}) = T(v^\circ + v^{\circ\prime}), \tag{5}$$

is lifted into the isocomposition of two isoboosts

$$\hat{T}(v^\circ) * \hat{T}(v^{\circ\prime}) = \hat{T}(v^\circ + v^{\circ\prime}). \tag{6}$$

The abstract identity of the Galilean and isogalilean relativities then follows from the manifest abstract identity of group (5) with its isotopic covering (6), that is, *isogeodesics in isospace coincide with the original geodesics in vacuum.* The same result can be directly reached via principle (5.7.17) which shows that, jointly with the deformation $b_k{}^2$ along the k–axis, the unit along the same axis is deformed of the inverse amount $b_k{}^{-2}$.

The re–emergence of the isocurvature tensor as part of the integrability

conditions of isoparallel transport, can then be considered as a confirmation of the achievement of a novel geometrical notion.

We now pass to the *isogeodesics motion*. Let s be an invariant parameter and consider the tangent $\dot{x}^i = \hat{d}x^i/\hat{d}s$ of the curve C on an n-dimensional isoriemannian space $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$. Its absolute isodifferential is given by

$$\hat{D}\dot{x}^i = \hat{d}\dot{x}^i + \hat{\Gamma}2^{\ i}_{r\ s}\, T^r_{\ p}\, \dot{x}^p\, T^s_{\ q} \hat{d}x^q. \qquad (5.7.15)$$

In accordance with Definition 5.6.3, $\hat{D}\dot{x}^i$ remains isoparallel along C iff $\hat{D}\dot{x}^i = 0$. We can therefore introduce the following

Definition 5.7.2 [loc. cit.]: *The "isogeodesics" of an n-dimensional isoriemannian manifold of Class I, $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$, are the solutions of the differential equations*

$$\frac{\hat{d}^2x^i}{\hat{d}s^2} + \hat{\Gamma}2^{\ i}_{r\ s}(x, \dot{x}, \ddot{x}..)\, T^r_{\ p}(x, \dot{x}, \ddot{x}..)\, \frac{\hat{d}x^p}{ds}\, T^s_{\ q}(x, \dot{x}, \ddot{x}..)\, \frac{\hat{d}x^q}{ds} = 0 . \qquad (5.7.16)$$

It is a simple but instructive exercise to prove the following

Lemma 5.7.2 [loc. cit.]: *The isogeodesics of an n-dimensional isoriemannian space $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ are the curves verifying the principle*

$$\delta\int \hat{d}s = \delta\hat{\int} [\, \hat{g}_{ij}(x, \dot{x}, \ddot{x}..)\, \hat{d}x^i\, \hat{d}x^j\,]^{\frac{1}{2}} = 0. \qquad (5.7.17)$$

We discover in this way a new important role of the isometric essentially similar to the corresponding role of conventional metric in geodesic motion. Also, the appearance of the isometric in the variational principle characterizing isogeodesic motion is a confirmation of the achievement of a novel geometry.

APPENDIX 5.A: ELEMENTS OF THE SYMPLECTIC GEOMETRY

In this appendix we shall outline the rudiments of the conventional symplectic geometry from ref.s [3,4,6] in its local-differential, canonical as well as Birkhoffian versions. The presentation will then result to be useful for reader not familiar with the field, not only for the nonlocal-integral extension of this chapter, but also for the isotopies of symplectic quantization of Vol. II.

As done in Sect. 5.4, all quantities are assumed to verify the needed continuity conditions, e.g., of being of Class $\mathcal{C}^\infty$, and all neighborhoods of given points are assumed to be star-shaped, or have a similar topology also ignored hereon for brevity.

Let M(R) be an n-dimensional manifold over the reals R(n,+,×) . A *tangent vector* X_m at a point $m \in M(R)$ is a linear function defined in the neighborhood

of m with values in R satisfying the rules

$$X_m(\alpha f + \beta g) = \alpha X_m(f) + \beta X_m(g), \tag{5.A.1a}$$

$$X_m(fg) = f(m) X_m(g) + g(m(X_m(f), \tag{5.A.1b}$$

for all f, g $\in C^\infty(M)$, $\alpha, \beta, \in R$.

The *tangent space* T_mM at m is the vector space of all tangent vectors at m. The *tangent bundle* is the 2n-dimensional space TM = $U_m T_mM$ equipped with a structure (see below). The cotangent bundle T^*M is the dual of TM given by the space of all linear functional on TM also equipped with a structure.

Let $x = \{ x^{1,\dots,} xn \}$ be a local chart in the neighborhood of m. Then it can be shown that the ordered set dx forms a basis of T^*M, while $\partial/\partial x$ forms a basis of TM. An element $\theta \in T^*M$ and $\chi \in$ TM can the be written in local coordinates

$$\theta = \theta_i(m)\, dx^i , \quad X = X^i(m)\, \partial / \partial x^i , \tag{5.A.2}$$

θ is then called the *canonical form*. The cotangent bundle T^*M equipped with θ is at times denoted $T^*M_1(R)$. The *fundamental (canonical) symplectic form* is then given by the two-form

$$\omega = d\theta, \tag{5.A.3}$$

which is nowhere degenerated, exact and therefore closed; i.e., such that $d\omega \equiv 0$. The manifold $T^*M(R)$, when equipped with two-form ω becomes an (exact) *symplectic manifold* $T^*M_2(R)$ in canonical realization. The *symplectic geometry* is the geometry of symplectic manifolds as characterized by exterior forms, Lie's derivative, etc.

Let H be a function on $T^*M_2(R)$ called the *Hamiltonian*. A vector-field X on $T^*M_2(R)$ is called a *Hamiltonian vector-field* when it verifies the condition

$$X \rfloor \omega = - dH. \tag{5.A.4}$$

The above equation provides a global, coordinate-free characterization of the conventional *Hamilton's equations* (those without external terms) for the case of *autonomous systems;* i.e., systems without an explicit dependent in the independent variable (time t).

Finally, we recall that the *Lie derivative* of a vector-field Y with respect to the vector field X on $T^*M_2(R)$ can be defined by

$$L_X Y = [X,Y], \tag{5.A.5}$$

where [X, Y] is the canonical commutator. The case of *nonautonomous systems* (those with an explicit dependence on time) requires the further extension to the *contact geometry* (see, e.g., ref. [3]). However, the Lie content is contained in the symplectic part of the geometry.

The Birkhoffian generalization of the above canonical geometry is straightforward, and was worked out in ref.s [5,6]. Introduce in the same cotangent bundle $T^*M_1(R)$ the most general possible one-form Θ, called the *Birkhoffian* or *Pfaffian one-form*. The *Birkhoffian two-form* is then given by

$$\Omega = d\Theta, \qquad (5.A6)$$

under the condition that it is nowhere degenerate. Ω is exact by construction and therefore closed, that is, symplectic. The manifold $T^*M(R)$, when equipped with the two-form Ω, becomes an *exact, Birkhoffian, symplectic manifold* $T^*M_2(R)$.

Let B be another function on $T^*M_2(R)$ called the *Birkhoffian.* Then, a non-Hamiltonian vector-field $\hat{X}$ on $T^*M_2(R)$ is called a *Birkhoffian vector-field* when it verifies the property

$$\hat{X} \rfloor \Omega = - dB. \qquad (5.A.7)$$

which provides a global, coordinate-free characterization of *Birkhoff's equations for autonomous systems.*

Similarly, we recall that the *Lie-isotopic derivative* of a vector-field $\hat{Y}$ with respect to a *nonhamiltonian* vector field $\hat{X}$ [5,6] can be written

$$\hat{L}_{\hat{X}}\hat{Y} = [\hat{X} \,\hat{,}\, \hat{Y}], \qquad (5.A.8)$$

where the brackets are now Birkhoffian (see below for the explicit form).

The realization of the above global structures in local coordinates is straightforward. Interpret the space M(R) as an Euclidean space $E(r,\delta,R)$ with local coordinates $r = (r_i)$, i= i, 2, ..., n. Then, the cotangent bundle T^*M becomes $T^*E(r,\delta,R)$ with local coordinates $(r,p) = (r_i, p_i)$, where $p = dr/dt$ represents the tangent vectors, and we ignore for simplicity of notation the distinction between contravariant and covariant indices in Euclidean spaces (but not in the cotangent bundle). The canonical one-form (5.A.2) then admits the local realization

$$\theta = p_i \, dr_i. \qquad (5.A.9)$$

The Hamiltonian two-form (5.A.3) admits the realization

$$\omega = d\theta = dp_i \wedge dr_i, \qquad (9.10)$$

from which one can easily verify that $d\omega \equiv 0$. A vector–field can then be written

$$X = A_i(r,p)\,\partial / \partial r_i + B_i(r,p)\,\partial / \partial p_i, \tag{5.A.a}$$

$$A_i\, dr_i + B_i\, dp_i = -dH, \tag{5.A.b}$$

which can hold iff Hamilton's equations are verified, i.e.,

$$\frac{dr_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = - \frac{\partial H}{dr_i}, \tag{5.A.12}$$

Finally, Lie's derivative (5.A.4) admits the simple realization

$$L_X Y = [X,Y] = \frac{\partial X}{\partial r_i}\,\frac{\partial Y}{\partial p_i} - \frac{\partial Y}{\partial r_i}\,\frac{\partial X}{\partial p_i}, \tag{5.A.13}$$

where one recognizes in the commutator the familiar Poisson brackets.

The realization of the Birkhoffian generalization of the above structures requires the introduction of the unified notation

$$a = (a^{\mu}) = (r, p) = (r_i, p_i), \quad \mu = 1, 2, ..., 2n, \quad i = 1, 2, ..., n, \tag{5.A.14}$$

where we preserve the distinction between contravariant and covariant indices. The canonical one–form can then be rewritten

$$\theta = R^{\circ}_{\mu}\, da^{\mu} \equiv p_i\, dr_i, \quad R^{\circ} = (p, 0), \tag{5.A.15}$$

and Hamiltonian two–form (5.A.10) becomes

$$\omega = d\theta = \tfrac{1}{2}\,\omega_{\mu\nu}\, da^{\mu} \wedge da^{\nu} \equiv dp_i \wedge dr_i, \tag{5.A.16}$$

where $\omega_{\mu\nu}$ is the *covariant, canonical, symplectic tensor* (5.A.15), i.e.,

$$(\omega_{\mu\nu}) = \left(\frac{\partial R^{\circ}_{\nu}}{\partial a^{\mu}} - \frac{\partial R^{\circ}_{\mu}}{\partial a^{\nu}}\right) = \begin{pmatrix} 0_{n\times n} & -I_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{5.A.17}$$

A vector–field can then be written

$$X = X_{\mu}(a)\,\partial / \partial a^{\mu}. \tag{5.A.18}$$

The conditions for a Hamiltonian vector–field become

$$\omega_{\mu\nu} X^\mu da^\mu = -dH, \tag{5.A.19}$$

and can hold iff

$$X = X_\mu \frac{\partial}{\partial a^\mu} = \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} \frac{\partial}{\partial a^\mu}, \tag{5.A.20}$$

where

$$\omega^{\mu\nu} = (|\omega_{\alpha\beta}|^{-1})^{\mu\nu}, \tag{5.A.21}$$

namely, iff Hamilton's equations (5.A.12) hold, which in the unified notation can be written

$$\dot{a}^\mu = \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu}. \tag{5.A.22}$$

Finally, Lie's derivative becomes

$$L_X Y = [X,Y] = \frac{\partial X}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial Y}{\partial a^\nu}, \tag{5.A.23}$$

The transition to the Birkhoffian realization [6] is now straight–forward [5,6]. In fact, it merely requires the transition from the canonical quantities $R^\circ(a) = (p, 0)$ to arbitrary quantities $R(a)$ on $T^*E_1(r,\delta,R)$ under which the Birkhoffian one–form (5.A.5) assumes the realization

$$\Theta = R_\mu(a)\, da^\mu, \tag{5.A.24}$$

while the Birkhoffian two–form (5.A.6) becomes

$$\Omega = d\Theta = \tfrac{1}{2} \Omega_{\mu\nu}(a)\, da^\mu \wedge da^\nu. \tag{5.A.25}$$

where $\Omega_{\mu\nu}$ is the *(covariant) symplectic Birkhoff's tensor*

$$\Omega_{\mu\nu}(a) = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \tag{5.A.26}$$

A Birkhoffian vector-field $\hat{X}$ can no longer be decomposed in the simple form (5.A.11), but can be written

$$\hat{X} = X^{\mu} \partial / \partial a_{\mu}. \tag{5.A.27}$$

The conditions for a vector-field $\hat{X}$ to be Birkhoffian, Eq.s (5.A.7), then become

$$\hat{X} \rfloor \Omega = \Omega_{\mu\nu} \hat{X}^{\nu} da^{\mu} = - dB, \tag{5.A.28}$$

and they hold iff

$$\hat{X} = \hat{X}^{\mu} \frac{\partial}{\partial a^{\mu}} = \Omega^{\mu\nu} \frac{\partial B}{\partial a^{\nu}} \frac{\partial}{\partial a^{\mu}}, \tag{5.A.29}$$

where

$$\Omega^{\mu\nu} = (|\Omega_{\alpha\beta}|^{-1})^{\mu\nu}, \tag{5.A.30}$$

which can hold iff the *autonomous Birkhoff's equations* hold, i.e.,

$$\dot{a}^{\mu} = \hat{X}^{\mu} = \Omega^{\mu\nu}(a) \frac{\partial B(a)}{\partial a^{\nu}}. \tag{5.A.31}$$

Similarly, the Lie-isotopic derivative (5.A.8) assumes the realization

$$\hat{L}_{\hat{X}}\hat{Y} = [\hat{X} \hat{,} \hat{Y}] = \frac{\partial \hat{X}}{\partial a^{\mu}} \Omega^{\mu\nu}(a) \frac{\partial \hat{Y}}{\partial a^{\nu}}, \tag{5.A.32}$$

For additional aspects, the reader may consult ref. [6], the appendices of Ch. 4.

Note that an arbitrary vector-field $\hat{X}$ is not Hamiltonian in a given local chart. A central result of ref. [6] can be reformulated as follows

Theorem 5.A.1 – Direct universality of the symplectic geometry for local nonhamiltonian Newtonian systems [6]: *An arbitrary, local-differential, nonhamiltonian, analytic and regular vector-field* $\hat{X}$ *on a given chart on* $T^*M_2(r,R)$ *always admits in a star-shaped neighborhood of the local variables a direct representation as a Birkhoffian vector-field, i.e., a representation via Birkhoff's equations directly in the chart considered.*

The physical implications are as follows. When considering conservative-potential systems of the exterior dynamical problem (Ch. I.1), the vector-fields are evidently Hamiltonian in the frame of the experimenter. However, when

considering the nonconservative systems of the interior dynamical problem, the vector-fields are generally nonhamiltonian in the frame of the experimenter.

Now, under sufficient topological conditions, the *Lie-Koening theorem* (see ref. [6] and quoted literature) ensures that a local-differential nonhamiltonian vector-field can always be transformed into a Hamiltonian form under a suitable change of coordinates.

However, since the original vector-field is nonhamiltonian by assumption, the transformations must necessarily be *noncanonical* and *nonlinear*, thus creating evident physical problems, e.g., conventional relativities become inapplicable because turned into *noninertial* formulations.

This creates the need of the "direct representation" of the physical systems considered; that is, their representation, first, in the frame of the experimenter, as per Theorem 5.A.1. Once this basic task is achieved, then the judicious use of the transformation theory may have some physical value.

Intriguingly, the identification of the Lie-Koening transformation $a \Rightarrow a'$ turning nonhamiltonian systems $\hat{X}(a)$ into Hamiltonian forms $\hat{X}(a(a')) = X(a')$, implies the Birkhoffian representation of Theorem 5.A.1 in the a-frame of the observer. In fact, Birkhoff's equations (5.A.31) in the a-frame can be characterized precisely via a *noncanonical* transformation $a' \Rightarrow a$ of Hamilton's equations (5.A.22) in the a'-frame, i.e.,

$$\omega_{\mu\nu} \dot{a}'^{\nu} - \frac{\partial H(a')}{\partial a'^{\mu}} = \frac{\partial a^{\rho}}{\partial a'^{\mu}} [\Omega_{\rho\sigma}(a) - \frac{\partial B(a)}{\partial a^{\rho}}] = 0, \qquad (5.A.33a)$$

$$H(a'(a)) = B(a), \qquad (5.A.33b)$$

(see ref. [6], p.130 for details).

As an introduction to the covering isosymplectic geometry (Sect. 5.4), the above canonical and Birkhoffian forms can be expressed in a yet more general way. Consider again the original cotangent bundle $T^*M(R)$, and let

$$I^{\circ} = (I_{n\times n}) = \text{diag.}(1, 1, \ldots., 1) \equiv T^{\circ\,-1} \qquad (5.A.34)$$

be its unit. Then, the canonical one form (5.A.2) can be identically written in terms of the factorization

$$\theta \equiv \hat{\theta}^{\circ} = \theta \times T^{\circ} : T^*\hat{M}_I{}^{\circ} \Rightarrow T^*(T^*\hat{M}_I{}^{\circ}), \qquad (5.A.35)$$

while the canonical two-form (5.A.3) becomes

$$\omega \equiv \hat{\omega}^{\circ} = d\hat{\theta}^{\circ} = (d\theta) \times T^{\circ} + \theta\, dT^{\circ} \equiv \omega \times T^{\circ} \qquad (5.A.36)$$

This implies that, in the realization $T^*E(r,\delta,R)$ of $T^*M(R)$ with local chart a =

(r, p), we can write

$$\hat{\omega}^{\circ}{}_{\mu\nu} = T^{\circ}{}_{\mu}{}^{\alpha}\,\omega_{\alpha\nu}, \tag{5.A.37}$$

Then, its contravariant version is exhibited in the Lie–tensor of the theory,

$$\hat{\omega}^{\circ\mu\nu} = \omega^{\mu\alpha}\, I^{\circ}{}_{\alpha}{}^{\nu}. \tag{5.A.38}$$

The transition to the isosymplectic geometry in Birkhoff–isotopic realization is then performed by assuming that the isotopic element and unit are no longer the trivial unit, but arbitrary integro–differential quantities.

In the latter generalization one central property persists: *the transition from the canonical to the Birkhoffian and Birkhoffian–isotopic formulations requires noncanonical transformations.* This is the geometric–analytic counterpart of the corresponding algebraic property. In fact, *the transition from the classical (operator) formulation of Lie's theory to its isotopic covering necessarily requires noncanonical transformations (nonunitary transformations)*

the above results imply that *quantum mechanics and its covering hadronic mechanics are inequivalent because not interconnected by a unitary transformations* (see Vol. II for details).

In closing we mention the so–called *multisymplectic generalization* of the content of this appendix, as presented in the recent monograph by Sardanashvily [42] and related jet manifolds which have intriguing possibilities for further isotopic formulation and application to interior dynamical problems.

APPENDIX 5.B: GRAVITATION IN ISOMINKOWSKIAN SPACE

Isotopic techniques permit *novel* approaches to gravitation, i.e., approaches not permitted by conventional Riemannian methods. One of them is *the equivalent study of gravitation on a isoflat geometry.*

This approach is not a mere mathematical curiosity, but resolves a rather old problematic aspect of current gravitational theories: the absence of weight in relativistic theories. Consider a test body experiencing a gravitational field at a space–time point x in a Riemannian space $\Re(x,g,R)$. As well known [10,11,37], *gravitation is entirely represented by the curvature in current theories,* i.e., by the metric g(x) (for null total charge whose gravitational effect is ignorable anyhow). In passing at the tangent Minkowski space $M(x,\eta,R)$ at the same point x, all gravitational effects disappear (equivalence principle), which is contrary to experimental evidence that a relativistic particle, such as a proton in a particle acceleration, does indeed verify gravity [21]. Weight is preserved in current theories in flat spaces, but only in the limit into the *Euclidean* space.

In the physical reality, weight is present irrespective of our treatment,

whether nonrelativistic, relativistic or gravitational. Any consistent treatment of gravitation must therefore have a well defined Minkowskian counterpart.

The above problematic aspect of current theories is resolved by the isotopies because of the geometric equivalence between the Riemannian and isominkowskian spaces of Sect. I.3.7,

$$\Re(x,g,R) \approx \hat{\Re}(x,\hat{g},\hat{R}) \equiv \hat{M}(x,\hat{\eta},\hat{R}) \approx M(x,\eta,R), \quad (5.B.1a)$$

$$g(x) = T(x)\,\eta = \hat{\eta}, \qquad \hat{1} = [\,\hat{T}(x)\,]^{-1}. \quad (5.b.1b)$$

where $\hat{1}$ ($\hat{T}$) is called the *gravitational isounit (gravitational isotopic element).*

In fact, all gravitational theories admit the decomposition $g = T\eta$ with $T > 0$ as a necessary condition to be locally Minkowskian. Then $\hat{1} > 0$ and the equivalence chain (5.B.1a) follows.

Current gravitational theories are formulated in a *curved space* with metric $g(x)$ with respect to the *conventional unit* $I = \text{diag.}\ (1, 1, 1, 1)$. Isotopic theories permit the treatment of exactly the same metric $g(x) \equiv \hat{\eta}(x)$ although referred to the gravitational isounit $\hat{1}$ in the isominkowskian space $\hat{M}(x,\hat{\eta},\hat{R})$.

Note that curvature is entirely contained in the isotopic element $T(x)$ of decomposition $g(x) = T(x)\eta$. The isominkowskian treatment therefore implies the study of the curvature via $g \equiv T\eta$ at x, while assuming at the same point x an isounit which is the inverse of the "curvature", $\hat{1}(x) = [\hat{T}(x)]^{-1}$. This is precisely the mechanism that renders the treatment of gravitation locally *isoflat.*

In Sect. 5.3 we have indicated that the isominkowskian geometry preserves all curved characteristics of the Riemannian geometry. It is an instructive exercise for the interested reader to reconstruct in $\hat{M}(x,\hat{\eta},\hat{R})$ all properties of the Riemannian geometry, including Ricci lemma, Einstein's tensor, field equations, etc. One can therefore see in this way that all the results on $\Re(x,g,R)$ equally hold on $\hat{M}(x,\hat{\eta},\hat{R})$.

Besides resolving the problematic aspect of the "disappearance of weight" at the tangent Minkowski space, isotopic methods permit a *novel* approach to gravitational singularities, which now become *the singularities of the isounit,*

$$\hat{T}(x) \rightarrow 0, \qquad \hat{1}(x) \rightarrow \infty, \quad (5.B.2)$$

or *the singularities of the isotopic element,*

$$\hat{T}(x) \rightarrow \infty, \qquad \hat{1}(x) \rightarrow 0. \quad (5.b.3)$$

As an example, the celebrated Schwarzschild's line element in spherical polar coordinates admits the isotopic factorization into

$$\hat{T} = \text{diag.}\ \{\,(1 - 2M/r)^{-1}, r^2, r^2 \sin^2\theta, (1 - 2M/r)\,\}. \quad (5.B.4)$$

where one should keep in mind our ordering (+, +, +, –). We then have the following

Proposition 5.B.1 [21]: *The Schwarzschild's singularity at the horizon $r = 2M$ is a zero of the isounit, while its singularity at the origin $r = 0$ is a zero of the isotopic element.*

The reader should be aware that the above novel perspectives on gravitational collapse are studied merely as a basis for the intended studies, their treatment via the interior nonlocal isoriemannian geometries. In fact, the equivalence chain (5.B.1a) can also be formulated at the fully isotopic level of Class I

$$\hat{R}(x,\hat{g},\hat{R}) \approx \hat{M}(x,\hat{\eta},\hat{R}), \tag{5.B.5a}$$

$$\hat{g} = \hat{T}(x, \dot{x}, \ddot{x}, \mu, \tau, n, ...)\, g(x) \equiv \hat{T}'(\dot{x}, \ddot{x}, \mu, \tau, n, ...)\, \eta. \tag{5.B.5b}$$

As a result, *gravitational singularities on the horizon are the zeros of the general isotopic element of the isoriemannian geometry*

$$T(x, \dot{x}, \ddot{x}, \mu, \tau, n, ...) = 0\,, \tag{5.B.6}$$

while the singularities at the origin ate the zeros of the isounit

$$\hat{1}(x, \dot{x}, \ddot{x}, \mu, \tau, n, ...) = 0\,. \tag{5.B.7}$$

As a matter of fact, the latter reformulation is done precisely to study the contributions to singularities expected from nonlinear–nonlocal–nonlagrangian interior effects.

The broadening of the scientific horizon from Eq.s (5.B.2)–(5.B.3) to (5.B.6)–(5.B.7))is evident, as we illustrate in more detail in Vol. II and III.

APPENDIX 5.C: ISOTOPIC LIFTINGS OF THE PYTHAGOREAN THEOREM, TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

5.C.1: Foreword. We indicated in the preceding chapters that the notion of angles, the conventional Pythagorean theorem, the trigonometric and hyperbolic functions and other familiar methods are inapplicable under isotopies for numerous independent reasons, such as: the loss of the conventional unit 1 in

favor of generalized isounits $\hat{1}$; the inapplicability of the Euclidean distance; the generally curved character of the lines which prohibit the preservation of conventional angles; etc.

In this appendix we study the rudiments of the liftings of the Pythagorean theorem, trigonometric and hyperbolic functions which are applicable under isotopies. These generalizations were studied for the first time in memoir [46] of 1989 under the respective names of *Isopythagorean Theorem, isotrigonometric* and *isohyperbolic functions*, presented in more details in Appendix 6.A, Vol. I, ref. [47] (the first edition of this monograph); and studied more recently in note [48]. These generalizations are a necessary pre-requisite for: the isotopies of the Legendre functions, spherical harmonics, and other special functions; the study of the isorepresentation theory of the Lie–Santilli isogroup $\hat{O}(3)$; the application to a scattering theory capable of incorporating the conventional action-at-a-distance, potential potential interactions as well as additional contact, nonpotential effects due to the extended, nonspherical and deformable character of the colliding particles; and other applications studied in Vol. II.

We shall continue to use the symbols $\hat{x}$, $\hat{A}$, $\hat{D}$, etc. to denote quantities computed in isospace and x, A, D, etc., to denote their projection in the original space.

5.C.2: Isopythagorean Theorem. Consider a conventional two-dimensional Euclidean space $E = E(r,\delta,R)$ with contravariant coordinates $r = \{r^k\} = \{x, y\}$ and metric $\delta = \mathrm{diag.}\,(1, 1)$ over the field $R = R(n,+,\times)$ of real numbers n with conventional sum + and multiplication × and respective additive unit 0 and multiplicative unit 1. The fundamental notion of this space is the assumption of the basic unit $I = \mathrm{diag.}\,(1, 1)$ which implies the assumption of the same basic (dimensionless) unit +1 for both x– and y–axes, resulting in the familiar *Euclidean distance* among two points $x, y \in E$

$$D = [(x_1 - x_2)(x_1 - x_2) + (y_1 - y_2)(y_1 - y_2)]^{1/2} \in R(n,+,\times). \qquad (5.C.1)$$

The quantity $D^2 = D{\times}D$, $\times \in R$, then represents the celebrated *Pythagorean theorem* expressing the hypothenuse D of a right triangle with sides A and B according to the familiar law $D^2 = A^2 + B^2$.

The flat geometry of the plane $E(r,\delta,R)$ permits the introduction of the trigonometric notion of "angle α" between two intersecting straight vectors, and of "cosinus of α" which, for the case when the vectors initiate at the origin $0 \in E$ and go to two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, is given by

$$\cos\alpha = \frac{x_1 x_2 + y_1 y_2}{(x_1 x_1 + y_1 y_2)^{1/2}(x_2 x_2 + y_2 y_2)^{1/2}}. \qquad (5.C.2)$$

From the above definition one can derive the entire conventional trigonometry. For instance, by assuming that the points are on a circle of unit radius, D = 1, for $P_1(x_1, y_1)$ and $P_2(1, 0)$ we have $\cos\alpha = x_1$, for $P_1(x_1, y_1)$ and $P_2(0, 1)$ we have $\sin\alpha = y_1$, with consequential familiar properties, such as $\sin^2\alpha + \cos^2\alpha = 1$, etc.

Consider now the *two-dimensional isoeuclidean space of Class I*, $\hat{E} = \hat{E}(\hat{r},\hat{\delta},\hat{R})$ (Sect. I.3.3) over the isofield $\hat{R} = \hat{R}(\hat{n},+,\hat{\times})$ of isoreal numbers $\hat{n} = n\times\hat{1}$, where the isounit $\hat{1}$ is a positive-definite 2×2-matrix whose elements have a well behaved but otherwise arbitrary dependence on time t, the local coordinates r and their derivatives of arbitrary order $\hat{1} = \hat{1}(t, r, \dot{r}, \ddot{r}, ...)$.

The realization of $\hat{E}$ studied in this appendix is the simplest possible one of Class I, that with diagonal isounit, of the type

$$\hat{E}(\hat{r},\hat{\delta},\hat{R}) : \hat{r} = \{ \hat{r}^k \} = (\hat{x}, \hat{y}) \equiv \{ r^k \} = \{ x, y \}, \quad \hat{r}_k = \hat{\delta}_{ki}\,\hat{r}^i \neq r_k = \delta_{ki} r^i, \tag{5.C.3a}$$

$$\hat{\delta} = \hat{T}(t, r, \dot{r}, \ddot{r}, \dots)\,\delta = \mathrm{diag.}\,(b_1^2, b_2^2), \quad b_k = b_k(t, r, \dot{r}, \ddot{r}, \dots) > 0, \tag{5.C.3b}$$

$$\hat{1} = \hat{T}^{-1} = \mathrm{diag.}\,(b_1^{-2}, b_2^{-2}), \quad k = 1, 2, \tag{5.C.3c}$$

The central notion of the isoeuclidean plane is the assumption of new (dimensionless) units, the quantities b_1^{-2} for the $\hat{x}$-axis and b_2^{-2} for the $\hat{y}$-axis. Thus, not only the unit is now different than +1, but different axes have different units and, in addition, each of them is a function of the local variables.

Consider now two points $\hat{P}_1(\hat{x}_1, \hat{y}_1)$, $\hat{P}_2(\hat{x}_2, \hat{y}_2) \in \hat{E}(\hat{r},\hat{\delta},\hat{R})$. Then the conventional distance is (uniquely) generalized into the *isoeuclidean distance* (Sect. I.5.2)

$$\hat{D} = [\,(x_1 - x_2)\,b_1^2\,(x_1 - x_2) + (y_1 - y_2)\,b_2^2\,(y_1 - y_2)\,]^{1/2}\times\hat{1} \in \hat{R}, \tag{5.C.4}$$

where one should note the final (ordinary) multiplication by $\hat{1}$ as a necessary condition for $\hat{D}$ to be an element of the isofield $\hat{R}$.

Despite the visible difference between D and $\hat{D}$, all conventional notions in E are preserved under isotopies *provided* that they are computed in $\hat{E}$ over $\hat{R}$. In this way, we have the notions of *isolines, isostraight line, isotriangle, isostraight triangle*, etc. studied in Sect. I.5.2. We then have the following:

Theorem 5.C.1 (Isopythagorean theorem) [46–48]: *The following property holds in the isoeuclidean plane* $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ *of Class I, Eq. (5.C.3),*

$$\hat{D}^{\hat{2}} = \hat{D}\,\hat{\times}\,\hat{D} = \hat{A}^{\hat{2}} + \hat{B}^{\hat{2}} = \hat{A}\,\hat{\times}\,\hat{A} + \hat{B}\,\hat{\times}\,\hat{B} \in \hat{R}, \tag{5.C.5}$$

with projection in the conventional plane E(r,δ,R)

$$\hat{D}^{\hat{2}} = [\,A\,b_1^2(t, r, \dot{r}, \dots)\,A + B\,b_2^2(t, r, \dot{r}, \dots)\,B\,]\times\hat{1}, \tag{5.C.6}$$

that is, the isosquare of the isohypothenuse of an isoright isotriangle is the sum of the isosquare of the isosides.

To understand the geometric meaning of the above theorem, we recall that all isotopic notions have, in general, *three* different interpretations, the first in isospace $\hat{E}(\hat{r},\hat{\delta},\hat{R})$, the second via the projection in the original space $E(r,\delta,R)$, and the third in a *conventional* Euclidean space $E(\bar{r},\delta,R)$ over the conventional reals $R(n,+,\times)$ whose interval coincides with that in isospace. The latter condition is easily verified by the assumption

$$\bar{x} = \hat{x}\, b_1(t, x, y, \dot{x}, \dot{y}, \ldots), \quad \bar{y} = y\, b_2(t, x, y, \dot{x}, \dot{y}, \ldots), \tag{5.C.7}$$

under which

$$[(x_1 - x_2)\, b_1^2 (x_1 - x_2) + (y_1 - y_2)\, b_2^2 (y_1 - y_2)]^{1/2} \equiv$$
$$\equiv [(\bar{x}_1 - \bar{x}_2)(\bar{x}_1 - \bar{x}_2) + (\bar{y}_1 - \bar{y}_2)(\bar{y}_1 - \bar{y}_2)]^{1/2}. \tag{5.C.8}$$

The properties in isospace follow the general rules of all isotopies, that is, the preservation of all original properties, including their numerical values. Thus, straight lines in conventional space are mapped into *isostraight isolines* in isospace, i.e., lines which coincide with their tangent when computed in isospace; perpendicular lines in conventional space are mapped into *isoperpendicular isolines* whose angle is indeed 90° when measured in isospace, that is, with respect to its own isounit (see below); etc.

In this sense, a right triangle in the conventional plane remains so in isoplane, and the conventional Pythagorean Theorem holds also in isospace.

To understand the remaining geometric meaning we also have to consider the projection of Theorem 5.C.1 in the original Euclidean plane. Recall from Sect. I.5.2 that the isotopic lifting of the circle C in E yields the *isocircle* $\hat{C}$ in $\hat{E}$ which preserves the original geometric character including the value of the radius.

We also recall that isotopic maps are not transitive, in the sense that the lifting of the circle C on E into the isocircle $\hat{C}$ on $\hat{E}$ is axiom-preserving, but the projection of the isocircle $\hat{C}$ on the original space E is not, being in fact an ellipse, because such a projection does not imply the return to the original unit I = diag. (1, 1).

By using the reformulation in conventional space $\bar{E}$, it is easy to see that lines which are *straight* in $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ become *curved* in $E(\bar{x},\delta,R)$, according to the rule:

$$\hat{a} \hat{\times} \hat{x} + \hat{b} \hat{\times} \hat{y} + \hat{c} = 0 \rightarrow$$
$$\rightarrow\ a\, \bar{x}\, b_1^{-1}(t, x, y, \ldots) + b\, \bar{y}\, b_2^{-1}(t, x, y, \ldots) = 0, \quad \hat{a}, \hat{b}, \hat{c} \in \hat{R}. \tag{5.C.9}$$

The projection of the Isopythagorean Theorem in a conventional plane then results in the map of a right triangle into a geometric figure in which the sides are curved, with one intersection per pair as in Figure 5.C.1.

ISOPYTHAGOREAN THEOREM

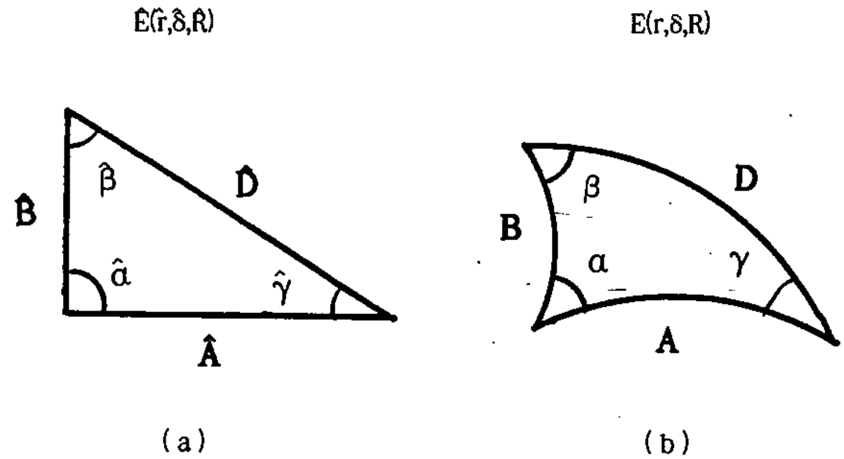

FIGURE 5.C.1. A schematic view of the *Isopythagorean Theorem,* first identified in [1], for an *isoright isotriangle* as in Diag (a), i.e., a triangle in isoeuclidean plane $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ (isotriangle) with a 90° angle measured with respect to its own isounit (*isoright angle* – see below for its identification), and its projection in the conventional plane $E(r,\delta,R)$ given by the Diag. (b).

A conjecture on the *Inverse Isopythagorean Theorem* is presented in the concluding remarks of Sect. I.5.A.5.

= **5.C.3: Isotrigonometric functions.** Let us use again the convention according to which the symbols $\hat{\alpha}$, $\hat{x}$, $\hat{y}$, etc., denote quantities computed in isospace $\hat{E}(\hat{r},\hat{\delta},\hat{R})$, the symbols $\bar{\alpha}$, $\bar{x}$, $\bar{y}$, etc., denote the corresponding quantities when computed in the plane $E(\bar{r},\delta,R)$, and the symbols α, x, y, etc., denote the projection in the conventional space $E(r,\delta,R)$.

Suppose that the two points $\hat{P}_1(\hat{x}_1, \hat{y}_1)$ and $\hat{P}_2(\hat{x}_2, \hat{y}_2)$ represent isostraight *isovectors* initiating from the origin $\hat{0} \in \hat{E}(\hat{r},\hat{\delta},\hat{R})$. Let us denote with $\hat{\alpha}$ the *isoangle* between these two isovectors to be identified below. Consider their *identical* reformulation in the conventional space $E(\bar{x},\delta,R)$, in which case the angle $\hat{\alpha}$ persists. We can then introduce the conventional cos $\hat{\alpha}$ in $\hat{E}(\hat{r},\hat{\delta},\hat{R})$

$$\cos\hat{\alpha} = \frac{\bar{x}_1\bar{x}_2 + \bar{y}_1\bar{y}_2}{(\bar{x}_1\bar{x}_1 + \bar{y}_1\bar{y}_2)^{1/2}(\bar{x}_2\bar{x}_2 + \bar{y}_2\bar{y}_2)^{1/2}} . \tag{5.C.10}$$

with projection in $E(r,\delta,R)$

$$\cos\hat{\alpha} = \frac{x_1 b_1^2 x_2 + y_1 b_2^2 y_2}{(x_1 b_1^2 x_1 + y_1 b_2^2 y_1)^{1/2} (x_2 b_1^2 x_2 + y_2 b_2^2 y_2)^{1/2}} \quad (5.C.11)$$

We now assume that the two points $\hat{P}_1(\hat{x}_1, \hat{y}_1)$ and $\hat{P}_2(\hat{x}_2, \hat{y}_2)$ are on the *unit isocircle*

$$\hat{D}^{\hat{2}} = (x b_1^2 x + y b_2^2 y) \times \hat{1} = \hat{1}, \text{ i.e.,} \quad (5.C.12a)$$

$$x b_1^2 x + y b_2^2 y = 1, \quad (5.C.12b)$$

which imply that for $y = 0$, $x = b_1^{-1}$ and for $x = 0$, $y = b_2^{-1}$.

By assuming the points $\hat{P}_1(\hat{x}_1, \hat{y}_1)$ and $\hat{P}_2(b_1^{-1}, 0)$, we have (for $0 < \hat{\alpha} < \pi/2$)

$$\cos\hat{\alpha} = x_1 b_1, \quad (5.C.13)$$

and for the points $\hat{P}_1(\hat{x}_1, \hat{y}_1)$ and $\hat{P}_2(0, b_2^{-1})$ we have

$$\sin\hat{\alpha} = y_1 b_2 \quad (5.C.14)$$

Definition 5.C.1: *The "isosinus", "isocosinus" and other isotrigonometric functions on the isoeuclidean plane* $E(\hat{r},\hat{\delta},\hat{R})$ *are defined by (for* $0 < \hat{\alpha} < \pi/2$*)*

$$\text{isosin}\,\hat{\alpha} = b_2^{-1} \sin\hat{\alpha}, \quad (5.C.15a)$$

$$\text{isocos}\,\hat{\alpha} = b_1^{-1} \cos\hat{\alpha}, \quad (5.C.15b)$$

$$\text{Isotan}\,\hat{\alpha} = \frac{\text{isosin}\,\hat{\alpha}}{\text{isocos}\,\hat{\alpha}}, \quad (5.C.15d)$$

$$\text{Isocot}\,\hat{\alpha} = \frac{\text{isocos}\,\hat{\alpha}}{\text{isocos}\,\hat{\alpha}}, \quad (5.C.15e)$$

$$\text{isosec}\,\hat{\alpha} = 1/\,\text{isocos}\,\hat{\alpha}, \quad \text{isocosec}\,\hat{\alpha} = 1\,/\,\text{isosin}\,\hat{\alpha}. \quad 5.C.(15f)$$

with basic property

$$\text{isocos}^{\hat{2}}\hat{\alpha} + \text{isosin}^{\hat{2}}\hat{\alpha} = b_1^2\,\text{isocos}^2\hat{\alpha} + b_2^{-2}\,\text{isosin}^{\hat{2}}\hat{\alpha} =$$

$$= \cos^2\hat{\alpha} + \sin^2\hat{\alpha} = 1, \quad (5.C.16)$$

and general rules for an isosquare isotriangle with isosides $\hat{A}$ *and* $\hat{B}$ *and isohypothenuse* $\hat{D}$ *as in Diag. (a) of Fig. 5.C.1*

$$\hat{A} = \hat{D}\ \mathrm{isocos}\,\hat{\gamma}\,, \quad \hat{B} = \hat{D}\ \mathrm{isosin}\,\hat{\gamma}\,, \quad \hat{A}/\hat{B} = \mathrm{isotan}\,\hat{\gamma}, \text{ etc.} \tag{5.C.17}$$

The isoangles have been identified from the representation theory of isorotations in a plane (see Vol. II, Ch. 6), and results to be given by

$$b_1\, b_2\, \alpha = \hat{\alpha}\,. \tag{5.C.18}$$

where the factor $b_1 b_2$ *is fixed for all possible isoangles of a given isoeuclidean space.* This means that the *isotopy of the trigonometric angles* is given by

$$\alpha \quad \rightarrow \quad b_1\, b_2\, \alpha = \hat{\alpha}\,, \tag{5.C.19}$$

with consequential *angular isotopic element*

$$\hat{T}_{\hat{\alpha}} = b_1\, b_2 = (\,\mathrm{Det}\ \hat{T}\,)^{1/2} \tag{5.C.20}$$

and *angular isounit*

$$\hat{1}_{\hat{\alpha}} = b_1^{-1}\, b_2^{-2} = (\,\mathrm{Det.}\ \hat{1}\,)^{1/2} \tag{5.C.21}$$

where $\hat{T}$ and $\hat{1}$ are the isotopic element and isounit, respectively, of the isoeuclidean plane, Eq.s (3).

Isoangles $\hat{\alpha}$ have a nonlinear and integro-differential dependence on the local coordinates and their derivatives when projected in the original Euclidean plane with expression

$$\hat{\alpha} = b_1(t, x, y, \dot{x}, \dot{y}, ...)\, b_2(t, x, y, \dot{x}, \dot{y}, ...)\, \alpha\,, \tag{5.C.22}$$

but they have *constant values* in isospace because measured with respect to the angle isounit $\hat{1}_{\hat{\alpha}} = b_1^{-1}\, b_2^{-1}$. We reach in this way the following property:

Proposition 5.C.1 [48]: *The isotopies of the plane geometry preserve the numerical value of the original angles, that is, if the original angle is* $\alpha = 90°$ *so is the value of the corresponding isoangle* $\hat{\alpha}$ *is isospace.*

In fact, a given isotopic deformation of the angle $\alpha \rightarrow b_1 b_2 \alpha$ occurs under the joint *inverse* deformation of the basic unit $I \rightarrow \hat{1} = b_1^{-1} b_2^{-1}$, thus leaving the original numerical value α unchanged.

With respect to Fig. 5.C.1 we therefore have $\hat{\alpha} = 90°$ and $\hat{\alpha} + \hat{\beta} + \hat{\gamma} = 180°$. However, after the lifting $\alpha = 90° \rightarrow \hat{\alpha} = 90°$, the projection of the latter in the original plane does not yield back the angle $\alpha = 90°$, but an angle α such that $\hat{\alpha} =$

$b_1 b_2 \alpha = 90°$ and similarly we have $\alpha + \beta + \gamma \neq 90°$ but $\hat{\alpha} + \hat{\beta} + \hat{\gamma} = b_1 b_2(\alpha + \beta + \gamma) = 180°$. It is then easy to see that the isotrigonometric functions are periodic as in the conventional case, i.e.,

$$\text{isosin}\,(\hat{\alpha} + 2\,k\,\pi) \equiv \text{isosin}\,\hat{\alpha}\,, \tag{5.C.23a}$$

$$\text{isocos}\,(\hat{\alpha} + 2\,k\,\pi) \equiv \text{isocos}\,\hat{\alpha}\,, \qquad k = 1, 2, 3, \ldots \tag{5.C.23b}$$

and preserve the conventional symmetry under the inversion of the angles

$$\text{isocos}\ -\hat{\alpha} \equiv \text{isocos}\,\hat{\alpha}\,, \quad \text{isosin}\,-\hat{\alpha} = -\,\text{isosin}\,\hat{\alpha}\,. \tag{5.C.24}$$

Similarly, we have the *Theorems of Isoaddition* [1]

$$\text{isosin}\,(\hat{\alpha} \pm \hat{\beta}) = b_1^{-1}\,(\,\text{isosin}\,\hat{\alpha}\ \text{isocos}\,\hat{\beta} \pm \text{isocos}\,\hat{\alpha}\ \text{isosin}\,\hat{\beta}\,)\,, \tag{5.C.25a}$$

$$\text{isocos}\,(\hat{\alpha} + \hat{\beta}) = b_1^{2}\,(\,b_2^{-2}\,\text{isocos}\,\hat{\alpha}\ \text{isocos}\,\hat{\beta} \pm b_1^{-2}\,\text{isosin}\,\hat{\alpha}\ \text{isosin}\,\hat{\beta}\,) \tag{5.C.25b}$$

$$\text{isosin}\,\hat{\alpha} + \text{isosin}\,\hat{\beta} = 2\,b_1^{-1}\,\text{isosin}\,\tfrac{1}{2}\,(\hat{\alpha} + \hat{\beta})\ \text{isocos}\,\tfrac{1}{2}\,(\hat{\alpha} - \hat{\beta})\,. \tag{5.C.25d}$$

The interested reader can then work out the isotopies of other trigonometric properties.

We are now equipped to introduce the following

Definition 5.C.2: *The "isopolar coordinates" are the polar coordinates of the unit isocircle in the isoeuclidean plane* $\hat{E}(\hat{r},\hat{\delta},\hat{R})$, *and can be written*

$$\hat{x} = \text{isocos}\,\hat{\alpha}\,, \qquad \hat{y} = \text{isosin}\,\hat{\alpha}\,, \tag{5.C.26}$$

with projection in the conventional Euclidean plane $E(r,\delta,R)$

$$x = b_1^{-1}\cos\,(\,b_1\,b_2\,\alpha\,)\,, \qquad \hat{y} = b_2^{-1}\ \text{isosin}\,(\,b_1\,b_2\,\alpha\,)\,. \tag{5.C.27}$$

and property

$$\hat{x}^{\hat{2}} + \hat{y}^{\hat{2}} = x\,b_1^{2}\,x + y\,b_2^{2}\,y =$$

$$= b_1^{2}\,\text{isocos}^2\,\hat{\alpha} + b_2^{2}\,\text{isosin}^2\,\hat{\alpha} = \cos^2\hat{\alpha} + \sin^2\hat{\alpha} = 1\,. \tag{5.C.28}$$

The exponential formulation of trigonometric functions also admits a simple, yet unique and effective isotopic image. It requires the lifting of the conventional enveloping associative algebras ξ and their infinite-dimensional basis with conventional unit I and product × (the Poincaré-Birkhoff-Witt

Theorem) into the *enveloping isoassociative algebras* $\hat{\xi}$ of Sect. I.4.3 with isotopic image of the original infinite basis characterized by the isounit $\hat{1}$ and the isotopic product $\hat{x} = x\hat{T}x$ (the *isotopic Poincaré-Birkhoff-Witt Theorem*).

ISOTRIGONOMETRIC FUNCTIONS ON THE ISOCIRCLE

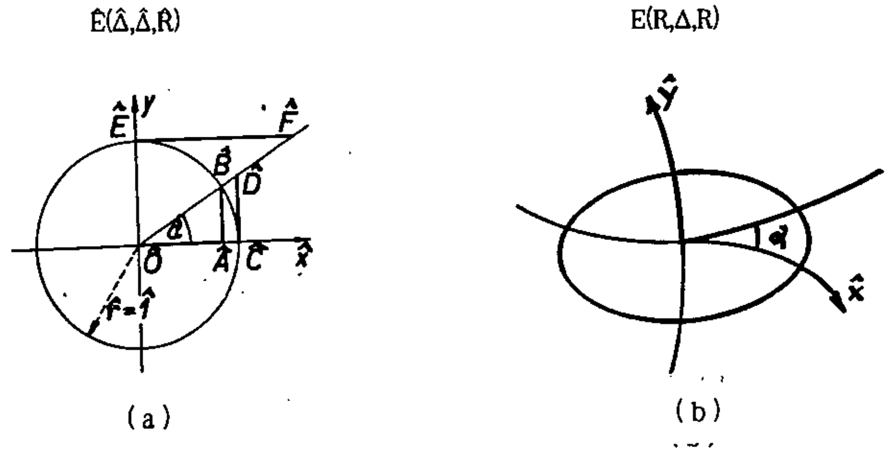

FIGURE 5.C.2: A schematic view of the isotrigonometric functions on the *isocircle* (Sect. I.5.2), that is, the circle in isospace, Diag. (a), and in its projection in conventional space, Diag. (b). Isotrigonometry shows that the the geometric structure of the circle is indeed axiomatic in the sense that it persists under isotopies. This is illustrated by the preservation under isotopy of the polar coordinates on the conventional circle (Diag. (a))

$$x = \cos\,\alpha \quad \rightarrow \quad \hat{x} = \text{isocos}\,\hat{\alpha}\,,$$

$$y = \sin\alpha \quad \rightarrow \quad \hat{y} = \text{isosin}\,\hat{\alpha}\,.$$

However, the projection of the above structure back to the conventional plane implies the deformation of the circle into the ellipse (Diag. (b)), with deformation of the polar coordinates

$$x = \cos\,\alpha \quad \rightarrow \quad x = b_1^{-1}\cos(\,b_1\,b_2\,\alpha\,)\,,$$

$$y = \sin\hat{\alpha} \quad \rightarrow \quad \hat{y} = b_2^{-1}\sin(\,b_1\,b_2\,\alpha\,)\,.$$

The reader is warned *not* to attempt the computation of *isotrigonometric* properties in the *conventional* Euclidean plane. This is due to the fact that the $\hat{x}$ and $\hat{y}$ isostraight axes in $\hat{E}$ are mapped into *curves* in E, as depicted in Diag. (b). Mathematical consistency of the isotrigonometry is then achieved only in isospace.

The isotrigonometric functions can then be expressed in term of the *isoexponentiation* according to the rule

$$\hat{e}^{\,i\hat{\alpha}} = \hat{1} + (i\hat{\alpha})/1! + (i\hat{\alpha})\hat{T}(i\hat{\alpha})/2! + \dots =$$

$$= \hat{1}_{\hat{\alpha}} \times e^{\,i\hat{T}_{\hat{\alpha}}\alpha} = (b_1 b_2)^{-1} \times e^{\,i(b_1 b_2)\alpha} =$$

$$= b_2^{-1} \text{ isocos } \hat{\alpha} + i\, b_1^{-1} \text{ isosin } \hat{\alpha}, \tag{5.C.29}$$

where $\hat{e}$ denotes isoexponentiation and e conventional exponentiation.

The interested reader can then work out additional properties of the isotrigonometric functions.

5.C.4: Isohyperbolic functions. The application of the preceding method to the lifting of the hyperbolic functions is straighforward, leading to the following:

Definition 5.C.3 [46–48]: *The "isohyperbolic functions" on isoeuclidean space $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ of Class I are given by*

$$\text{isocosh } \hat{\alpha} = b_1^{-1} \cosh(b_1 b_2 \alpha), \tag{5.C.30a}$$

$$\text{isosinh } \hat{\alpha} = b_2^{-1} \sinh(b_1 b_2 \alpha), \tag{5.C.30b}$$

with basic property

$$b_1^2 \text{ isocosh}^2 \hat{\alpha} - b_2^2 \text{ isosinh}^2 \hat{\alpha} = 1, \tag{5.C.31}$$

and derivation via the isoexponentiation

$$\hat{e}^{\,\hat{\alpha}} = \hat{1}_{\hat{\alpha}}\, e^{\hat{T}_{\hat{\alpha}}\alpha} = (b_1 b_2)^{-1} e^{(b_1 b_2)\alpha} =$$

$$= b_1^{-1} \text{ isocosh } \hat{\alpha} + b_2^{-1} \text{ isosinh } \hat{\alpha}\,. \tag{5.C.32}$$

The interested reader can then work out the remaining properties of the isohyperbolic functions.

We now show the property that the distinction between trigonometric and hyperbolic functions is essentially due to the excessive simplicity of the basic unit customarily used in contemporary mathematics, while such a distinction is lost under more general units.

In fact, the use of a more general unit under isotopies allows the following result.

Lemma 5.C.1 [48]: *Isotrigonometric and isohyperbolic functions lose any distinction on isoeuclidean planes $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ of Class III*

Proof. Assume the realization of the isounits $\hat{1}$ and $\hat{1}_{\hat{\alpha}}$ of Class III,

$$\hat{1} = \mathrm{diag.}(g_{11}^{-1}, g_{22}^{-1}), \quad \hat{1}_{\hat{\alpha}} = (g_{11}\, g_{22})^{-1/2}, \tag{5.C.33}$$

where the functions $g_{kk} = g_{kk}(t, x, y, \dot{x}, \dot{y}, ...)$ are smooth, real–valued and nowhere null but otherwise arbitrarily positive or negative. Then, the isoexponential realization of the isotrigonometric functions (29) and of the isohyperbolic functions (32) are unified into the form

$$\hat{e}^{\hat{\alpha}} = \hat{1}_{\hat{\alpha}}\, e^{\hat{T}_{\hat{\alpha}}\alpha} = (g_{11}\, g_{22})^{-1/2}\, e^{(g_{11}\, g_{22})^{1/2}\alpha}, \tag{5.C.34}$$

where the isotrigonometric functions occur when the product $g_{11}g_{22}$ is positive and the isohyperbolic functions occur when the same product is negative. **q.e.d.**

Lemma 1 also unifies the *conventional* trigonometric and hyperbolic functions, the former occurring for $\hat{1} = I = \mathrm{diag.}(1, 1)$ or Dig. $(-1, -1)$ and the latter for $\hat{1} = \mathrm{diag.}(+1, -1)$ or Diag. $(-1, +1)$, the second alternatives being the isodual of the first ones.

5.C.5: Open problems. In this appendix we have studied the rudiments of the isotopies of the Pythagorean Theorem, trigonometric and hyperbolic functions for the simplest possible case of Class I in which the isounit is *positive–definite and diagonal,* Eq. (3c). Numerous problems remain open for the interested reader, among which we indicate the study of the Isopythagorean Theorem, isotrigonometric and isohyperbolic functions for:

1) Isotopies of Class II, requiring the study of the *isostraight lines, isoangles, isotriangle and isocircles with negative unit.*

2) Isotopies of Class III, requiring the study of *isostraight lines, isoangles, isotriangle and isocircles with units of undefined signature.*

3) Isotopies of Class IV, requiring the study of *isostraight lines, isoangles, isotriangle and isocircles with singular units.*

4) The isotopies of Class V, requiring the study of *isostraight lines, isoangles, isotriangle and isocircles with unrestricted – e.g., discontinuous – units.*

All the above studies are referred to *diagonal isounits* of the type

$$\hat{1} = \begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & g_{22}^{-1} \end{pmatrix}, \tag{5.C.35}$$

Additional open problems are given by the study of the Isopythagorean Theorem, isotrigonometric and isohyperbolic functions of Classes I–V with *nondiagonal isounits* of the type

$$\hat{1} = \begin{pmatrix} 0 & g_{33}^{-1} \\ g_{33}^{-1} & 0 \end{pmatrix}, \tag{5.C.36}$$

as well as those *with* general isounits of the type

$$\hat{1} = \begin{pmatrix} g_{11}^{-1} & g_{33}^{-1} \\ g_{33}^{-1} & g_{22}^{-1} \end{pmatrix}, \tag{5.C.37}$$

which are unknown at this writing.

The study of the following conjecture may also be of some interest:

Conjecture 5.C.1 (Inverse Isopythagorean Theorem) [48]**:** *Given a geometric figure consisting of three smooth but otherwise arbitrary curves in a conventional Euclidean plane intersecting each other as per Diagram (b) of Fig. 1, there always exists an isotopy of the unit of Class I,* $I \rightarrow \hat{1}$, *under which said geometric figure is mapped into the isoright isotriangle in isoeuclidean space for which the isopythagorean theorem holds.*

If correct, the above conjecture would establish that the abstract geometric structure of the historical Pythagorean theorem applies to a class of figures much broader of what considered until now, and it is in fact universal for all "triangles" with "curved sides".

Note that the proof of Conjecture 5.C.1 appears to be possible for the case of nondiagonal isounits of type (37) because they contain *three* arbitrary functions $g_{kk}(t, x, y, ...)$ as needed to characterize the three independent curves of the "triangle". A more difficult case is whether the isotopic lifting of Diag. (b) into (a) of Fig. 1 exists also for a *diagonal* isounit with *two* independent functions g_{kk} while we have *three* independent curves.

The author hopes to have illustrated in this appendix once more that the removal of the current restriction of our entire mathematical knowledge to the trivial unit identified since biblical times, and the use of structurally more general units, implies a rather vast broadening of all of mathematics, beginning with the most elementary ones such as angles, and then following with all remaining structures, permitting basically novel applications in a variety of fields (Vol.s II and III).

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6: FUNCTIONAL ISOANALYSIS AND ITS ISODUAL

6.1: STATEMENT OF THE PROBLEM

The transition from Newtonian to quantum mechanics implies the preservation of the basic mathematical notions such ash numbers, angles, metric spaces, special functions, etc., and only the reformulation of observable on a Hilbert space.

The transition from quantum to hadronic mechanics is much deeper because it requires a suitable generalization of all basic mathematical notions of quantum mechanics, beginning with numbers, angles, metric spaces, special functions, etc., and then passing to a generalization of Hilbert spaces themselves.

The above occurrence can be expressed by the fact that *functional analysis remained unchanged in the transition from classical to quantum mechanics.* On the contrary, *the transition from quantum to hadronic mechanics requires a structural generalization of functional analysis into a new discipline called "functional isoanalysis".*

The need for an isotopic lifting of numbers, angles and trigonometric functions has been indicated earlier in this volume, jointly with that for the generalization of other ordinary functions, such as exponentiation, hyperbolic functions, logarithm, etc. The need for a lifting of special functions is then consequential, as studied in this chapter.

Let us identify here the need for lifting Hilbert spaces themselves. As recalled earlier, hadronic mechanics was originally submitted [1] in 1978 as an isoassociative enveloping algebra $\hat{\xi}_T$ (Sect. 4.3)[43] of operators A, B, ... with isotopic product

$$\hat{\xi}_T = A * B := A\, T\, B, \quad \hat{1} = T^{-1}, \tag{6.1.1}$$

on a *conventional* Hilbert space $\mathcal{H}$ with elements ψ , ϕ , ... with familiar inner

[43] For clarity due to the subsequent analysis, in this chapter we shall identify with a subscript the isotopic element of a given structure, such as in $\hat{\xi}_T$.

product

$$\mathcal{H}: \; <\psi \,|\, \phi> \; = \; \int d^3r \, \psi^\dagger(r) \, \phi(r) \quad \in \; C(c,+,\times) \, . \tag{6.1.2}$$

which is indeed a mathematically correct formulation.

However, this original formulation had the physical problematic aspect that operators of the original envelope ξ which are Hermitean on $\mathcal{H}$ did not remain necessarily Hermitean under lifting $\xi \rightarrow \hat{\xi}_T$. This is due to the fact that, as we shall see in this chapter, the condition of Hermiticity of an operator $H \in \hat{\xi}_T$ on $\mathcal{H}$ is given by

$$H^{\hat{\dagger}} \; = \; T \; H^\dagger \; T^{-1} \, . \tag{6.1.3}$$

where $H^\dagger$ is the conventional Hermiticity. Since the operators T and H do not necessarily commute, we have in general that $H \neq H^\dagger$.

This implied that observable of quantum mechanics, such as the total energy H, the linear momentum p, etc., do not necessarily remain observable under under isotopies $\xi \rightarrow \hat{\xi}_T$ over $\mathcal{H}$.

This clearly called for an appropriate generalization of the underlying Hilbert space $\mathcal{H}$ in such a way to preserve observability under isotopies. These studies were initiated by this author immediately after proposal [1], e.g., in ref. [2] of 1979. The resolution of the problem received a first rigorous treatment by Myung and Santilli in ref. [3] of 1983 via the introduction of the notion of *isotopic Hilbert space* $\hat{\mathcal{H}}_T$ or *isohilbert space* for short, which is essentially the image of $\mathcal{H}$ under the lifting of the composition

$$\begin{aligned} \hat{\mathcal{H}}_T: \; <\psi \hat{|} \phi> \; &:= \; <\hat{\psi} \,|\, T \,|\, \hat{\phi}> \hat{1} \; = \; <\hat{\psi} \,|\, * \,|\, \hat{\phi}> \hat{1} \; = \\ &= \; \hat{1} \int d^3r \, \psi^\dagger(r) \, T(t, r, \dot{r}, \ddot{r}, \ldots) \, \phi(r) \quad \in \; \hat{C}(\hat{c},+,*) \, . \end{aligned} \tag{6.1.4}$$

where, as one can see, the assumption of the positive–definiteness of the isounit (Class I) implies the preservation of the inner character of the composition and, thus, of the Hilbert character of the space $\hat{\mathcal{H}}_T$.

Isohilbert space (6.1.4) did indeed achieve the desired objective because, as we shall see better in this chapter, the condition of Hermiticity of an operator $H \in \hat{\xi}_T$ on $\hat{\mathcal{H}}_T$ coincides with the conventional Hermiticity,

$$H^{\hat{\dagger}} \; \equiv \; H^\dagger \, , \tag{6.1.5}$$

thus permitting the preservation of Hermiticity under isotopies.

The importance of this result should be indicated for readers not familiar with isotopic techniques. A central objective of hadronic mechanics is to complement conventional quantum mechanical descriptions of interacting

particles at large mutual distances ($\gg$ 1 fm), with additional internal, short range, nonlinear–nonlocal–nonhamiltonian interactions when in conditions of mutual penetration of their wavepackets at very short distances ($<$ 1 fm) (see Fig. I.1.1.1)

This implies that the operators such as the energy $H = K + V$ of the particle in exterior conditions does not change in the transition to the condition of total mutual immersion of the particles considered because the additional interactions have no potential by conception. Still in turn, this implies that *a necessary condition for the physical consistency of the isotopies is the preservation of the <observability> of the original energy* H, *i.e., the preservation of its Hermiticity.*

Thus, ref. [3] identified the fundamental carrier space of (the Lie–isotopic branch of) hadronic mechanics, the space $\hat{\mathcal{H}}_T$, which fulfills the fundamental task of preserving the observability of conventional physical quantities. However, the lifting $\mathcal{H} \rightarrow \hat{\mathcal{H}}_T$ implies a structural revision of the conventional Hilbert space theory into five classes, as we shall see.

The subsequent studies by Mignani, Myung and Santilli [4] of 1983 indicated that formulation (6.1.4) is still restrictive because the enveloping isoassociative algebra (6.1.1) could be consistently formulated also in the different isospace

$$\hat{\mathcal{H}}_G : \ <\psi\hat{\uparrow}\phi> \ : \ <\hat{\psi}|G|\hat{\phi}>\hat{1} \ = \ <\hat{\psi}|\odot|\hat{\phi}>\hat{1} \ =$$

$$= \ \hat{1}\int d^3r\, \psi^{\dagger}(r)\, G(t, r, \dot{r}, \ddot{r}, ...)\, \phi(r) \quad \in \ \hat{C}(\hat{c},+,*) , \qquad (6.1.6)$$

where G is an operator *independent* of T. The lifting $\hat{\mathcal{H}}_T \rightarrow \hat{\mathcal{H}}_G$ implies again the general loss of Hermiticity because, as we shall see in details in this chapter, the condition of Hermiticity of an operator $H \in \hat{\xi}_T$ on $\hat{\mathcal{H}}_G$ is given by

$$H^{\hat{\dagger}} \ = \ G^{-1}\, T\, A^{\dagger}\, G\, T^{-1} , \qquad (6.1.7)$$

which includes as particular case condition (6.1.3).

Subsequent studies indicated that, despite the general loss of the original Hermiticity, the formulation of hadronic mechanics via isoenvelopes $\hat{\xi}_T$ on the isohilbert space $\hat{\mathcal{H}}_G$ with $T \neq G$ is important in certain specific cases in which the formulation on $\hat{\mathcal{H}}_T$ is not sufficient. In fact, the introduction of an isotopic element G in the Hilbert space different than T represents an additional "hidden degree of freedom" of the theory.

The motivations are linked to *the reconstruction of exact Lie symmetries at the isotopic level of hadronic mechanics when believed to be broken at the simpler quantum mechanical level.* The use of only one isotopic element T for both the envelope and the Hilbert space is sufficient for the reconstruction of the exact symmetry in a number of cases, such as the reconstruction of the exact rotational symmetry when believed to be broken by ellipsoidical deformations of the sphere [5–7], the reconstruction of the exact Lorentz symmetry when believed to be broken by signature–preserving deformations of the Minkowski metric [8],

the reconstruction of the exact isospin symmetry in nuclear physics with equal proton and neutron masses in isospace, and others.

However, there exist cases in which one sole degree of freedom is insufficient, and two different isotopic elements T and G are needed. This is the case for the ongoing attempts (see the initial effort [9] studied in more detail in Vol. II) to *reconstruct parity at the isotopic level as an exact symmetry for "weak" interactions* via the embedding of all symmetry breaking terms in the isotopic elements.

The technical issue is the identification of which isotopic element should incorporate all symmetry violating terms. Recall from Sect. I.4.5 that in the lifting of continuous symmetries we have the appearance of the isotopic element T in the isoexponentiation. Thus, *the embedding of the symmetry breaking terms in the isotopic element* T *of the isoenvelope* $\hat{\xi}_T$ *and isofields* $\hat{F}_T$ *is generally sufficient for the reconstruction of exact "continuous" symmetries.*

The case for discrete transformation is different because they admit no isoexponentiation, and actually admit the reduction to the corresponding conventional transformations (Sect. I.4.7), e.g.,

$$\hat{\pi} * \psi(r) = \pi\,\psi(r) = \psi(-r)\,, \qquad \hat{\pi} = \pi\, T^{-1}\,. \tag{6.1.8}$$

The general insuffiency of the isotopic element T is then evident. As a result, *the reconstruction of exact "discrete" symmetries generally requires the embedding of the symmetry breaking terms in both the isotopic element* T *of the isoenvelopes* $\hat{\xi}_T$ *and of the isofield* $\hat{F}_T$ *as well as in the isotopic element* G *of the isohilbert space* $\hat{\mathcal{H}}_G$.

In summary, the part of functional isoanalysis dealing with isohilbert spaces implies a rather broad lifting of conventional quantum mechanical formulations consisting of a double generalization, the first via the same isotopic element of the envelope and the second based on the differentiation between the isotopy of the envelope and that of the Hilbert space.

We now pass to a few comments on the lifting of the remaining aspects of functional analysis. Recall that the first step that lead to hadronic mechanics was the isotopy of the Poincaré–Birkhoff–Witt theorem resulting in a generalized notion of exponentiation (Sect. I.4.3).It was then known since the original proposal [1] that the isotopies of the enveloping operator algebra, $\xi \rightarrow \hat{\xi}_T$, imply a generalization of all familiar structures of quantum mechanics such as Dirac's δ–function, the Fourier transforms, Gauss distributions, etc.

A first formulation of the isotopic δ–function appeared in ref. [3], while its systematic study was presented in memoirs [10,11], jointly with the first formulations of the isotopies of Fourier series and transforms isotopies studied in this chapter.

The full implications of these studies for conventional functional analysis (see, e.g., ref.s [12–13] and quoted references) was however identified only recently

by Kadeisvili [14] who understood that *the isotopies of fields* $F(\alpha,+,\times) \rightarrow \hat{F}_T(\hat{\alpha},+,*)$, *enveloping algebras* $\xi \rightarrow \hat{\xi}_T$ *and Hilbert spaces* $\mathcal{H} \rightarrow \hat{\mathcal{H}}_T$ *imply a nontrivial, nonlinear–nonlocal–noncanonical isotopic generalization of the totality of functional analysis, that is, not only of square integral, Banach, Hilbert and other spaces, but also of conventional special polynomials (such as the Legendre polynomial), special functions (such as Bessel and Legendre functions), transforms (such as Fourier and Laplace transform), etc.* In fact, the terms "functional isoanalysis" appeared for the first time in ref. [14].

The mathematical relevance of these isotopies will be evident during the analysis of this chapter. Their physical relevance can be best illustrated with the fact that, in the subsequent paper [15], Kadeisvili reinspected the isotopies of the Fourier transforms of ref. [10] and discovered that they imply a necessary generalization of Heisenberg's uncertainties precisely into the form submitted by this author [16] back in 1981

$$\Delta x \ \Delta k \geqq \tfrac{1}{2} < \hat{1} > , \tag{6.1.9}$$

where $\hat{1}$ is the isounit and < ..... > is a certain form of the expectation value to be studied in Vol. II.

In fact, Kadeisvili [15] showed that the Fourier isotransform, when applied to a Gaussian distribution, implies the map (in term of the isoexponentiation of Sect. I.4.3)

$$\psi(x) \approx e_\xi^{\ -x^2/2a} \equiv e_\xi^{\ -x^2 T \ 2a^2} \Rightarrow \phi(k) \approx e_\xi^{\ -k^2 a^2/2} \equiv e_\xi^{\ -k^2 T\, a^2/2} \tag{6.1.10}$$

as a result of which we have the isotopic behaviour

$$\Delta x \approx a / T^{\frac{1}{2}}, \quad \Delta k \approx 1 / a\, T^{\frac{1}{2}}, \tag{6.1.11}$$

yielding precisely isouncertainties (6.1.9).

We reach in this way the first illustration of the fact that *the isotopies imply such a generalization of the mathematical structure of quantum mechanics for the exterior problem in vacuum to result in fundamentally more general physical laws for the interior problem.*

For future need in Ch. I.7, note the *uniqueness* of the generalizations originating from the uniqueness of the exponentiation (Sect. I.4.3).

In App. 6.A we outline the notions of isomanifolds and related isotopology first derived by Tsagas and Sourlas [30]. As we shall see, these studies have identified a new *integro-differential* topology which is everywhere local-differential except at the unit.

In App. 6B we point out a different generalization of special functions, the so-called *q-special functions.* The latter generalizations are different than those needed for hadronic mechanics for numerous reasons, such as:

A) the q-special functions are deformations preserving the original unit while in hadronic mechanics, as now familiar, we have deformations under the joint lifting of the unit;

B) q-deformations are *q-number* deformations, while hadronic mechanics requires *Q-operator* deformations;

C) q-deformations are defined on an ordinary space, while the Q-operator deformations are defined on an isospace;

and others.

As we shall see in Vol. II, q-special functions are not invariant, because the q-number becomes Q-operator under the time evolution of the theory. This is one of the reasons why *the use of q-special functions in hadronic mechanics leads to a host of generally hidden inconsistency.* By comparison, our Q-special functions remain invariant at all times.

We learn in this way that a fundamental condition for the consistent applicability of isotopic special functions is their invariance under Lie-Santilli isotransformation groups, whether in classical or operator realizations depending on the case at hand.

In App. 6.C we reprint a recent article by Aringazin, Kirukhin and Santilli [31] on the construction of the isolegendre, isojacobi and isobessel functions, which may serve as a basis for the study of other isospecial functions.

In this chapter we present the rudiments of functional isoanalysis with the understanding that this discipline too is at its first infancy and so much remains to be done. In particular, our presentation is intended for graduate students in physics and all mathematical profiles are left to interested mathematicians. Additional aspects, such as special isofunctions needed for specific applications, will be worked out in Vol.s II and III.

6.2: ISOHILBERT SPACES AND THEIR ISODUALS

It is significant for this chapter to recall that functional analysis (see, e.g., ref.s [12,13]) was born and developed primarily because of specific physical motivations, rather than abstract mathematical needs.

In fact, the French mathematician J. B. J. Fourier identified his celebrated series and transforms during his study on heat conduction; Freedholm worked on integral equations because of specific problems in classical electromagnetism; von Neumann conducted most of his studies on operator algebras because of specific physical needs; not to mention the fundamental physical role of Hilbert studies in quantum mechanics.

It is intriguing to note that, much along the same lines, the new discipline of *functional isoanalysis* , was also born out, specifically, of physical problems, given this time by the author's studies of nonlinear, nonlocal and noncanonical

systems of the interior dynamical problem.

Conventional functional analysis can be seen as the discipline which is and will remain fundamental for the *exterior* dynamical problem of particles in vacuum (see Sect. 1.1), while functional isoanalysis is a covering discipline specifically conceived for the more general *interior* dynamical problem of extended particles moving within physical media.

Despite its rather vast current dimension, contemporary functional analysis remains based on conventional notions, such as conventional fields, conventional vector spaces, conventional operations, etc. It is then inevitable that the isotopic generalizations of these structural foundations imply the existence of a consequential, corresponding generalization of the entire theory.

It is also significant to note that functional isoanalysis was born and completely developed in physical publications until very recently. In fact, Kadeisvili papers [14,15] are the first papers appeared very recently in a mathematical Journal, to the author's best knowledge.

The foundations of functional isoanalysis are those reviewed in the preceding chapters, and consist of the isotopies of fields, vector spaces, transformation theory, algebras, groups, geometries, etc. This section is solely devoted to the isotopies of Hilbert spaces, while additional aspects will be studied in the following sections.

The first notion of isoanalysis is the isofield $\hat{F}(\hat{\alpha},+,*)$ with isonumbers $\hat{\alpha} = \alpha\hat{1}$, conventional sum +, isoproduct $* = \times T \times$, and isounit $\hat{1} = T^{-1}$. For simplicity, we shall restrict $\hat{F}$ to have isocharacteristic zero and to represent the isofields of real isonumbers $\hat{R}(\hat{n},+,*)$ and of complex isonumbers $\hat{C}(\hat{c},+,*)$. More general formulations of isoanalysis on isoquaternions are left to the interested reader.[44]

The second fundamental notion is a generic, finite-dimensional vector isospace $\hat{S}(x,\hat{C})$ on the isofield $\hat{C}$. The abstract identity of $\hat{C}(\hat{c},+,*)$ and $C(c,+,\times)$ and that of $\hat{S}(x,\hat{C})$ and $S(x,C)$ should be kept in mind to anticipate that *functional isoanalysis coincides with the conventional formulation at the abstract level by construction* (although only for the case of isounits of Class I, see below).

Recall that conventional complex numbers c can be reinterpreted as being complex isonumbers under the isotopy of the multiplication. Along similar lines, a conventional function f(x) on $S(x,C)$ can be reinterpreted as being a function on $\hat{S}(x,\hat{C})$. In fact, it is not the value of the function f(x) which identifies the distinction between $S(x,C)$ and $\hat{S}(x,\hat{C})$, but rather the operations on it.

Finally, the reader should recall that the isotopies automatically generalize a linear, local and canonical theory into an axiom-preserving, nonlinear, nonlocal and noncanonical form because of the arbitrary functional dependence of the isounit $\hat{1} = \hat{1}(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, ...)$.

The first isotopic operation among functions on $\hat{S}(x,\hat{C})$ is the *isoscalar*

43 It should be recalled that, on strict mathematical grounds, even the formulation on isocomplex numbers is inessential owing to the unification of all numbers and isonumbers in the abstract field of isoreals (Sect. 2.7).

product (or *isoproduct* for short) of two functions $f_1(x)$ and $f_2(x)$, which is given by

$$f_1(x) * f_2(x) \;=\; f_1(x)\, G(x, ...)\, f_2(x) \;\in\; \hat{S}(x,\hat{C}), \qquad (6.21)$$

where the isotopic element G is fixed, and different than T.

The *isoinner product* of two functions $f_1(x)$ and $f_2(x)$ on $\hat{S}(x,\hat{C})$ is the composition with elements in $\hat{C}$ introduced in ref. [3][45]

$$(f_1 \hat{,} f_2) \;=\; \int_a^b dx\, \bar{f}_1(x)\, G\, f_2(x) = \int_a^b dx\, \bar{f}_1(x) \odot f_2(x) \;\in \hat{C}(c,+,*), \qquad (6.2.2)$$

where $\bar{f}$ denotes ordinary complex conjugation and $\hat{C}(c,+,*)$ is the isofield of Proposition 2.3.1 (that without the lifting of the numbers in which case the isounit must necessarily be an element of the original field).

The above foundations then imply the lifting of the conventional quantity $|\, f(x)\, |$ into the *isoabsolute value* $\uparrow f(x) \uparrow$ characterized by

$$\uparrow f(x) \uparrow^{\hat{2}} \;=\; (\, \overline{f(x)}\, G\, f(x)\,)\,\hat{1} \;=\! (\; \overline{f(x)} \odot f(x)\,)\,\hat{1}\,, \qquad (6.2.3)$$

and given by

$$\uparrow f(x) \uparrow \;=\; (\,\bar{f}\, G\, f\,)^{\frac{1}{2}}\;\hat{1} \;=\; (\,\bar{f} \odot f\,)^{\frac{1}{2}} \times \hat{1}\,. \qquad (6.2.4)$$

where $\hat{1}^{\frac{1}{2}}$ is a conventional square root. The *isonorm* $|\!\uparrow f(x) \uparrow\!|$ of a function f(x) is then defined by the element of the isoreals

$$|\!\uparrow f(x) \uparrow\!|^{\hat{2}} \;=\; (f \hat{,} f)\hat{1} \;=\; \hat{1} \int_a^b dx\, \bar{f}(x)\, G\, f(x) \qquad \in\; \hat{R}(n,+,*)\,, \qquad (6.2.5)$$

and given by

$$|\!\uparrow f(x) \uparrow\!| \;=\; (f \hat{,} f)^{\hat{\frac{1}{2}}} \;=\; (\, f_1\,, f_2\,)^{\frac{1}{2}}\;\hat{1}\;. \qquad (6.2.6)$$

It should be indicated from the outset that the above definitions are not unique, owing to the degrees of freedom of the isotopies. In fact, one can consider the maps

$$f \;\rightarrow\; \hat{f} = f\,\hat{1} \;\in \hat{S}(\hat{x},\hat{C}), \quad c \rightarrow \hat{c} \;=\; c\,\hat{1} \;\in\; \hat{C}(\hat{c},+,*)\,, \qquad (6.2.7)$$

in which case we have the map of the isoproduct

$$f_1\, G\, f_2 \quad\rightarrow\quad \hat{f}_1\, G\, \hat{f}_2 \;=\; f_1 \hat{1}\, G\, f_2 \hat{1} \;=\; f_1\, f_2\, \hat{1}\,, \qquad (6.2.8)$$

[45] It should be indicated that, as shown in Sect. 6.7, the measure dx is lifted into the form $\hat{d}x = d(\hat{1}x)$. However, for $\hat{1}$ independent of x, we have $\int \hat{d}x * \bar{f}_1(x) * f_2(x) \;=\; \int dx\; \bar{f}_1(x) * f_2(x)$.

with corresponding definitions for isoabsolute value

$$\hat{\uparrow}\hat{f}(x)\hat{\uparrow} = (\bar{\hat{f}}\ \hat{f}\hat{1})^{\frac{1}{2}}\hat{1}, \tag{6.2.9}$$

isoinner product

$$(\hat{f}\,\hat{,}\,\hat{f}) = \hat{1}\int_a^b dx\,\bar{\hat{f}}(x)\,\hat{f}(x)\,\hat{1}(x,..) \in \hat{R}(\hat{n},+,*). \tag{6.2.10}$$

and isonorm

$$|\hat{\uparrow}\hat{f}(x)\hat{\uparrow}| = (\hat{f}\,\hat{,}\,\hat{f})^{\frac{1}{2}} = (\hat{f}_1, \hat{f}_2)^{\frac{1}{2}}\hat{1}. \tag{6.2.11}$$

The transition from the preceding formulation in terms of ordinary numbers and functions to the latter one was introduced by the author in ref. [10] for the particular case of $T \equiv G$ under the name of *reciprocity transformation* because based on the replacement

$$T \rightarrow \hat{1}, \qquad \hat{1} \rightarrow \hat{1}^{-1}, \tag{6.2.12}$$

the case $T \neq G$ being a simple generalization. The formulation on isocomplex numbers $\hat{C}(\hat{c},+,*)$ is that primarily used in physics because it implies that the isotopic eigenvalues are the conventional ones (see below in this section), although both formulations emerge rather naturally, e.g., in the lifting of Dirac delta–function (see Sect. 6.4).

Needless to say, maps (6.2.7) are, by far, nonunique and a number of additional maps implying nontrivial alterations of the isoproduct are possible. Nevertheless the above two alternatives are sufficiently to identify the foundations of isoanalysis.

From these rudimentary notions it is sufficient to see the need to use again Kadeisvili classification:

Primary classification: based on the characteristics of the isounit (Sect. 1.5):
Class I: Functional isoanalysis properly speaking;
Class II: Isodual functional isoanalysis;
Class III: Indefinite functional isoanalysis ;
Class IV: Singular functional isoanalysis ;
Class V: General functional isoanalysis .

Secondary classification: based on the assumed realization of isofields and isovector spaces
Subclass A: characterized by $\hat{F}(\hat{\alpha},+,*)$ and $\hat{S}(x,\hat{F})$, i.e., isofields whose elements are ordinary numbers and with ordinary functions f(x) on

$\hat{S}(x,\hat{F})$.

Subclass B: characterized by $\hat{F}(\hat{\alpha},+,*)$ and $\hat{S}(\hat{x},\hat{F})$, i.e., isofields with elements elements $\hat{\alpha} = \alpha \hat{1}$ and with isofunctions $\hat{f}(x) = f(x)\hat{1}$ on $\hat{S}(x,\hat{F})$.

By no means the above classification is complete. In fact, a further structural generalization is that suggested by the more general, one-sided, Lie-admissible formulations of the next chapter. Nevertheless, the above classification is sufficient to identify the new discipline and initiate its systematic study.

A first purpose of the above classification is to separate the axiom-preserving liftings from the more general ones. As an example, an "inner" product remains inner for Classes I , but not necessarily for Class III.

The mathematician can now see the novel concepts implied by isoanalysis, such as [10]: negative-definite composition (Class II); functional analysis based on a singular isounit (Class IV); isohilbert space whose unit is a lattice, or a distribution (Class V), etc.

Note that *the isoinner product is invariant under isoduality,*

$$(f_1 , f_2)^d : = 1^d \int_a^b dx\, \overline{f}_1(x)\, 1^d\, f_2(x) \;\equiv \int_a^b dx\, \overline{f}_1(x)\, f_2(x) . \qquad (6.2.13)$$

However, one should recall that positive numbers are negative when referred to isodual fields, evidently because their unit is negative-definite. This point is clarified below when studying the isodual isohilbert spaces.

From now on, unless otherwise stated, we shall study in this section only the isoanalysis of Class IA, and IB, and their isoduals IIA and IIB. The study of the remaining classes must be deferred for brevity to the individual researcher.

Let us consider first Class IA. The problem of *isocontinuity* , that is, continuity on an isospace, was first studied by Kadeisvili in ref. [14] via the *isocontinuity of a function* f(x) *at a point* $x \in \hat{S}(x,\hat{F})$, which occurs when $|\,\hat{\uparrow} f(x) \hat{\uparrow}\,| \to 0$ implies $\hat{\uparrow} f(x + \epsilon) - f(x) \hat{\uparrow} \to 0$.

Note that all conventionally continuous functions are also isocontinuous for Class IA, although the viceversa is not necessarily true under relaxed properties of the isounits. As a matter of fact, *functions that are conventionally discontinuous can be turned into isocontinuous forms via suitable selection of the isounit.*

The *isoschwartz inequality* , introduced in ref. [3] for the case T = G, is given by the simple isotopy of the conventional expression

$$\hat{\uparrow}(f_1 \hat{,} f_2)\hat{\uparrow} \quad \leq \quad \hat{\uparrow} f_1 \hat{\uparrow} * \hat{\uparrow} f_2 \hat{\uparrow} \; , \qquad (6.2.14)$$

and its validity (again, for Class I) can be easily proved.

A function f(x) on $\hat{S}(x,\hat{C})$ is said to be *isosquare integrable* [14] in the interval [a, b] when the integral

$$\int_a^b dx\,|f(x)|^{\hat{2}} = \hat{1}\int_a^b dx\,\bar{f}(x)\,G\,f(x)\,, \qquad (6.2.15)$$

exists and is finite. The set of all isosquare integrable functions in [a, b] will be denoted with $\hat{\mathcal{L}}^{(2)}[a,b]$. One can now begin to see some of the novel applications of isoanalysis. In fact, *a function which is not square integrable in a given interval, can be turned into an isosquare integrable form via a suitable selection of the isotopic element* with evident computational advantages (see below for an example).

A sequence $f_1, f_2, ...$ is said to be *strongly isoconvergent to f* when

$$\mathrm{Lim}_{k\to\infty} |\hat{|} f_k - f \hat{|}| = 0 \qquad (6.2.16)$$

with a similar definition holding for series. Again, for Class IA, strong convergence implies the strong isoconvergence, which is a trivial occurrence.

A nontrivial property is that the opposite is not necessarily true, namely, *a sequence (or, more generally, a series) which is strongly isoconvergent is not necessarily conventionally convergent*. This property has fundamental physical relevance that motivated this authors and several independent researchers to study hadronic mechanics,

In fact, as well known, electromagnetic interactions do have a convergent perturbative theory due to the low value of the coupling constant, which permits several numerical calculations suitable for experimental tests. On the contrary, strong interactions do not have such a convergent perturbative theory in their current formulation within the context of ordinary functional analysis, with evident consequential limitations of the theory.

As we studied in detail in Vol. II, the fundamental physical point here is that the covering functional isoanalysis offers real possibilities for the construction of a *convergent isoperturbation theory for strong interactions.*

The *isocauchy condition* is the isotopic property verified by every strong isoconvergence

$$|\hat{|} f_m - f_n \hat{|}| < \delta \qquad (6.2.17)$$

with $\delta > 0$ real arbitrary and for all m and n greater than a suitably chosen $N(\delta)$.

It is easy to see that, again for Class IA, when the isoinner product is isocontinuous, the isonorm is isocontinuous. The extension of the preceding results to Class IB is evident and will be tacitly implied hereon.

We now present the following notion introduced in ref.s [3,4,10]

Definition 6.2.1: *An "isohilbert space" $\hat{\mathcal{H}}_{IB,G}$ of Class I.B and isotopic element G is an isospace over the isofield $\hat{C}(\hat{c},+,*)$ characterized by the following axioms:*
A.1: $\hat{\mathcal{H}}_{IB,G}$ *is an isolinear and isolocal space (Sect. 4.2), i.e., for given elements* $\hat{\Phi}_1$,

$\hat{\psi}_2$ *of* $\mathcal{H}_{IB,G}$, *complex numbers* $\hat{c}_1$, $\hat{c}_2 \in \hat{C}$ *and operator* $\hat{U}$ *acting on* $\mathcal{H}_{IB,G}$, *we have*

$$\hat{U} * (\hat{c}_1 * \hat{\psi}_1 + \hat{c}_2 * \hat{\psi}_2) = \hat{c}_1 * \hat{U} * \hat{\psi}_1 + \hat{c}_2 * \hat{U} * \hat{\psi}_2 ; \qquad (6.2.18)$$

where the isotopic product is given by $* = \times T \times$, *and the isounit is* $\hat{1} = T^{-1}$;
A.2: $\mathcal{H}_{IB,G}$ *is equipped with an isoinner product defined for every pair of elements* $\hat{\psi}_1 , \hat{\psi}_2 \in \mathcal{H}_{IB,G}$ *by*

$$(\hat{\psi} \hat{,} \hat{\psi}) := \hat{1} \int_a^b dx\, \hat{\psi}^\dagger(x)\, G(x, \dot{x}, ...)\, \hat{\psi}(x) \in \hat{R}(\hat{n},+,*) , \qquad (6.2.19a)$$

$$(\hat{\psi}_1 \hat{,} \hat{\psi}_2) = \overline{(\hat{\psi}_2 \hat{,} \hat{\psi}_1)} \in \hat{C}(\hat{c},+,*) , \qquad (6.2.19b)$$

$$(\hat{c} * \hat{\psi}_1 \hat{,} \hat{\psi}_2) = \bar{\hat{c}} * (\hat{\psi}_1 \hat{,} \hat{\psi}_2), (\hat{\psi}_1 \hat{,} \hat{c} * \hat{\psi}_2) = (\hat{\psi}_1 \hat{,} \hat{\psi}_2) * \hat{c}, \qquad (6.2.19b)$$

$$(\hat{\psi}_1 + \hat{\psi}_2 \hat{,} \hat{\psi}) = (\hat{\psi}_1 \hat{,} \hat{\psi}) + (\hat{\psi}_2 \hat{,} \hat{\psi}) , \qquad (6.2.19c)$$

$$\hat{\psi}_k \in \mathcal{H}_{IB,G} , \ \hat{c} = c\,\hat{1} \in \hat{C}(c,+,*), \ G \neq T,$$

A.3: *The isonorm* $||\, \hat{\uparrow}(x)\, ||$ *is always positive definite, or null for* $\hat{\uparrow} = 0$, *and verifies the isoschwartz inequality (6.3.14), thus implying that both isotopic elements are of Class I (sufficiently smooth, bounded, nowhere degenerate, Hermitean and positive definite)*,

$$T > 0, \qquad G > 0; \qquad (6.2.20)$$

A.4: $\mathcal{H}_{IB,G}$ *is countable, i.e., there exists a countable set of elements* $\hat{e}_1$, $\hat{e}_2$, ..., $\hat{e}_n$ *approximating every element* $\hat{\psi} \in \mathcal{H}_{IB,G}$,

$$\hat{\psi} = \sum_{k=1,...,n} \hat{c}_k * \hat{e}_k \in \mathcal{H}_{IB,G} , \ \hat{c} \in \hat{C} , \qquad (6.2.21)$$

with arbitrary accuracy, i.e.,

$$||\,\hat{\psi} - \sum_{k=1,...,n} \hat{c}_k * \hat{e}_k \hat{\uparrow}|| < \delta \qquad (6.2.22))$$

for arbitrary $\delta > 0$ *and sufficiently large* n. *The elements* $\hat{\psi}$, $\hat{\phi}$, *etc. of an isohilbert space are called "isostates".*
A.5: $\mathcal{H}_{IB.G}$ *is conventionally complete* [12,13].

The reason for the formulation of isohilbert spaces for Class IB is now evident. In fact, for Class IA, we have in general $G = G(t, x, \dot{x}, \psi, \bar{\psi}, ...)$, as a result of which, in general,

$$(c * \psi_1 \hat{,} \psi_2) \neq \bar{c} * (\psi_1 \hat{,} \psi_2), \quad (\psi_1 \hat{,} c * \psi_2) \neq (\psi_1 \hat{,} \psi_2) * c. \qquad (6.2.23)$$

As a result, we have the following

Proposition 6.2.1 [14]: *Isohilbert spaces of Class IB are Hilbert, but those of Class IA are generally not.*

However, in most physical applications, we have the single isotopic element $T = G$ which can be assumed to be independent of x and ψ. In this latter case isohilbert spaces of Class IA do verify all axioms of Definition 6.2.1, including the axioms

$$(c * \psi_1 \hat{,} \psi_2) = \bar{c} * (\psi_1 \hat{,} \psi_2), \quad (\psi_1 \hat{,} c * \psi_2) = (\psi_1 \hat{,} \psi_2) * c. \qquad (6.2.24)$$

by therefore being Hilbert.

Definition 6.2.2 [10]: *Two elements $\hat{\psi}_1$ and $\hat{\psi}_2$ of an isohilbert space $\hat{\mathcal{H}}_{IB,G}$ over the isofield $\hat{C}$ are said to be "isoorthogonal" when*

$$(\hat{\psi}_1 \hat{,} \hat{\psi}_2) \equiv 0; \qquad (6.2.25)$$

an element $\hat{\psi}$ is said to be "isonormalized" when

$$(\hat{\psi} \hat{,} \hat{\psi}) \equiv \hat{1}; \qquad (6.2.26)$$

and a basis $\hat{e}_1, \ldots, \hat{e}_n$ is said to be "isoorthonormal" when it verifies the rules

$$(\hat{e}_i \hat{,} \hat{e}_j) = \hat{\delta}_{ij} = \hat{1}\,\delta_{ij} \qquad (6.2.27)$$

The corresponding expression for spaces of Class IA are given by

$$(\psi_1 \hat{,} \psi_2) \equiv 0, \qquad (\psi \hat{,} \psi) = 1, \qquad (e_i \hat{,} e_j) = \delta_{ij}. \qquad (6.2.28)$$

Definition 6.2.3 [14]: *An isobanach space $\hat{B}_{IB}$ of class IB is an isospace over an isofield $\hat{C}(\hat{c},+,*)$ characterized by the following axioms:*
A.1: *$\hat{B}_{IB}$ is an isolinear space;*
A.2: *For every element $\hat{f} \in \hat{B}_{IB}$ there is an isonorm $|\hat{\uparrow} \hat{f} \hat{\uparrow}|$ with values in $\hat{R}(\hat{n},+,*)$ verifying the properties*

$$|\hat{\uparrow}\hat{c} * \hat{f}\hat{\uparrow}| = \hat{\uparrow}\hat{c}\hat{\uparrow} * |\hat{\uparrow}\hat{f}\hat{\uparrow}|, \quad |\hat{\uparrow}\hat{f}_1 + \hat{f}_2\hat{\uparrow}| \leq |\hat{\uparrow}\hat{f}_1\hat{\uparrow}| + |\hat{\uparrow}\hat{f}_2\hat{\uparrow}| \qquad (6.2.29)$$

$|\hat{\uparrow}\hat{f}\hat{\uparrow}|$ is positive-definite, or null for $\hat{f} = 0$; and
A.3: *$\hat{B}_{IB}$ is (conventionally) complete as for the isohilbert space.*

Again, one can see that *an isobanach space of Class IB is Banach, but one of Class IA is not necessarily so, unless the isounit is independent from the local coordinates*.

The classification given above for functional isoanalysis evidently applies also to square integrable, Hilbert, Banach and other spaces, resulting in isospaces of Class IA, IB, IIA, IIB, IIIA, IIIB, etc.

To study the isodual image of isohilbert spaces it is best to use Dirac's notation via bras and kets. Recall that the elements of a conventional Hilbert space $\mathcal{H}$ are the *states* $|\psi\rangle$ with familiar inner product and normalization

$$\langle\psi|\phi\rangle = \int d^3r\, \psi^\dagger(r)\, \phi(r) \;\in\; C(c,+,\times), \qquad \langle\psi|\psi\rangle = 1\,. \tag{6.2.30}$$

The *dual Hilbert space* $\bar{\mathcal{H}}$ is then the space with *dual states* $\langle\psi|$ equipped with the same composition (6.2.30) over $C(c,+,\times)$. As well known, $\mathcal{H}$ and $\bar{\mathcal{H}}$ are not independent, but interconnected with the conjugation

$$\langle\psi| = (|\psi\rangle)^\dagger. \tag{6.2.31}$$

In the above formulation, the isohilbert space $\hat{\mathcal{H}}$ is an isolinear space of *isostates* $|\hat\psi\rangle$ (with $\hat\psi$ generally different from ψ) equipped with the isoinner product and isonormalization

$$\langle\hat\psi \hat{|} \hat\phi\rangle = \hat{1}\int d^3r\, \hat\psi^\dagger(r)\, T(r,\ldots)\, \phi(r) \;\in\; \hat{C}(\hat{c},+,*)\,, \tag{6.2.32a}$$

$$\langle\hat\psi \hat{|} \hat\psi\rangle = \hat{1}\,. \tag{6.2.32b}$$

The *isodual isohilbert space* $\hat{\mathcal{H}}^d$ can then be defined As the isolinear space with *isodual isostates* $\langle\hat\psi|^d$ equipped with the same composition (6.3.32a) *but now referred over the isodual isofield* $\hat{C}^d(\hat{c}^d,+,*^d)$, with calls for an isonormalization with respect to $\hat{1}^d = -\hat{1}$.

This implies that $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}^d$ are interconnected by the conjugation

$$\langle\hat\psi|^d = -(|\hat\psi\rangle)^\dagger, \tag{6.2.33}$$

which is the extension to Hilbert spaces of the isodual conjugation for complex numbers $c \to c^d = -\bar{c}$.

The *isodual isoinner product* and *isodual isonormalization* can then be written

$$\langle\hat\psi \hat{|} \hat\phi\rangle^d = (\langle\hat\psi|^d)|\hat\psi\rangle = -\hat{1}^d\int d^3r\, \hat\psi^\dagger(r)\, T^d(r,\ldots)\, \phi(r) \in \hat{C}(\hat{c},+,*)\,, \tag{6.2.34a}$$

$$\langle\hat\psi \hat{|} \hat\psi\rangle^d = \hat{1}^d = -\hat{1}\,. \tag{6.2.34b}$$

In summary, the following four spaces will have a primary relevance for our analysis:

A) Conventional Hilbert spaces $\mathcal{H}$, which are and will remain at the foundation of *particles in exterior conditions*;

B) Isodual Hilbert spaces $\mathcal{H}^d$, occurring for $\hat{1}^d = -1$, which are assumed as the basic spaces to represent *antiparticles in exterior conditions*;

C) Isohilbert spaces $\hat{\mathcal{H}}$ (generally assumed of Class I), which are the basis of the representation of *particles in interior conditions*; and

D) Isodual isohilbert spaces $\mathcal{H}^d$(generally assumed of Class II), which are assumed at the basis of *antiparticles in interior conditions*.

We leave to the interested reader for brevity the study of the isohilbert spaces of Classes III, IV and V, as well as the *isodual square integrable spaces* $\hat{\mathcal{L}}^{2d}[a, b]$ and the *isodual isobanach spaces* $\hat{\mathcal{B}}^d$.

The fundamental character of the isotopy of the unit $I \Rightarrow \hat{1}$ is evident from the preceding structures. Note that the integral realizations of $\hat{1}$ mentioned above characterizes the particular type of integral topology of Fig. I.1.4.1. In this sense, functional isoanalysis constitutes an integral generalization of the conventional analysis.

Numerous examples of integral isounits will be given in Vol.s II and III. They essentially represent the overlapping of the wavepackets as a necessary condition to have an interior dynamical system, in such a way that, when the overlapping is null, the isounits $\hat{1}$ recover the conventional unit I. In this way, functional isoanalysis recovers the conventional functional analysis identically, by construction at the limit $\hat{1} \rightarrow I$.

Whenever needed for clarity, isospaces will be denoted with symbols of the type $\hat{\mathcal{L}}^{(2)}{}_{IAT}[a, b]$, $\hat{\mathcal{H}}_{IAT}$, $\hat{\mathcal{B}}_{IAT}$, etc., identifying the class as well as the selected isotopic element.

All conventional operations and properties of linear–local operators on Hilbert and other spaces (such as determinant, trace, Hermiticity, unitarity, etc.) admit a consistent isotopic generalization studied in the next section.

At this point we indicate that the conventional eigenvalue equation $H\psi = E\psi$ on $\mathcal{H}$ is lifted on $\hat{\mathcal{H}}_{IBT}$ into the *isoeigenvalues equations* [1,2,3]

$$H * \hat{\psi} = \hat{E} * \hat{\psi} \equiv E\,\hat{\psi}\ , \quad \hat{E} = E\,\hat{1} \in \hat{C}(\hat{c},+,*),\ E \in C(c,+,\times)\,. \qquad (6.2.35)$$

This illustrates the reasons indicated earlier for the preference in physical calculations of formulations of Class IB. In fact, the identity $\hat{E} * \hat{\psi} \equiv E\hat{\psi}$ implies that the "numbers" of the theory are the conventional values E, rather than the isovalues $\hat{E} = E\hat{1}$ even when $\hat{1}$ is an operator.

We can now indicate the nontriviality of the isotopies of Hilbert spaces. To begin, *the lifting* $\mathcal{H} \rightarrow \hat{\mathcal{H}}_{IB}$ *implies the alteration of the eigenvalues of an operator*, as clearly illustrated by Eq.s (6.2.35). Moreover, *Hilbert and isohilbert spaces are not unitarily equivalent*, that is, there exist no (conventionally) unitary

transformation mapping $\mathcal{H}$ into $\hat{\mathcal{H}}_{IB}$. However, $\mathcal{H}$ and $\hat{\mathcal{H}}_{IB}$ are indeed interconnected by a *conventionally nonunitary* transformation [1]. In fact, the maps

$$|\psi> \rightarrow |\psi'> = U|\psi>, \quad <\psi| \rightarrow <\psi'| = <\psi|U^\dagger, \qquad U\,U^\dagger = \hat{1} \neq I\,, \quad (6.2.36)$$

implies the map of the inner product into the isotopic form

$$<\psi|\psi> \;\rightarrow\; <\psi'|T|\psi'>\,,\;\; T = (U\,U^\dagger)^{-1} \equiv T^\dagger\,. \qquad (6.2.37)$$

while, jointly, the unit is mapped into the isounit $I \rightarrow \hat{1} = T^{-1} = UU^\dagger$.

The physical inequivalence of the Hilbert and isohilbert formulations is then established. Note that the isotopic element T emerging from mapping (6.2.37) is Hermitean, as it should be for Class IA or IB.

The remarkable properties of the isotopies is that, despite these physical and structural differences, *Hilbert and isohilbert spaces coincide at the abstract level.* In fact, for the particular case in which T = G = cost. or independent from the integration variables for Class IB,[46] the isoinner product has been constructed in such a way to coincide with the conventional product, $<\hat{\psi}|\hat{\psi}> = \hat{1}<\hat{\psi}|T|\hat{\psi}> = \hat{1}T<\hat{\psi}|\hat{\psi}> \equiv <\hat{\psi}|\hat{\psi}>$. Nevertheless, eigenvalues and isoeigenvalues remain different even for a constant isounit $\hat{1} \neq 1$.

As a result, *functional analysis and its isotopic covering also coincide at the abstract level by construction.*

An example of functions which are not square integrable but are isosquare integrable is given by

$$f(x) = 1 / \sqrt{x}\,, \qquad (6.2.38)$$

which is known not to be square integrable in the interval [0, 1]. In fact, function (6.2.38) becomes isoquare integrable in the same interval for the isotopic element $T(x) = x^{1/6}$. A significance of the isospaces is therefore given by the fact that if a functional space does not constitute a conventional $\mathcal{L}^{(2)}$, Hilbert or Banach space, there may exist an isotopic element T such that the same sets does indeed form an $\hat{\mathcal{L}}^{(2)}$, isohilbert or isobanach space.

In any case, functional isoanalysis establishes that statements such as "a given function f(x) is or is not square integrable" need, for necessary mathematical consistency, the joint identification of the unit of the underlying space.

A simple example of a set of functions isoorthonormal on $\hat{\mathcal{H}}_{IA,T}$ is given by

$$f_n(x) \;=\; (2\pi)^{-1/2}\, e^{\,inTx}\,, \quad n \;=\; 0, \pm 1, \pm 2, \ldots \qquad (6.2.39)$$

---

[46]Note that for Class IA, $\hat{1}$ is an element of the original field, that is, a constant.

for $x \in [-\pi/T, +\pi/T]$ and T independent of x (but dependent on $\dot{x}$ and other variables). In fact, we can write

$$(f_n \hat{,} f_m) = (1/2\pi) \int dx\, e^{-inTx}\, T\, e^{imTx} =$$

$$= (1/2\pi) \int dx\, e^{-inz}\, e^{+imz}\, dz = \delta_{nm} . \qquad (6.2.40)$$

It also important to have an idea of the physical applications of functional isoanalysis in general, and of hadronic mechanics in particular. An example is the representation of unstable hadrons as "synthesis" of other hadrons which implies the inverse possibility of stimulating their decay artificially, with an apparent new technology. These possibilities are strictly precluded for the conventional functional analysis, and require instead the covering isoanalysis for their quantitative treatment.

A number of other important applications of isoanalysis also exist with a simpler structure which, as such, can be outlined here. We mention in this respect the possibility indicated earlier of achieving *a convergent perturbation theory of the strong interactions.* In fact, we have the following

Theorem 6.2.1 [10]: *Given a perturbative series which is conventionally divergent on a Hilbert space $\mathcal{H}$, there always exist an isotopy under which the series becomes isoconvergent on isohilbert space $\hat{\mathcal{H}}_T$.*

The proof is so simple to be trivial. Consider, e.g., a divergent canonical expansion of an operator A(k), $k \in R(n,+,\times)$, on $\mathcal{H}$ in terms of a Hermitean Hamiltonian $H = H^\dagger$ for a large value of k

$$A(k) = A(0) + k\,[A, H] / 1! + k^2\,[[A, H], H]\, 2! + \ldots\ldots \to \infty, \quad k \gg 1, \qquad (6.2.41)$$

where $[A, H] = AH - HA$ is the Lie product. Theorem 6.2.1 then establishes that there *always* exists an isotopy of the unit $I \Rightarrow \hat{1} = T^{-1}$ and a reinterpretation of A(k) and H on $\hat{\mathcal{H}}_T$ under which the series becomes isoconvergent

$$A(k) = A(0) + k\,[A\hat{,}H] / 1! + k^2\,[[A\hat{,}H], H]\, 2! + \ldots\ldots \to K < \infty, \quad k \gg 1 \qquad (6.2.42)$$

where $[A\hat{,}H] = ATH - HTA$ is the Lie-isotopic product. In fact, a solution exists even for a constant isotopic element T when sufficiently smaller than k, e.g.,

$$T = k^{-n}, \qquad (6.2.43)$$

with n a sufficiently large positive integer.

Yet another important application of functional isoanalysis in physics occurs when the conventional Hilbert space $\mathcal{H}$ and its isotopic image $\hat{\mathcal{H}}_{IB}$ are

incoherent, in the sense that the transition probability among states belong to $\mathcal{H}$ and $\hat{\mathcal{H}}_{IB}$ is identically null.

This mathematically simple property implies the possibility of resolving a vexing problem of contemporary particle physics, the lack of exact confinement of quarks beginning at the discrete nonrelativistic level. In fact, as preliminarily studied in ref. [11], and studied in detail in Vol. III, quarks possess an *exact confinement* when treated via hadronic mechanics, i.e., when belonging to $\hat{\mathcal{H}}_{IB}$, because they have an identically null probability of escaping to the exterior world represented by the conventional space $\mathcal{H}$ *even for collision with infinite energies or in the absence of a potential barrier.* In addition, as we shall see, the isotopy $\mathcal{H} \rightarrow \hat{\mathcal{H}}_{IB}$ preserves all axiomatic properties and quantum numbers (for the case of standard isorepresentations) of SU(3), while permitting convergent isoseries.

Intriguingly, it appears that the lack of exact confinement is essentially due to the insistence of current quark theories of using conventional, rather than isotopic, functional analysis.

Further *novel* applications of isoanalysis (that is, applications which cannot be formulated with the conventional analysis, let alone treated quantitatively) are possible via isotopies of the remaining classes. For instance, the singular isoanalysis of Class IV is given by the isotopic element characterizing the space component in spherical coordinates $\{r, \theta, \phi\}$ of Schwarzschild's metric for the exterior gravitational problem (Sect. 5.4)

$$T = \mathrm{diag}\,(r\,/\,(r - 2M),\, r^2,\, r^2 \sin\theta)\,. \tag{6.2.44}$$

The singular character of the isoanalysis at the limit when the astrophysical bodies collapse into a singularity with $T = 0$ is evident.

To close this section with a few comments of historical character, let us recall that the appearance of the isotopic element G in composition (6.2.2) has considerable connections with the known *weight function* of the conventional functional analysis [12,13]. As a matter of fact, the techniques known for the latter are extendable to the former.

The extension of Hilbert spaces $\mathcal{H}$ to the form $\mathcal{H}_T$ with a weight function T and composition on ordinary fields C

$$(\,f_1\,\hat{;}\,f_2\,)\ =\ \int_a^b dx\ \overline{f_1(x)}\ T(x)\ f_2(x)\ \in C\,, \tag{6.2.51}$$

is known since the first part of this century in both mathematical and physical literature [12,13]

The novelty of the isotopies here studied is the introduction of the isotopic function G *jointly* with the lifting of the underlying fields $F \rightarrow \hat{F}$. The nontriviality of the latter as compared to the former is easily illustrated by the fact that the basic unit remains unchanged for the former although it is generalized for the latter, or by the fact that the latter has a generally nonlocal–

integral topology as compared to the local–differential topology of the former, or by the fact that the isohilbert spaces $\hat{\mathcal{H}}_{IB,G}$ coincide with the conventional ones $\mathcal{H}$ at the abstract level, which is not generally the case for structures (6.2.51). In turn this is an additional illustration of the remarkable implications of the isotopies of the unit.

6.3: OPERATIONS ON ISOHILBERT SPACES

A property of functional analysis with fundamental physical implications is the linearity of the operations on a Hilbert space $\mathcal{H}$, from which superposition principle, causality, measurement theory, and other physical laws follow (see, e.g., ref.s [17,18] and literature quoted therein).

A property of functional isoanalysis of equally fundamental physical character is the capability of representing the most general known nonlinear, nonlocal and nonhamiltonian interactions via an operator theory on isohilbert spaces which preserves the original axioms of linearity, thus permitting the achievement of consistent generalizations of conventional physical laws.

To state it differently, the isotopic methods disprove the rather widespread belief that a nonlinear and nonlocal formulation of the strong interactions implies the loss of superposition principle, causality, measurement theory, and all that.

These physical aspects will be studied in Vol. II. In this section we shall study the elements of the *isooperator theory* , i.e., the theory of operators on isohilbert spaces.

The understanding of this section requires a knowledge of the preceding parts, with particular reference to the notions of: isolinear and isolocal transformations (Sect. 4.2); isomodules (App. 4.A); isoexponentiations (Sect. 4.3); etc.

Proposition 6.3.1 [10]: *Lei* $\hat{\xi}_T$ *be an isoassociative enveloping algebra of operators* A, B, C, *with isoproduct* $A*B = ATB$ *and isounit* $\hat{1} = T^{-1}$ *acting on an isohilbert space* $\hat{\mathcal{H}}_{IB,G}$ *over an isofield* $\hat{F}(\hat{\alpha},+,*)$ *of isoreal or isocomplex numbers. Then,* $\hat{\xi}_T$ *is isolinear and isolocal on* $\hat{\mathcal{H}}_{IB,G}$ *, i.e., it verifies the properties*

$$A*(\hat{\alpha}*\psi + \hat{\beta}*\phi) = \hat{\alpha}*(A*\psi) + \hat{\beta}*(A*\phi), \qquad (6.3.1a)$$

$$(\hat{\alpha}*A + \hat{\beta}*B)*\psi = \hat{\alpha}*(A*\psi) + \hat{\beta}*(B*\psi), \qquad (6.3.1b)$$

$$(A*B)*\psi = A*(B*\psi), \qquad (6.3.1c)$$

$$\hat{1}*\psi = \psi, \quad \forall\ A, B, \in \hat{\xi}_T,\ \psi \in \hat{\mathcal{H}}_{IB,G}, \quad \hat{\alpha}, \hat{\beta} \in \hat{F}, \qquad (6.3.1d)$$

and $\mathcal{H}_{IB,G}$ is a one-sided, left or right isomodule of $\hat{\xi}_T$.

We now study of the isotopies of conventional operations of quantum mechanics. Let $\hat{A}$ be an isolinear and isolocal operator on $\hat{\xi}_T$ over $\hat{F}$. Then the n-th *isopower* $A^{\hat{n}}$ of A is given by

$$\hat{A}^{\hat{n}} = A * A * ... * A \text{ (n times)}. \tag{6.3.2}$$

In particular, the *isoquare* of an operator A on $\mathcal{H}_{IB,G}$ is $A^{\hat{2}} = A*A$. Thus, conventional powers, such as that of the linear momentum "p^2" = pp, or of the angular momentum "J^2" = JJ, etc. have no mathematical or physical meaning for hadronic mechanics and their use actually leads to a host of easily demonstrable (but often undetected) inconsistencies.

The *isoinverse* $A^{\hat{-1}}$ of $A \in \hat{\xi}_T$ on $\hat{F}(\hat{\alpha},+,*)$ is defined by the conditions

$$A * A^{\hat{-1}} = A^{\hat{-1}} * A = \hat{1}, \tag{6.3.3}$$

and given by

$$A^{\hat{-1}} = \hat{1}\, A^{-1}\, \hat{1}, \tag{6.3.4}$$

where A^{-1} is the conventional quantum mechanical inverse.

By ignoring hereon the isolinearity and isolocality for simplicity, we then have the following

Definition 6.3.1 [3,4,10]: *Let $\mathcal{H}_{IB,G}$ be an isohilbert space with isoinner product (6.2.19) over an isofield $\hat{F}_T$. Then, the "isohermitean conjugate" $A^{\hat{\dagger}}$ of an operator $A \in \hat{\xi}_T$ on $\mathcal{H}_G$ is defined by*

$$[(\psi * \hat{A}^{\hat{\dagger}}) \hat{\,} \psi) = (\psi \hat{,} [A * \psi)] \tag{6.3.5}$$

Proposition 6.3.2 [4,10]: *Necessary and sufficient condition for an operator $A \in \hat{\xi}_T$ on $\mathcal{H}_{IB,G}$ to be isohermitean is that* [47]

$$A^{\hat{\dagger}} = G^{-1}\, T\, A^{\dagger}\, G\, T^{-1}, \tag{6.3.6}$$

The following properties can also be readily proved

$$(\hat{\alpha} * A + \hat{\beta} * B)^{\hat{\dagger}} = \tilde{\alpha} * A^{\hat{\dagger}} + \tilde{\beta} * B^{\hat{\dagger}}, \tag{6.3.7a}$$

$$(A * B)^{\hat{\dagger}} = B^{\hat{\dagger}} * A^{\hat{\dagger}}, \tag{6.3.7b}$$

---

[47] Note that the isotopic elements T and G are inverted in ref. [4] as compared to their use in this presentation, which is the notation now widely adopted in the literature.

$$(A^{\hat\dagger})^{\hat\dagger} \equiv A, \quad \forall\ \hat\alpha, \hat\beta \in \hat F,\ A \in \hat\xi_T. \tag{6.3.7c}$$

where the upper bar denotes complex conjugation.

An example of isohermitean operator on $\hat{\mathcal{H}}_G$ is given by

$$A = G^{-1} \,|\,\psi > < \psi\,|\, T^{-1} = A^{\hat\dagger}. \tag{6.3.8}$$

Definition 6.3.2 [3,4,10]: *Let $\hat{\mathcal{H}}_{IB,G}$, $\hat F_T$ and $\hat\xi_T$ be as in Definition 6.3.1. Then, an operator $\hat U \in \hat\xi_T$ on $\hat{\mathcal{H}}_{IB,G}$ is called "isounitary" if it verifies the condition*

$$(\psi \hat{,}\ \hat U^{\hat\dagger} * \hat U * \psi) \equiv (\psi \hat{,}\ \psi), \tag{6.3.9}$$

i.e.,

$$\hat U * \hat U^{\hat\dagger} = \hat U^{\hat\dagger} * \hat U = \hat 1, \quad \text{or} \quad \hat U^{\hat\dagger} = \hat U^{\hat{-1}}. \tag{6.3.10}$$

Proposition 6.3.3 [loc. cit.]: *Let $\hat U$ be a isounitary operator and A an isohermitean operator on $\hat{\mathcal{H}}_{IB}$. Then, the transformation*

$$A' = \hat U * A * \hat U^{\hat{-1}} \tag{6.3.11}$$

is isohermitean.

It is an instructive exercise for the interested reader to prove the following property (see ref. [3] for a detailed treatment)

Proposition 6.2.4 [loc. cit.]: *Isounitary operators $\hat U \in \hat\xi_T$ on $\hat{\mathcal{H}}_{IB,G}$ over $\hat F_T$ always admit the following realization via the isoexponentiation of an isohermitean operator $X = X^{\hat\dagger} \in \hat\xi_T$, $\hat w\ \hat F_T$,*

$$\hat U(w) = e_{\hat\xi}^{\,i\,\hat w * X} \equiv e_{\hat\xi}^{\,i\,w\,X} = \hat 1\,\{e^{-wTX}\} = \{e^{iXTw}\}\,\hat 1, \tag{6.3.12}$$

As we shall seee in Vol. II, the above property has fundamental character for the Lie–isotopic theory of Ch. 4 because it permits the realization of continuous Lie–isotopic transformation groups via isounitary operator on a isohilbert space with isocomposition rules (Sect. 4.5)

$$\hat U(\hat 0) = \hat U(\hat w) * \hat U(-\hat w) = \hat 1, \tag{6.3.13a}$$

$$\hat U(\hat w) * \hat U(\hat w') = \hat U(\hat w') * \hat U(\hat w) = \hat U(\hat w + \hat w'), \tag{6.3.13b}$$

In particular, the time evolution in hadronic mechanics is characterized precisely by a isounitary transformation admitting of the above isoexponentiation and,

thus, forming a Lie-isotopic group.

A further important property is given by (see ref. [3], p. 1304 for proof)

Proposition 6.3.5 : *Any isolinear, isolocal and isohermitean operator* $A = A^{\hat{\dagger}}$ *is bounded.*

We now pass to a study of the isotopies of eigenvalues equations.

Definition 6.3.3 [loc. cit.]: *Let* H *be a (not necessarily isohermitean) operator on an isohilbert space* $\hat{\mathcal{K}}_{IB,G}$. *Then a generally isocomplex number* $\hat{c} \in \hat{C}_T$ *is called an "isoeigenvalue" of* H *if there exist an isostate* $\psi \in \hat{\mathcal{K}}_G$ *such that*

$$H * \psi = \hat{c} * \psi \equiv c\,\psi. \qquad (6.3.14)$$

We therefore confirm that *the isoeigenvalues* $\hat{c}$ *of an operator* H *on* $\hat{\mathcal{K}}_{IB,G}$ *coincide with the conventional eigenvalues* c *of the operator* $\hat{H} = HT$. Thus, the "numbers" predicted by hadronic mechanics for measurements are conventional numbers.

The following property is important for the applications of hadronic mechanics.

Proposition 6.3.6 [3]: *A set of isocomplex numbers* $\hat{c} = c\,\hat{1}$ *are the isoeigenvalues of an operator* $H \in \hat{\xi}_T$ *on* $\hat{\mathcal{K}}_{IB,G}$ *iff they are the solution of the so-called "isocharacteristic equation" of* H

$$\mathrm{Det}\,(HT - c) = 0. \qquad (6.3.15)$$

A number of conventional properties of the eigenvalue theory (see, e.g., ref.s [17,18]) persist under isotopies, thus implying that they are indeed genuine axioms of quantum mechanics. This is the case for the following important property (see ref. [3], p. 1310, or ref. [4], p. 1922).

Proposition 6.3.7 [loc. cit.]: *All isoeigenvalues of isohermitean operators* $H \in \hat{\xi}_T$ *on* $\hat{\mathcal{K}}_{IB,G}$ *are real.*

The above property establishes that the reality (observability) of the eigenvalues of Hermitean operators is a true axiom of quantum mechanics because it persists under isotopies. Another important property which also persists under isotopy is expressed by the following

Proposition 6.3.8 [loc. cit.]: *The isoeigenvalues of isohermitean operators are invariant under isounitary transformations.*

However, there are a number of properties of quantum mechanics which are not invariant under isotopy and, as such, they cannot be considered as true

axioms of the theory. The first is the rather popular belief that Hermitean operators admit a unique set of eigenvalues which is disproved by the following :

Proposition 6.3.9 [10]: *A Hermitean operator does not admit a unique set of real eigenvalues, but admits instead an infinite number of different sets of eigenvalues, each of which is real, depending on the assumed basic unit.*

Let H be conventionally Hermitean and consider for this purpose the conventional eigenvalues $H\psi = E_0\psi$. Consider now an isotopy of the preceding equations under which H remains Hermitean (as anticipated in Sect. 6.1, this is always the case when T = G). Then, we have *different* isoeigenvalues for the *same* operator H, i.e., the isotopies imply for the eigenvalues equations the lifting

$$H\psi = E_0 \rightarrow H * \hat{\psi} = E_T\hat{\psi}, \quad E_T \neq E_0, \tag{6.3.16}$$

which is inherent in the basic isotopy of these volumes, Eq.s (1.1.1), i.e.

$$I \rightarrow \hat{1}. \tag{6.3.17}$$

But an infinite number of different isotopic elements T are possible. This proves that a Hermitean operator H can have an infinite number of different sets of eigenvalues E_T depending on the selected isounit $\hat{1}$ or isotopic element T.

As we shall see in Vol. II, *expression (6.3.16) permits an explicit realization and operator generalization of the so-called "hidden variables"*. We shall also see that *Bell's inequality, von Neumann's theorem and other properties are not preserved under isotopies and, as such, they are not true axioms of quantum mechanics.* As we shall see, these and other intriguing occurrences permit an isotopic completion of quantum mechanics into hadronic mechanics which is intriguingly close to the historical argument of Einstein, Rosen, Podolsky and others.

Definition 6.3.4 [3,4,100]: *Let A be an operator on a finite-dimensional isohilbert space $\hat{\mathcal{H}}_{IB,G}$, and let $\hat{c}_1$, $\hat{c}_2$, ..., $\hat{c}_n$ be its isoeigenvalues. Then the "isotrace" $\hat{T}\hat{r}A$ of A is given by*

$$\hat{T}\hat{r}A = \hat{c}_1 + \hat{c}_2 + ... + \hat{c}_n. \tag{6.3.18}$$

The "isodeterminant" $\hat{D}\hat{e}t$ A of a matrix A is the isoscalar defined by

$$\hat{D}\hat{e}t\, A = [\,Det\,(AT)\,]\,\hat{1} \in \hat{C}_T. \tag{6.3.19}$$

A further instructive exercise for the interested reader is to prove the

following properties:

Proposition 6.3.10 [loc. cit.]: *Isolinear and isolocal operators* A, B, C $\in \hat{\xi}$ *on a finite-dimensional isohilbert space verify the following properties*

$$\hat{Tr}\, A = (Tr\, A)\hat{1}, \tag{6.3.20a}$$

$$\hat{Tr}\, (A \times B) = (\hat{Tr} A) * (\hat{Tr} B), \tag{6.3.20b}$$

$$\hat{Tr}\, (B * A * B^{-\hat{1}}) = \hat{Tr}\, A, \tag{6.3.20c}$$

$$\hat{Det}\, (A * B) = (\hat{Det}\, A) * (\hat{Det}\, B), \tag{6.3.20d}$$

$$\hat{Det}\, (A^{-\hat{1}}) = (\hat{Det}\, A)^{-\hat{1}}, \tag{6.3.20e}$$

$$\hat{Det}\, \{ e_{\hat{\xi}}^{\ A} \} = e_{\hat{\xi}}^{\ \hat{Tr}\, A}, \tag{6.3.20f}$$

Definition 6.3.5 [3]: *Let* A *be an isolinear and isolocal operator on* $\hat{\mathcal{H}}_{IB,G}$. *Then the "isospectrum"* $\hat{Sp}A$ *of* A *is defined as the set of isocomplex numbers* $\hat{c} = c\,\hat{1}$ *which are such that the quantity* $(A - \hat{c})$ *is not invertible in* $\hat{\xi}_T$, *and admits the realization in term of the conventional spectrum* SpA *of* A

$$\hat{Sp}\, A = (\, Sp\, AT\,)\hat{1} \in \hat{C}_T. \tag{6.3.21}$$

We now pass to the study of the isotopies of another important notion of conventional quantum mechanics, that of projection operators.

Definition 6.3.6 [4]: *Two "isosubspaces"* $\hat{\mathcal{H}}^1{}_{IB,G}$ *and* $\hat{\mathcal{H}}^2{}_{IB,G}$ *of* $\hat{\mathcal{H}}_{IB,G}$ *are said to be "isoorthogonal" when all their isostates are isoorthogonal (Definition 6.2.2). For any given subspace* $\hat{\mathcal{H}}^o{}_{IB,G}$ *of* $\hat{\mathcal{H}}_{IB,G}$ *the isoorthogonal complement* $\hat{\mathcal{H}}^c{}_{IB,G}$ *is the isoorthogonal subset for which we have the direct sum decomposition*

$$\hat{\mathcal{H}}_{IB,G} = \hat{\mathcal{H}}^o{}_{IB,G} + \hat{\mathcal{H}}^c{}_{IB,G}\,. \tag{6.3.22}$$

One can then study the isotopies of similar properties of conventional quantum mechanics [17,18].

Definition 6.3.7 [4,10]: *An operator* $\hat{P}$ *on* $\hat{\mathcal{H}}$ *is called "isoidempotent" when it verifies the property*

$$\hat{P}^{\hat{2}} = \hat{P} * \hat{P} \equiv \hat{P}, \tag{6.3.23}$$

An isoidempotent operator $\hat{P}$ *is an "isoprojection" of* $\hat{\mathcal{H}}_{IB,G}$ *onto* $\hat{\mathcal{H}}^o{}_{IB,G}$

when it verifies the properties

$$\hat{P} * \psi \; = \; \psi_0 \; , \tag{6.3.24}$$

for all $\psi \in \hat{\mathcal{H}}_{IB,G}$, *with* $\psi_0 \in \hat{\mathcal{H}}_{IB,G}{}^{o}$.

The following property is important for the applications of hadronic mechanics (see ref. [3,4] for its proof).

Proposition 6.3.11: *An isolinear and isolocal operator* $P \in \hat{\xi}_T$ *acting on a finite–dimensional isohilbert space* $\hat{\mathcal{H}}_{IB,G}$ *is an isoprojection operator iff it is isohermitean and isoidempotent.*

The explicit realization of isoprojection operators is given by the following

Proposition 6.3.12 [3,4,10]: *Let* $\hat{\mathcal{H}}_{IB,G}{}^{o}$ *be a closed subspace of a (finite–dimensional) isohilbert space* $\hat{\mathcal{H}}_{IB,G}$, *and let* $\psi_o{}^k$ *be the isoorthogonal basis of* $\hat{\mathcal{H}}_{IB,G}{}^{o}$. *Then, an operator* $\hat{P}$ *is an isoprojection operator of* $\hat{\mathcal{H}}_{IB,G}$ *onto* $\hat{\mathcal{H}}^{o}{}_{IB,G}$ *if it has the explicit realization*

$$\hat{P} \; = \; \sum^k \; | \psi_o{}^k > < \psi_o{}^k | \; G \, T^{-1} \, , \tag{6.3.25}$$

Corollary 6.3.12A: *Under realization (6.3.25) the isoprojection operator of* $\hat{\mathcal{H}}^{o}{}_{IB,G}$ *onto the complement* $\hat{\mathcal{H}}^{c}{}_{IB,G}$ *is given by*

$$\hat{P}^c \; = \hat{1} \; - \; \hat{P} \, . \tag{6.3.26}$$

This completes the notions of isooperator algebras on isohilbert spaces that are minimally sufficient for the initiation of physical applications of Vol. II. Additional, more detailed aspects will be studied when needed. The reader interested in acquiring a technical knowledge of isotopic methods is however suggested to work out a systematic study of the isotopies of conventional operator algebras [17,18].

We consider now the isodual isohilbert spaces of Class II B first studied in ref. [10]. For this purpose let us recall the isodual image $\mathcal{H}^d$ of the conventional Hilbert space $\mathcal{H}$, called *isodual Hilbert space*, which must be defined for consistency over the isodual field $C^d(c^d,+,\times^d)$ and with isodual states given by

$$\psi^d \; = \; \psi^\dagger \, 1^d = - \, \psi^\dagger \; . \tag{6.3.27}$$

Its most salient property is that the isodual norm, i.e., the image under duality of quantity (6.2.6) is now negative–definite

$$|\uparrow\psi^d\uparrow|^d < 0. \tag{6.3.28}$$

The isodual inner product is then given by

$$\mathcal{H}^d : (\psi \hat{,} \psi)^d = 1^d \int d^3r\, \psi^d(r) \times^d \psi(r) \equiv -(\psi, \psi). \tag{6.3.29}$$

As now familiar, this property is important for a study of antiparticles via isoduality.

The *isodual isohilbert spaces* are then isospaces defined over $\hat{F}^d(\hat{\alpha}^d,+,*^d) = \hat{R}^d$ or $\hat{C}^d$ with isodual isostates $\hat{\psi}^d = \psi\dagger\,1^d = -\hat{\psi}^\dagger$

Definition 6.3.8 [10]: *An operator* H *is said to be "isodual isohermitean" on* $\mathcal{H}^d_{IIB,G}$ *when it verifies the condition*

$$H^{\dagger d} = T^{-1}\, G\, A^\dagger\, T\, G^{-1}, \tag{6.3.30}$$

The isodual isoprojection operators on $\hat{\mathcal{H}}^d_{IIB,G}$ *are then given by*

$$\hat{P}^d = T\, G^{-1} \sum^k |\psi_o{}^k\rangle < \psi_o{}^k|. \tag{6.3.31}$$

By comparing the above definition with Proposition 6.3.3, we have the following intriguing property of hadronic mechanics in its general formulation under consideration here with $T \neq G$.

Proposition 6.3.13 [10]: *An operator* H *which is isohermitean on* $\mathcal{H}_{IB,G}$ *is not necessarily isohermitean in its isodual* $\mathcal{H}^d_{IIB,G}$. [48]

In summary we have four primary mathematical structures at the foundation of the Lie–isotopic branch of hadronic mechanics:

A) **Linear operator theory** for the representation of particles in exterior conditions;

B) **Isodual operator theory** for the representation of antiparticles in exterior conditions;

C) **Isolinear operator theory** for the representation of particles in interior conditions; and

D) **Isodual isolinear operator theory** for the representation of

[48] This property, however, is dependent on the assumed notion of duality, that based on a *conventional* conjugation. It is evident that a more general notion of duality is possible in such a way to preserve the operation of isohermiticity, but this approach has other undesirable implications (e.g., for normalizations) and it has not been adopted until now in the applications of hadronic mechanics.

antiparticles in interior conditions.

As we shall see in Vol.s II and III, each of the above parts of functional isoanalysis will have significant applications.

6.4: ISODELTA FUNCTIONS

As indicated in Sect. 6.1, the isotopies imply a generalization not only of main structural foundations of functional isoanalysis, as outlined in Sect.s 6.2 and 6.3, but also of all conventional distributions, special functions and transforms.

This is a topic of such a dimension to require a separate volume for its detailed treatment. In this section we shall merely illustrate these generalizations for the case of Dirac's delta function. In the remaining sections we shall then provide examples of isotopic generalizations of special series and transforms.

As well known (see, e.g., ref. [19] and quoted bibliography), the conventional *Dirac delta function* is not a function, but a distribution representing a rather delicate limit procedure in a conventional functional space, such as the Hilbert space $\mathcal{H}$, with a mathematically well defined meaning only when it appears under an integral.

When the singularity is at the point x = 0, the δ–function can be defined in terms of a well behaved function f(x) on a one–dimensional space S(x,R) over the reals R by [loc. cit.]

$$\int_{-\infty}^{+\infty} f(x)\, \delta(x)\, dx \;=\; f(0), \qquad \int_{-\infty}^{+\infty} \delta(x)\, dx \;=\; 1 \tag{6.4.1}$$

This essentially means that δ(x) = 0 everywhere except at x = 0 where it is singular. Nevertheless, what is mathematically and physically significant is the behaviour near that point, which permits explicit realizations, such as the familiar integral form

$$\delta(x) \;=\; (1\,/\,2\pi) \int_{-\infty}^{+\infty} e^{\,ixy}\, dy\,. \tag{6.4.2}$$

If the singularity is at a point x ≠ 0, then we can write [loc. cit.]

$$f(x) \;=\; \int_{-\infty}^{+\infty} f(x')\, \delta(x' - x)\, dx'\;. \tag{6.4.3}$$

Finally, the δ–function verifies the basic properties

$$\delta(x) \;=\; \delta(-x) \quad , \quad \delta(x - x') \;=\; \int_{-\infty}^{+\infty} dz\, \delta(x - z)\, \delta(z - x')\,. \tag{6.4.4}$$

The delta function is evidently inapplicable when dealing with functional

isospaces, such as the isohilbert spaces $\mathcal{H}$. In particular, exponentials of the type appearing in the integrand of Eq. (6.4.2) are no longer defined in isospaces, and must be replaced by the isoexponentials.

These occurrences rendered mandatory the studies of the isotopies of the delta function. Their origin can therefore be traced back to the isotopies by this author of the Poincaré–Birkhoff–Witt theorem reviewed in Sect. 4.3. The existence of a consistent isotopic generalization of Dirac's delta function was indicated in ref. [1], first studied in detail in ref. [3], subjected to systematic studies and classification in ref. [10], and finally applied to a number of cases reviewed in Vol.s II and III. Inspection of the recent treatment by Kadeisvili [15] is also recommended.

In particular, six mathematically and physically distinguishable isotopies of the Dirac delta function are identified in ref. [10] under the name of *isodelta functions*. Their outline is recommendable as an application of functional isoanalysis, and as a pre–requisite for the isotopies of the Fourier transforms studied in the subsequent sections.

Consider a one–dimensional isospace of Class I, denoted $\hat{S}_I(x,\hat{R})$ with (conventional) real coordinates x over the isofield of real numbers $\hat{R}(n,+,*)$ with conventional elements n and sum +, but isotopic multiplication $n_1 * n_2 := n_1 T n_2$, where T is the isotopic element and $\hat{1} = T^{-1}$ is the multiplicative isounit of Class I.

Let f(x) be an ordinary function defined on $\hat{S}_{IA}(x,\hat{R})$ which verifies the conditions of strong isocontinuity of Sect. 6.2 in all possible subintervals of [$-\infty$, $+\infty$]. Recall that the isotopic element T of Class I is a strongly isocontinuous, bounded, real valued, and positive–definite function of the coordinate x as well as its derivatives with respect to an independent variable of arbitrary order and any other needed quantity, $T = T(x, \dot{x}, \ddot{x}, ...)$.

Then, the *isodelta function of the first kind*, denoted $\hat{\delta}_I$, can be defined in terms of the expression

$$\int_{-\infty}^{+\infty} f(x) * \hat{\delta}_I(x)\, dx \;=\; f(0), \tag{6.4.5}$$

from which we obtain for f = 1

$$\int_{-\infty}^{+\infty} T(x, \dot{x}, \ddot{x}, ...)\, \hat{\delta}_I(x)\; dx \;=\; 1\ . \tag{6.4.6}$$

The isotopic image of (6.4.3) is then given by

$$f(Tx) = \int_{-\infty}^{+\infty} f(Tx') * \hat{\delta}_I(x' - x)\, dx' . \tag{6.4.7}$$

namely, it is not possible any longer to map the dependence on x to the dependence at x', but rather the dependence on Tx to Tx'. This confirms the very peculiar nonlocality of the topology underlying the isotopies discussed earlier.

In fact, the isotopic element T can have an integral dependence on the

interval $x \in [a, b]$ centered at x. In this case the singularity of the Dirac δ at x can be spread over the interval $[a, b]$ by the isodelta function for a suitable selection of T.

In several cases of physical interest, T can be assumed as having an explicit dependence only on the derivatives $T = T(\dot{x}, \ddot{x}, \ldots)$, with consequential identity $Tdx \equiv d(Tx)$. In this case, the projection of the $\hat{\delta}_1$–function into the original functional space $S(x,R)$ implies the equivalence

$$\hat{\delta}_1(x) \approx \delta(Tx). \tag{6.4.8}$$

It is easy to see that, under the assumption of T being independent from x (which is the case for Class IA), the $\hat{\delta}_1$–function admits the integral representation

$$\hat{\delta}_1(x) = (1/2\pi)\int_{-\infty}^{+\infty} T\, e_{\xi}^{\,ixy}\, dy = (1/2\pi)\int_{-\infty}^{+\infty} e^{\,ixTy}\, dy, \tag{6.4.9}$$

(where we have used the fundamental Theorem 6.3.1 on isoexponentiation), and verifies the properties

$$\hat{\delta}_1(x) = \hat{\delta}_1(-x) \quad , \quad \hat{\delta}_1(x - x') = \int_{-\infty}^{+\infty} dz\, \hat{\delta}_1(x - z) * \hat{\delta}_1(z - x'). \tag{6.4.10}$$

For the case of an isospace of Class IB, $\hat{S}_{IB}(\hat{x},\hat{R})$, with isofunctions $\hat{f}(x) = f(x)\,\hat{1}$, a different isotopic expression emerged in ref. [10], called *isodelta function of the second kind,* and denoted $\hat{\delta}_2$, which is characterized by the property

$$\int_{-\infty}^{+\infty} \hat{f}(x) * \hat{\delta}_2(x)\, dx = \int_{-\infty}^{+\infty} f(x)\, \hat{\delta}_2(x)\, dx = \hat{f}(0) = f(0)\,\hat{1}, \tag{6.4.12}$$

In this case the $\hat{\delta}_2$–function must necessarily be an isofunction, i.e., admitting a structure of the type $\hat{\delta}_2(x) = \hat{\delta}_2(x)\,\hat{1}(x, \dot{x}, \ddot{x}, \ldots)$. Then, for $\hat{f} = \hat{1}$, we have

$$\int_{-\infty}^{+\infty} \hat{\delta}_2(x)\, dx = \int_{-\infty}^{+\infty} \hat{\delta}_2(x)\,\hat{1}(x, \dot{x}, \ldots)\, dx = \hat{1} \tag{6.4.13}$$

and the isotopic image of (6.4.3) is given by

$$\hat{f}(x) = \int_{-\infty}^{+\infty} \hat{f}(x') * \hat{\delta}_2(x' - x)\, dx'. \tag{6.4.14}$$

One can see that the projection of the $\hat{\delta}_2$–function in the original functional space $S(x,R)$ implies the equivalence (again for isounits independent of the integration variable)

$$\hat{\delta}_2(x) \approx \delta(x)\,\hat{1}(\dot{x}, \ddot{x}, \ldots). \tag{6.4.15}$$

It is easy to see that, under the same assumptions, the $\hat{\delta}_2$–function admits the integral representation [6]

$$\hat{\delta}_2(x) = 1 / 2\pi \int_{-\infty}^{+\infty} T\, e_{\xi}^{\ ixy}\, dy = \hat{1} / 2\pi \int_{-\infty}^{+\infty} e_{\xi}^{\ ixTz}\, d(Tz) \qquad (6.4.16)$$

and verifies the properties

$$\hat{\delta}_2(x) = \hat{\delta}_2(-x) \ , \quad \hat{\delta}_2(x - x') = \int_{-\infty}^{+\infty} dz\ \hat{\delta}_2(x - z) * \hat{\delta}_2(z - x') . \qquad (6.4.17)$$

It is an intriguing exercise for the reader interested in learning the isotopic techniques to prove that *the first and second kind isodelta functions can be interconnected by the reciprocity transformation* $T \rightarrow \hat{1}$.

To present the *isodelta function of the third kind* $\hat{\delta}_3$, let us recall [10] that the separation on a generic, n-dimensional isospace $\hat{S}(x,\hat{g},\hat{R})$, $\hat{g} = Tg$, $\hat{R} \approx R\hat{1}$, $\hat{1} = T^{-1}$ (see Sect. 3.2 for details), can be formally written as that of a fictitious conventional space in the same dimension $S(\tilde{x},g,R)$, according to the simple rule

$$x^{\hat{2}} = x^t \hat{g}\, x \equiv \tilde{x}^t \tilde{x} = \tilde{x}^2 , \qquad \tilde{x} = T^{\frac{1}{2}} x . \qquad (6.4.18)$$

This implies that a number of problems in isospaces can be worked out in this fictitious conventional space in the $\tilde{x}$-variables, and the results then re-expressed in the x-variables.

The $\hat{\delta}_3$-function emerged precisely from reduction of this type. It can be defined via the conditions [10]

$$\int_{-\infty}^{+\infty} f(\tilde{x})\, \hat{\delta}_3(\tilde{x})\, d\tilde{x} = \int_{-\infty}^{+\infty} f(T^{\frac{1}{2}} x)\, \hat{\delta}_3(T^{\frac{1}{2}}x)\, d(T^{\frac{1}{2}} x) = f(0), \quad T^{\frac{1}{2}} = T^{\frac{1}{2}}(x, \dot{x}, ...) \qquad (6.4.19)$$

from which we obtain for f = 1

$$\int_{-\infty}^{+\infty} \hat{\delta}_3(T^{\frac{1}{2}}x)\, d(T^{\frac{1}{2}}x) = 1 . \qquad (6.4.20)$$

with realization in terms of the conventional δ-function

$$\hat{\delta}_3(x) \approx \delta(\tilde{x}) = \delta(T^{\frac{1}{2}}x) . \qquad (6.4.21)$$

It should be stressed that, while the isodelta functions of the first and second kind are bona-fide isotopies of the conventional expression, this is not the case for $\hat{\delta}_3$ which is merely a pragmatic tool for simplifying calculations, rather than a mathematically rigorous structure.

The above expressions have been presented for the case of one-dimensional coordinates x. The extension to three-dimensions is trivial, and given by isotopic products of the type

$$\hat{\delta}_1(r) = \hat{\delta}_1(x) * \hat{\delta}_1(y) * \hat{\delta}_1(z) . \qquad (6.4.22)$$

Consider now the isodual isospace $\hat{S}_{IIA}{}^{d}(x^d, \hat{R}^d)$ over the isodual isoreals $\hat{R}^d$, with isotopic element $T^d = -T$ and isounit $\hat{1}^d = -\hat{1}$. The isodelta functions on isodual isospaces can then be defined accordingly, by reaching three additional quantities $\hat{\delta}^d{}_1$, $\hat{\delta}^d{}_2$ and $\hat{\delta}^d{}_3$ called *isodual isodelta functions.* The following property then holds.

Proposition 6.4.1 [10]: *The isodual isodelta functions of the first and second kind change their overall sign under isoduality.*

In fact, by recalling that $x^d = -x$, $y^d = -y$, $i^d = -i$, we have the *isodual isodelta function of the first kind*

$$\hat{\delta}^d{}_1(x^d) = (1/2\pi)\int_{-\infty}^{+\infty} T^d\, e_{\xi^d}{}^{i^d x^d y^d}\, dy^d = -(1/2\pi)\int_{-\infty}^{+\infty} T\, e_{\xi}{}^{ixy}\, dy\,, \quad (6.4.23)$$

with a similar expression for the second kind. Note that $\hat{\delta}_3$ has no isoselfdual structure, evidently because it is not an isotopic structure.

The properties of the isodelta functions for all the remaining Classes III, IV and V are vastly unknown at this writing. Additional generalizations of the delta functions are expected in the one–sided Lie–admissible formulations of the next chapter.

Note that, while the Dirac delta function is unique, there exist infinitely possible isodelta functions for each of the above six kinds, evidently because of the infinitely possible isounits or isotopic elements. The reader may have noted the intriguing character of the general case of isodelta functions (6.4.5) and (6,4,12) for $T = T(x,...)$, which are hoped to receive an attention in the literature much needed for physical advances.

As well known, the locality of quantum mechanics is precisely expressible via the Dirac delta function. The nonlocality of the isotopies of quantum mechanics is then expressed by the isodelta functions. In turn, such nonlocality is necessary for a quantitative treatment of the extended character of hadrons with consequential nonlocal components in the strong interactions due to mutual overlapping of the wavepackets and charge distributions of the particles.

While the Dirac delta is a *bona fide* distribution, the isodelta functions are not necessarily so because the original singularity at x can be spread over an interval of which x is the center. Nevertheless, in specific cases, such as when $T =$ cost., the isotopic $\hat{\delta}$–functions are distributions similar to $\delta(x)$.

6.5: ISOSERIES

We indicate in Sect. 6.1 that all conventional series of functional analysis admit significant isotopic generalizations. In this section we shall illustrate this occurrence via the isotopies of the *Fourier series* (see, e.g., ref. [20,21]). The

interested reader can then compute the isotopies of other series.

As well known a sufficiently smooth function $f(\theta)$ on a conventional one-dimensional space $S(\theta,R)$ over the reals R, which is periodic in $[0, 2\pi]$, admits the representation in term of the Fourier series [loc. cit.]

$$f(\theta) = \sum_{n=-\infty,+\infty} a_k \, e^{in\theta}, \tag{6.5.1}$$

where

$$a_n = (1/2\pi)\int_0^{2\pi} e^{-in\theta} f(\theta)\, d\theta . \tag{6.5.2}$$

If the function is periodic in the interval $[0,L]$, we have instead

$$f(x) = \sum_{n=-\infty,+\infty} b_n \, e^{i2\pi nx/L}, \tag{6.5.3}$$

in which case

$$b_n = (1/L)\int_0^{L} e^{-i2\pi nx/L} f(x)\, dx . \tag{6.5.4}$$

When the underlying functional space is lifted into a functional isospace of Class IA, the above Fourier series are no longer applicable, again, because of the loss of basic definitions, such as that of exponential. For this reason this author [10] studied the isotopies of the Fourier series, resulting in a generalization called *isofourier series* [10] which can be defined for a function $f(\theta)$ also periodic in $[0, 2\pi]$ on an isospace of Class IA, $\hat{S}(\theta,\hat{R})$ over the isoreals $R_{IA}(n,+,\hat{*})$, via the expression

$$f(\theta) = \sum_{n=-\infty,+\infty} A_n * e_\xi^{\;in\theta} = \sum_{n=-\infty,+\infty} A_n \, e^{inT\theta}, \tag{6.5.5}$$

where we have again used the properties of isoexponentiation of Sect. 4.3. Then, for T independent of θ, by using the isoorthogonality of the isoexponentials, Eq.s (6.2.34), we have

$$A_n = (1/2\pi)\int_0^{2\pi} \{ T\, e_\xi^{\;-in\theta} \} * f(\theta)\, d\theta . \tag{6.5.6}$$

If the function is periodic in the interval $[0, L]$, we have instead

$$f(x) = \sum_{n=-\infty,+\infty} B_n * e_\xi^{\;i2\pi nx/L} = \sum_{n=-\infty,+\infty} B_n \, e^{i2\pi nxT/L}, \tag{6.5.7}$$

in which case

$$B_n = 1/L\int_0^{L} \{T\, e_\xi^{\;-i2\pi nx/L}\} * f(x)\, dx . \tag{6.5.8}$$

When one deals with functional isospaces of Class IB, the preceding results are essentially multiplied by the isounit $\hat{1}$. The extension of the results to the isodual isospaces of Classes IIA and IIIB is equally simple, and will be implied hereon.

An important application of the above isoseries occurs in the transition from Cartesian to polar coordinates in isospaces. This transition is linked to a

central property of isorelativities, their capability to represent particles with their actual, generally nonspherical shape, jointly with all their infinitely possible deformations. In turn, this formulation originates from the isotopies of the rotational symmetry [19,20] (see also review [21]).

Consider a two-dimensional Euclidean space

$$E(r,\delta,\Re): r = (x, y), \ \delta = \mathrm{diag.}(1, 1), \ r^2 = xx + yy \in R. \qquad (6.5.9)$$

The transition to polar coordinates is provided by the familiar expressions

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad (6.5.10)$$

A sufficiently smooth function g(x,y) on E(r,δ,R) can then be represented in the unit circle r = 1 via the expansion

$$g(x,y) = \mathrm{Lim}_{N\to\infty} \Sigma_{n,m=0,\dots,N}\, a_{nm}\, x^n\, y^m =$$

$$= g(\cos\theta, \sin\theta) = f(\theta) = \Sigma_{n,m=0,\infty}\, a_{nm} \cos^n\theta \sin^m\theta \qquad (6.5.11)$$

The use of the expressions

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \cos\theta = (e^{i\theta} + e^{-i\theta}) / 2, \quad \sin\theta = (e^{i\theta} - e^{-i\theta}) / 2i \qquad (6.5.12)$$

then implies the well known Fourier series [20,21]

$$f(\theta) = \Sigma_{n=-\infty,+\infty}\, a_n e^{in\theta} = c_0 / 2 + \Sigma_{n=1,\infty} [\, c_n \cos n\theta + d_n \sin n\theta\,],$$

$$c_n = (1/\pi) \int_{-\pi}^{+\pi} d\theta\, f(\theta) \cos n\theta, \quad n = 0, 1, 2, \dots$$

$$d_n = (1/\pi) \int_{-\pi}^{+\pi} d\theta\, f(\theta) \sin n\theta, \quad n = 1, 2, \dots \qquad (6.5.13)$$

On physical grounds, the central geometric object is the perfect and rigid circle, as requested by the fact that *the rotational symmetry is a symmetry for rigid bodies*.

Conceptual, mathematical and physical advances are permitted by the transition to the covering isoeuclidean space of Class IA (Sect. 3.3)

$$\hat{E}(r,\hat{\delta},\hat{R}): r = (x, y), \quad \hat{\delta} = T\delta = \mathrm{diag.}(b_1^2, b_2^2), \quad \hat{1} = T^{-1}, \qquad (6.5.14)$$

whose separation

$$r^{\hat{2}} = x\, b_1^2\, x + y\, b_2^2\, y = \mathrm{inv}. \qquad (6.5.15)$$

represents all possible signature preserving deformations of the circle, i.e, an infinite family of possible ellipses with semiaxes $a = b_1^{-2}$ and $b = b_2^{-2}$. In this case, the invariance is provided by the isotopic covering $\hat{O}(2)$ of the rotational symmetry O(2) [6,7] we shall study in detail in Vol. II.

What is important for this section is that the conventional transformations to polar coordinates, Eq.s (6.5.10), is no longer applicable in isoeuclidean space (6.5.14) and must be generalized into the *isopolar coordinates* of App. 6.A [10]

$$x = r\, b_1^{-1} \operatorname{isocos} \Delta^{\frac{1}{2}} \theta\,, \quad y = r\, b_2^{-1} \operatorname{isosin} \Delta^{\frac{1}{2}} \theta\,. \tag{6.5.16}$$

where isocos $\hat{\alpha}$ and isosin $\hat{\alpha}$ are the *isotrigonometric functions* and $\Delta = \det T = b_1^2 b_2^2$.

We can therefore study the isotopy of expansion (6.5.11) for the unit case

$$r^2 = x\, b_1^2\, x + y\, b_2^2\, y = 1\,, \tag{6.5.17}$$

in the form

$$g(x,y) = \operatorname{Lim}_{N\to\infty} \sum\nolimits_{n,m=0,\dots N} A_{nm} * x^{\hat{n}} * y^{\hat{m}} =$$

$$= g(b_1^{-1}\operatorname{isocos}(\Delta^{\frac{1}{2}} \theta), b_2^{-1}\operatorname{isosin}(\Delta^{\frac{1}{2}} \theta)) = f(\theta) =$$

$$= \sum\nolimits_{n,m=0,\infty} A_{nm} * (b_1^{-1}\operatorname{isocos}(\Delta^{\frac{1}{2}} \theta))^n (b_2^{-1} \operatorname{isosin}(\Delta^{\frac{1}{2}} \theta))^m\,. \tag{6.5.18}$$

A physical significance of isoseries is in the expansion of an intensity in isospace, i.e., in expressions of the type

$$(1/L)\int_0^L |f(x)|^{\hat{2}}\, dx = (1/L)\int_0^L \overline{f(x)} * f(x)\, dx =$$

$$= (1/L)\int_0^L dx \{\sum\nolimits_n \bar{B}_n\, e^{-i2\pi nxT/L}\} * \{\sum\nolimits_m B_m\, e^{i2\pi mxT/L}\} =$$

$$= \sum\nolimits_{n=-\infty,+\infty} \bar{B}_n * B_n = \sum\nolimits_{n=-\infty,+\infty} |B_n|^{\hat{2}}\,. \tag{6.5.19}$$

In the simple case here considered, the original intensity is reduced to the sum of the individual isocontributions $\bar{B}_n * B_n$ without interference terms $\bar{B}_n * B_m$.

However, a rather complex interference pattern occurs for the case of T explicitly dependence on the integration variable, or merely when T is a nondiagonal matrix [10]. This is a representation of the nonlocal character of the isotopic wave-theory, namely, the representation of wavepackets of particles in conditions of mutual penetration.

Note that additional generalizations of isoseries are possible for liftings of the addition, but their would imply the loss of the distributive law (Sect. 2.3).

Note also that the same function $f(\theta)$ can be expanded in Fourier series (6.5.1), when dealing with ordinary functional spaces, as well as in the isoseries (6.5.18), when dealing with isospaces. The selection of which series holds is therefore relinquished again to the basic multiplicative unit.

6.6: ISOTRANSFORMS

We also indicated in Sect. 6.1 that conventional transforms of functional analysis admit nontrivial isotopic generalizations. In this section we shall illustrate this occurrence via the isotopies of the Fourier and Laplace transforms (see, e.g., ref.s [20,21] for their conventional forms). The reader can then work out any needed additional isotransform with the same techniques.

Six different isotopies of the Fourier transforms were identified by this author in ref. [10] in correspondence with the six different types of isodelta function of Sect. 6.4. They apply for correospondently different mathematical and physical conditions, and can be presented as follows.

Consider a one-dimensional functional isospace $\hat{S}_{IA,T}(x,\hat{R})$ over the isoreals $\hat{R}$, with the isotopic element T and isounits $\hat{1} = T^{-1}$, and the Fourier isoseries in the interval [–L, L] for strongly isocontinuous functions f(x) on $\hat{S}_{IA,T}(x,\hat{R})$ with 2L periodicity

$$f(x) = (2L)^{-\frac{1}{2}} \sum\nolimits_{n=-\infty,+\infty} g_n * e_{\xi}^{\,in\pi X/L} . \qquad (6.6.1a)$$

$$g_n = (1L)^{-\frac{1}{2}} \int_{-L}^{+L} f(x) * e_{\xi}^{\,-in\pi x/L}\, dx . \qquad (6.6.1b)$$

As in the conventional case [20,21], set $(\pi/L)^{\frac{1}{2}}\, x = y$ and $(n\pi L)^{\frac{1}{2}} = k_n$, so that $(n\pi/L)\, x = k_n y$, $\Delta k_n = k_{n+1} - k_n = (\pi/L)^{\frac{1}{2}}$, and $(2L)^{-\frac{1}{2}} = \Delta k_n (2\pi)^{-\frac{1}{2}}$. Then Eq.s (6.6.1) become

$$f(y) = (2\pi)^{-\frac{1}{2}} \sum\nolimits_{k_n=-\infty,+\infty} g_{k_n} * e_{\xi}^{\,ik_n y}\, \Delta k_n . \qquad (6.6.2a)$$

$$g_{k_n} = (2\pi)^{-\frac{1}{2}} \int_{-(\pi L)^{\frac{1}{2}}}^{+(\pi L)^{\frac{1}{2}}} f(y) * e_{\xi}^{\,-ik_n y}\, dy . \qquad (6.6.2b)$$

At the limit $L \rightarrow \infty$, we have the *Fourier isotransforms of the first kind* [10]

$$f_1(x) = (1/2\pi) \int_{-\infty}^{+\infty} g_1(k) * e_{\xi}^{\,ikx}\, dk =$$

$$= (1/2\pi) \int_{-\infty}^{+\infty} g_1(k)\, e^{ikTx}\, dk , \qquad (6.6.3a)$$

$$g_1(k) = (1/2\pi) \int_{-\infty}^{+\infty} f_1(x) * e_\xi^{-ikx} dx =$$
$$= (1/2\pi) \int_{-\infty}^{+\infty} f_1(x)\, e^{-ikTx} dx . \qquad (6.6.3b)$$

The reason why the above isotransforms are called of the first kind is that they are linked to the isodelta function of the same kind, Eq.s (6,4,9), as illustrated by the following

Theorem 6.6.1 [10]: *The isotopic Fourier integral theorem reads*

$$f_1(x) = (1/2\pi) \int_{-\infty}^{+\infty} dk\, e_\xi^{ikx} * (\int_{-\infty}^{+\infty} f_1(x') * e_\xi^{ikx'} dx') =$$
$$= \int_{-\infty}^{+\infty} dx'\; f_1(x') * \{ (1/2\pi) \int_{-\infty}^{+\infty} e_\xi^{i(x-x')k} dk =$$
$$= \int_{-\infty}^{+\infty} dx'\; f_1(x') * \delta_1(x - x') \qquad (6.6.4)$$

In particular, it is easy to prove the following isotopy of the corresponding conventional property

$$\int_{-\infty}^{+\infty} dx\, |\, f_1(x)\, |^{\hat{2}} = \int_{-\infty}^{+\infty} dx\, \overline{f_1(x)}\, T\, f_1(x) \equiv$$
$$\equiv \int_{-\infty}^{+\infty} dk\, |\, g_1(k)\, |^{\hat{2}} = \int_{-\infty}^{+\infty} dk\, \overline{g_1}(k)\, T\, g_1(k) . \qquad (6.6.5)$$

The entire theory of Fourier transforms can therefore be subjected to step-by-step isotopic liftings. Studies along these general lines have been initiated by Kadeisvili [15] and their continuation is left to the interested reader.

The *Fourier isotransforms of the second kind* are defined on isospaces $\hat{S}_{IB,T}(x,\hat{R})$, that is, for isofunctions $\hat{f}(x) = f(x)\hat{1}$, are given by

$$\hat{f}_2(x) = (1/2\pi) \int_{-\infty}^{+\infty} \hat{g}_2(k) * e_\xi^{ikx} dk , \qquad (6.6.6a)$$
$$\hat{g}_2(k) = (1/2\pi) \int_{-\infty}^{+\infty} \hat{f}_2(x) * e_\xi^{-ikx} dx \qquad (6.6.6b)$$

and can be written for isounits independent on the integration variable

$$\hat{f}_2(x) = (\hat{1}/2\pi) \int_{-\infty}^{+\infty} g_2(k)\, e_\xi^{ikTx} dk , \qquad (6.6.7a)$$
$$\hat{g}_2(k) = (\hat{1}/2\pi) \int_{-\infty}^{+\infty} f_2(x)\, e_\xi^{-ikTx} dx . \qquad (6.6.7b)$$

Note that, again, the isotransforms of first and second kind are interconnected by the reciprocity transforms $T \to \hat{1}$.

The *Fourier isotransforms of the third kind* are defined on an ordinary space $S(\tilde{x},C)$ with local coordinates $\tilde{x} = T^{\frac{1}{2}} x$, and can be written [10]

$$f_3(x) = (1/2\pi)\int_{-\infty}^{+\infty} g_3(k)\, e_\xi^{ikT^{\frac{1}{2}}x}\, d(T^{\frac{1}{2}}k)\ , \tag{6.6.8a}$$

$$g_3(k) = (1/2\pi)\int_{-\infty}^{+\infty} f_3(x)\, e_\xi^{-ikT^{\frac{1}{2}}x}\, d(T^{\frac{1}{2}}x)\,. \tag{6.6.8b}$$

The remaining three cases are of isodual character. The *isodual isotransform of the first kind* on isospaces $\hat{S}_{IIA,T}(x,\hat{R})$ are given by [10]

$$f^d{}_1(x^d) = (1/2\pi)\int_{-\infty}^{+\infty} g^d{}_1(k^d)\, e^{\,i^d k^d T^d x^d}\, dk^d =$$

$$= (1/2\pi)\int_{-\infty}^{+\infty} g_1(k)\, e^{\,ikTx}\, dk \tag{6.6.9a}$$

$$g^d{}_1(k^d) = (1/2\pi)\int_{-\infty}^{+\infty} f^d{}_1(x^d)\, e^{\,-i^d k^d T^d x^d}\, dx^d =$$

$$= (1/2\pi)\int_{-\infty}^{+\infty} f_1(x)\, e^{\,-ikTx}\, dx \tag{6.6.9b}$$

from which we have the following simply but significant

Proposition 6.6.1 [10]: *The Fourier isotransforms of the first and second kind are isoselfdual.*

The isotransforms of the third kind are not isoselfdual, as it is the case for the corresponding isodelta function, because they are not genuine isotopies. As a matter of fact, isoduality turns the structure of $\hat{\delta}_3$ into a *Laplace isotransform* of the next section.

The reader has noted the simplicity of the isotransforms for isounits independent from the local coordinates, $\hat{1} = \hat{1}(\dot{x}, \ddot{x}, ...)$ which will be used in the great majority of physical applications of Vol.s II and III. However, their general expression, e.g., for isounits of gravitational type $\hat{1} = \hat{1}(x, ...)$, is nontrivial and substantially unexplored at this writing.

The extension of the above analysis to more than one dimension is trivial and shall be tacitly implied. The formulation and properties of the Fourier isotransforms for Classes III, IV and V are also unknown at this writing.

Note that despite their abstract equivalence, isotransforms and conventional transforms are inequivalent, as directly shown by the appearance of the isotopic element T in the exponent of the isotransforms or by the fact that ordinary transforms are linear and local, while the isotransforms are isolinear and isolocal.

Note that the Fourier transform is unique for a given function f(x). On the contrary, the same function can be subjected to an infinite variety of Fourier isotransforms, evidently depending on the infinitely possible isounits. This degree of freedom is necessary for physical consistency. In fact, empty space (the vacuum) is unique, and represented by the trivial unit I = diag. (1, 1, 1). A unique

transform is then fully consistent. On the contrary, there exist infinitely possible physical media due to infinitely possible densities, pressures, temperatures, etc., which are represented by the infinitely possible isounits. A corresponding infinite class of isotransforms is then necessary.

The Fourier isotransforms can evidently be applied to a large variety of nonlinear, nonlocal and nonhamiltonian problems. Their relevance was elegantly established by Kadeisvili [15] by proving that *the isotopies of the Fourier transform imply a necessary generalization of Heisenberg's uncertainty relations for particle in vacuum* (exterior dynamical problem of Sect. 1.1)

$$\Delta x \, \Delta k \;\approx\; 1 , \qquad (6.6.10)$$

into the *isouncertainties* for particles moving within physical media (interior problem)

$$\Delta x \, \Delta k \;\approx\; < \hat{1} > , \qquad (6.6.11)$$

first proposed by this author in ref. [16] and then re–examined in ref.s [10,11] (see Vol. II for detailed studies).

The proof of conventional uncertainties via Fourier transforms within the context of functional analysis is well known (see, e.g., ref. [18], p. 47–49), although it is worth reviewing for comparative purposes. Consider an ordinary Hilbert space $\mathcal{H}$ with states $\psi(x)$ depending on a variable x belonging to an ordinary one–dimensional space $S(x,C)$, which are normalized to one.

Gauss distribution within the above context can be written

$$\psi(x) \;=\; n \, e^{-x^2 / 2 a^2} , \qquad n = a^{-1/2} \, \pi^{-1/4} \in \Re \qquad (6.6.13)$$

The conventional Fourier transform of the above expression is given by

$$\phi(k) \;=\; (1/2\pi)^{\frac{1}{2}} \int_{-\infty}^{+\infty} \psi(x) \, e^{-ikx} \, dx \;=\; n' \, e^{-k^2 a^2 / 2} ,$$

$$n' \;=\; an \;=\; a^{1/2} \, \pi^{-1/4} \in \Re . \qquad (.6.6.13)$$

Now, the width of distribution (6.6.13) is of the order of $\Delta x \approx a$, while the width of transform (6.6.14) is of the order of $\Delta k \approx 1/a$. The conventional Heisenberg's uncertainties then follow,

$$\Delta x \approx a , \quad \Delta k \approx 1/a , \quad \Delta x \, \Delta k \;\approx\; 1 . \qquad (6.6.14)$$

We now reinspect the above formulation under isotopies within the context of functional isoanalysis. Consider an isohilbert space $\hat{\mathcal{H}}_{IA,T}$ with states $\Psi(x)$, where x is the local coordinates on an isospace $\hat{S}_{IA,T}(x,\hat{C})$ on the isofield $\hat{C}(c,+,*)$

with isounit $\hat{1}$ and isotopic element T which are then independent of x, and suppose that Ψ is isonormalized (Definition 6.2.2)

$$(\Psi \hat{,} \Psi) \quad := \hat{1} \int_{-\infty}^{+\infty} dx\, \overline{\Psi(x)} * \Psi(x) = \hat{1}\,. \qquad (6.6.15)$$

The conventional Gauss distribution cannot any longer be consistently defined under isotopies, e.g., because of the lack of meaning of the conventional exponentiation (Sect. 4.3). Its image in $\hat{\mathcal{H}}_{IA.T}$ is instead given by the *Gauss isodistribution* [10]

$$\Psi(x) = N * e_{\xi}^{-x^2 / 2 a^2} = N\, e^{-x^2 T / 2 a^2}\,, \qquad (6.6.16)$$

where

$$N = T^{-1/2} a^{-1/2} \pi^{-1/4} \qquad (6.6.17)$$

The conventional Fourier transform has no mathematical or physical meaning in isohilbert spaces, and must be replaced by the Fourier isotransform of the first kind which yields after simple algebra

$$\Phi(k) = (1/2\pi)^{\frac{1}{2}} \int_{-\infty}^{+\infty} \Psi(x) * e_{\xi}^{-ikx}\, dx =$$

$$= N' e^{-k^2 T a^2 / 2}, \qquad N' = aN = T^{-1/2} a^{1/2} \pi^{-1/4}\,. \qquad (6.6.18)$$

Now, the width of isodistribution (6.6.16) is given by $\Delta x \approx a/T^{\frac{1}{2}}$, while the width of its isotransforms (6.6.18) is $\Delta k \approx 1/(aT^{\frac{1}{2}})$, and this establishes isouncertainties (6.6.11),

$$\Delta x \approx a/T^{\frac{1}{2}}, \quad \Delta k \approx 1/(aT^{\frac{1}{2}}), \quad \Delta x\, \Delta k \approx \hat{1}\,. \qquad (6.6.19)$$

The implications of the above findings are manifestly far reaching. In fact, they confirm the existence and consistency of a step-by-step isotopic generalization of quantum mechanics into hadronic mechanics, which has been conceived and worked out for physical conditions of particles (those of the interior dynamical problem) fundamentally different than those of conception, applicability and experimental verification of quantum mechanics (those of the exterior dynamical problem).

Heisenberg's uncertainties are mathematically and physically valid in the arena of their conception and experimental verification, e.g., for an electron moving in an atomic orbit in vacuum. The isouncertainties have instead been conceived for the same electron when moving within hyperdense physical media, such as the core of a collapsing star. In this latter case, the isotopy $I \Rightarrow \hat{1}$ is expected to provide a quantitative treatment suitable for experimental verifications of *integral* corrections to Heisenberg's uncertainties due to: the total immersion of the wavepacket of the electron within those of the

surrounding particles; the inhomogeneity and anisotropy of the medium; and other *physical differences* with respect to motion in vacuum (see Vol. II for details).

Note that isouncertainties (6.6.11) depend on the preservation of the isogaussian character under Fourier isotransforms. In turn, this is dependent on the basic isoexponentiation of the Lie–isotopic theory. In fact, starting from the isoexponential

$$e_{\xi}^{\,-a^2/2a^2} = \hat{1}\, e^{-r^2 T / 2 a^2}, \qquad (6.6.20)$$

we also end up with the isoexponential

$$e_{\xi}^{\,-k^2 a^2/2} = \hat{1}\, e^{-k^2 T a^2 / 2}, \qquad (6.6.21)$$

In turn, the preservation of this isoexponential character is precisely the mechanism that alters Heisenberg's uncertainties via the isotopy $I \Rightarrow \hat{1}$.

The isotopic techniques used in this section for the Fourier transforms are easily extendable to other transforms. We mention as an example, the *Laplace isotransform* also studied, apparently for the first time, in ref. [10]

$$\hat{F}(z) = \int_0^{\infty} dx\, \hat{f}(x) * e_{\xi}^{-zx}, \quad z = \text{cost.} + iy, \qquad (6.6.22)$$

The same techniques are then applicable to the *isotopies of Hankel, Mellin, Hilbert and other transforms* [see ref.s [20,21] for their conventional forms).

6.7: ISOFUNCTIONS AND THEIR OPERATIONS

We shall now study the isotopies of a few representative elementary functions and the primary operations on them.

Definition 6.7.1: *Let* f(x) *be an ordinary function verifying the needed regularity and continuity conditions on a given closed interval of the real variable* $x \in R(n,+,\times)$. *Then the "isotopic image"* $\hat{f}(\hat{x})$ *of* f(x), *is a function of the corresponding closed isointerval of the isoreal number* $\hat{x} = x\hat{1} \in \hat{R}(\hat{n},+,*)$ *generally given by the rule*

$$\hat{f}(\hat{x}) = \hat{1}\, f(x). \qquad (6.7.1)$$

We have already encountered several *elementary isofunctions* during our analysis, such as the *isopower*

$$\hat{f}(\hat{x}) = \hat{x}^n = \hat{x} * \hat{x} * \ldots\ldots * \hat{x} \text{ (n-times)} = \hat{1}(x^n). \tag{6.7.2}$$

A most fundamental isofunction is the *isoexponentiation* of Sect. I.4.3. When written in terms of an isonumber $\hat{x}$, it also follows rule (6.7.1),

$$\hat{e}^{\hat{x}} := e_{\xi}{}^{\hat{x}} = \hat{1}\{e^{T\hat{x}}\} = \hat{1}\{e^{T\hat{1}x}\} = \hat{1}\, e^x, \tag{6.7.3}$$

where e^x is the ordinary exponentiation.

The *isologarithm* of an isonumber $\hat{a} \in \hat{F}(\hat{a},+,*)$ on isobasis $\hat{e} = e\hat{1}$ can be then defined as the quantity $\hat{\log}_{\hat{e}}\, \hat{a}$ such that

$$\hat{e}^{\hat{\log}_{\hat{e}} \hat{a}} = \hat{a}, \tag{6.7.4}$$

with evident(and unique) solution

$$\hat{\log}_{\hat{e}}\, \hat{a} = \hat{1} \log_e a . \tag{6.7.5}$$

where $\log_e a$ is the ordinary logarithm on basis e of the ordinary number a.

It is easy to see that the above definition of the isologarithm characterizes a correct isotopy because it preserves all the conventional properties of log a, such as (we ignore in the following the subscripts $\hat{e}$ and e for simplicity)

$$\hat{\log}\, \hat{e} = \hat{1}, \quad \hat{\log}\, \hat{1} = 0, \tag{6.7.5a}$$

$$\hat{\log}\, \hat{a} * \hat{b} = \hat{\log}\, \hat{a} + \hat{\log}\, \hat{b}, \quad \hat{\log}\, \hat{a}\, \hat{/}\, \hat{b} = \hat{\log}\, \hat{a} - \hat{\log}\, \hat{b}, \tag{6.7.6b}$$

$$\hat{\log}\, \hat{a}^{-\hat{1}} = - \hat{\log}\, \hat{a}, \quad \hat{b} * \hat{\log}\, \hat{a} = \hat{\log}\, \hat{a}^{\hat{b}}, \text{ etc.} \tag{6.7.6c}$$

A similar situation occurs for the isotopy of most, but not all functions. In fact, two exceptions are given by the isotopy of the trigonometric and hyperbolic functions, which were preliminarily identified in Ch. I.5, and are studied in more detail in App. I.5.C.

The isotopies of derivatives and integrals are intriguing because of the *variety* of the emerging novel notions. In fact, by again assuming T independent of x for simplicity, we can introduce the *three* different isodifferentials $\hat{d}_1 x = dx$, $\hat{d}_2 x = d(Tx) = T dx$ and $\hat{d}_3 = d(Ix) = \hat{1} dx$. We then have the *isoderivatives of the first, second and third kind*

$$\frac{\hat{d}_1}{\hat{d}_2 x} \hat{f}(\hat{x}) = \frac{\hat{d}_2}{\hat{d}_2 x} f(x) = \hat{1} \frac{d}{dx} f(x), \quad \frac{\hat{d}_3}{\hat{d}_3 x} \hat{f}(\hat{x}) = \frac{d}{dx} f(x) \tag{6.7.7}$$

as the reader can derive via more rigorously via *isolimits* here omitted for brevity, but studied in detail in ref.s [31,32]. Similarly, we have the three indefinite

isointegrals of the first, second and third kind $\hat{\int}_1 = \int$, $\hat{\int}_2 = \hat{1}\int$ and $\hat{\int}_3 = T\int$ which verify the axioms $\hat{\int}_k \hat{d}_k x = x$, $k = 1, 2, 3$. *Definite isointegrals* can be defined accordingly, e.g., for $\sigma = [a, b]$ being a closed isointerval of x,

$$\hat{\int}_{\hat{k}_\sigma} \hat{f}(\hat{x}) * \hat{d}_k x \quad = \int_{\hat{\sigma}} f(x)\, dx\,, \quad k = 1, 2, 3. \tag{6.7.8}$$

A virtually endless number of isotopic liftings of conventional treatments (see, e.g., ref. [22]) can be introduced, but its study is here left to the interested mathematician.

APPENDIX 6.A: ISOMANIFOLDS AND THEIR ISOTOPOLOGY

The notion of an N-dimensional *isomanifold* was first studied by Tsagas and Sourlas [30]. In this presentation we use the following simplest possible realization. Since an N×N-dimensional isounit is positive-definite, it can always be diagonalized into the form

$$\hat{1} \;=\; \text{diag.}\,(\, b_1{}^{-2}, b_2{}^{-2}, ..., b_N{}^{-2}\,) > 0\,, \quad b_k > 0\,, k = 1, 2, ..., N\,, \tag{6.A.1}$$

Consider then N isoreal isofields $\hat{R}_k(\hat{n},+,\hat{\times})$ each characterized by the isounit $\hat{1}_k = b_k{}^{-2}$ with (ordered) Cartesian product

$$\hat{R}^N \;=\; \hat{R}_1 \times \hat{R}_2 \times ... \times \hat{R}_N\,. \tag{6.A.2}$$

Since $\hat{R}_k \approx R$, it is evident that $\hat{R}^N \approx R^N$, where R^N is the Cartesian product of N conventional fields $R(n,+,\times)$. But the total unit of $\hat{R}^N$ is expression (6.A.1). Therefore, one can introduce a topology on $\hat{R}^N$ via the simple isotopy of the conventional topology on R^N,

$$\hat{\tau} \;=\; \{\, \emptyset\,,\; \hat{R}^N, \hat{B}_i\,\}\,, \tag{6.A.3}$$

where $\hat{B}_i$ represents the subset of $\hat{R}^n$ defined by

$$\hat{B}_i \;=\; \{\, \hat{P} = (\,\hat{a}_1, \hat{a}_2, ..., \hat{a}_n\,) \,/\, \hat{n}_i < \hat{a}_1, \hat{a}_2, ..., \hat{a}_n < \hat{m}_i,\, \hat{n}_i\,, \hat{m}_i\,, a_k \in \hat{R}\,\}\,. \tag{6.A.4}$$

As one can see, the above topology coincides everywhere with the conventional topology τ of R^n *except at the isounit* $\hat{1}$. In particular, $\hat{\tau}$ is everywhere local-differential, except at $\hat{1}$ which can incorporate integral terms. Such a topology shall be referred to as *Tsagas-Sourlas isotopology* or an *integro-differential topology* [30].

Definition 6.A.1 [loc. cit.]: *A "topological isospace"* $\hat{\tau}(\hat{R}^N)$ *is the isospace* $\hat{R}^N$

equipped with the isotopology $\hat{\tau}$. A "Cartesian isomanifold" $\hat{M}(\hat{R}^N)$ is the topological isospace $\hat{\tau}(R^N)$ equipped with a vector structure, an affine structure and the mapping

$$\hat{1}: \hat{R}^n \rightarrow \hat{R}^n, \quad \hat{1}: \hat{a} \rightarrow \hat{1}(\hat{a}) = \hat{a}, \quad \forall\ \hat{a} \in \hat{R}. \tag{6.A.5}$$

An "isoeuclidean isomanifold" $\hat{M}(\hat{E}(\hat{x},\hat{\delta},\hat{R}))$ occurs when the N-dimensional isospace $\hat{E}$ is realized as the Cartesian product

$$\hat{E}(x,\hat{\delta},\hat{R}) \approx \hat{R}_1 \times \hat{R}_2 \times \ \dots\ \times \hat{R}_N\ , \tag{6.A.6}$$

and equipped with the isotopology $\hat{\tau}$ with isounit (6.A.1).

The extension of the above definition to nondiagonal isounits $\hat{1}$ can be achieved, e.g., by assuming that the individual isounits $\hat{1}_k$ are positive-definite N×N-dimensional nondiagonal matrices such to yield the assumed total unit $\hat{1}$ via the ordered Cartesian product

$$\hat{1} = \hat{1}_1 \times \hat{1}_2 \times \ \dots\ \times \hat{1}_N\,. \tag{6.A.7}$$

For all additional aspects of isomanifolds and related topological properties we refer the interested reader to Tsagas and Sourlas [30]. It should be noted that their study is referred to $M(\hat{R}^N)$, rather than to $\hat{M}(\hat{R}^N)$ because of the use of the *conventional* topology τ (i.e. a topology with the conventional N×N-dimensional unit I). The extension to $\hat{M}(\hat{E})$ with the isotopology $\hat{\tau}$ has been introduced apparently for the first time in papers [32,33].

APPENDIX 6.B: OTHER GENERALIZATIONS OF FUNCTIONAL ANALYSIS

A considerable number of generalizations of functional analysis of non-isotopic type exist in the literature, some of which dating back to the past century. They are all *independent* from the isotopic generalization because derived from different assumptions. As such, they all have their own value. Regrettably, we cannot review them here for brevity, and must limit ourselves to indicate those most significant for our studies.

The generalization of functional analysis based on the so-called *q-deformations* (see, e.g., ref.s [23]) is particularly relevant for hadronic mechanics, and includes q-number-generalizations of ordinary and special functions, the operations defined on them, etc.

The differences between the isotopic and q-functional analysis are

numerous, such as the central dependence of the former is on th lifting of th unit and the preservation of the conventional unit for the latter, the validity of the former for arbitrary integro-differential operators T, and that of the latter for q-numbers, etc. (see App. I.7.A). Nevertheless, a knowledge of the q-functional analysis is unquestionably useful for the construction of the isospecial functions, as we shall see in Vol. II.

Among the great variety of q-number theories, a significant realization was conceived by Dirac (see the review [24], p. 320 ff.) whose study is also recommended here. Unfortunately, the differences between Dirac's q-numbers and the others q-numbers are so great to be misleading.[49] For this reason we shall refer to them via the alternative name of *queer numbers* suggested by Dirac himself.

Yet another generalization, this time, of the conventional differential calculus is the so-called *Helmholtz's calculus* (see, ref. [25]). This generalization too is significant for these volumes because it leads to an inevitable generalization of conventional relativities although different and independent from the isotopic one.

Additional special forms of differential calculus exist in the literature, depending on the needs at hand. We indicate, for instance, the *small derivative calculus* developed by Gonzalez-Diaz and Jannussis [26], which is specifically conceived for small distances and exhibits rather intriguing properties.

By no means the above indications exhaust all existing generalizations of conventional functional analysis. Additional novel possibilities can be found in the monograph by Lôhmus, Paal and Sorgsepp [28]. A further approach is presented in the monograph by Vougiouklis [29] via the *algebraic hyperstructures* (also called *multivalued algebras*) and the so-called H_v-*structures*, in which associativity, distributivity and commutativity are replaced by their weak forms. The latter approach also implies the chain of generalized *hyperfields, hyperspaces, hyperalgebras, hypergroups*, etc. with intriguing possibilities for isotopic reformulation and application to interior dynamical problems.

Nevertheless, none of these generalizations require a lifting of the basic unit, thus illustrating the uniqueness as well as independence of the isoanalysis of this section.

Also, q-theories are mathematically impeccable, but afflicted by rather serious problems of physical consistencies studied in Vol. II, such as: 1) lack of invariance of the unit, with consequential impossibility to apply q-theories to realistic measurements; 2) lack of conservation in time of Hermiticity, with consequential lack of observability; 3) lack of form-invariance of the special functions, with consequential invalidity of numerical results, e.g., originating from partial wave q-analysis; and others.

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[49] Dirac's q-formulation of quantum mechanics is a truly "quantum" theory, while the other q-deformations of quantum mechanics do not admit a "quantum" because they do not admit the unit (App. I.7.A).

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ISOTOPIC GENERALIZATION OF THE LEGENDRE, JACOBI, AND BESSEL FUNCTIONS

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Abstract

In this paper we consider the Lie-isotopic generalizations of the Legendre, Jacobi, and Bessel functions.

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1 Introduction

In this paper, we consider the Lie-isotopic generalization of the Legendre, Jacobi, and Bessel functions.

We describe in detail the group of rotations of three-dimensional isoEuclidean space, and the group locally isomorphic to it, $\hat{SU}(2)$, consisting of isounitary isounimodular 2×2 matrices. Also, we study the group $\hat{QU}(2)$ of quasiunitary matrices and the group $\hat{M}(2)$ of isometric transformations of isoEuclidean plane.

These studies are of interest both in mathematical and physical points of view. We refer the interested reader to monographs [3] for comprehensive review on the Lie-isotopic formalism and its applications.

The isotopic generalizations of the groups $SO(3)$, $SU(2)$, and $M(2)$ are of continuing interest in the literature. From physical point of view, our interest is that the Lie-isotopic generalizations of the Legendre functions as well as the other special functions, such as Jacobi and Bessel functions, can be used in formulating the nonpotential scattering theory [1, 2, 6, 7] when one considers non-zero isoangular momenta.

The paper is organized as follows.

Sections 2-7 are devoted to representations of the group $\hat{SU}(2)$ and isoLegendre functions. Namely, in Sec.2, we consider the group $\hat{SU}(2)$. In Sec.3, we consider unitary irreducible representations (irreps) of the group $\hat{SU}(2)$. In Sec.4, we present matrix elements of the unitary irreps of $\hat{SU}(2)$, and isoLegendre functions $\hat{P}^l_{mn}(\hat{z})$. In Sec.5, we present basic properties of the isoLegendre functions. In Sec.6, we present functional relations satisfied by the isoLegendre functions. In Sec.7, we present recurrency relations satisfied by the isoLegendre functions.

Sections 8-14 are devoted to representations of the group $\hat{QU}(2)$ and isoJacobi functions.

Sections 15-20 are devoted to representations of the group $\hat{M}(2)$ and isoBessel functions.

2 The group $\hat{SU}(2)$

In this Section, we consider representations of the group $\hat{SU}(2)$, elements of which are isounitary isounimodular 2×2 matrices, and its relation to the group $\hat{SO}(3)$ of rotations of three dimensional isoEuclidean space.

2.1 Parametrizations

Denote $\hat{SU}(2)$ the set of isounitary isounimodular 2×2 matrices, namely, of the matrices

$$\hat{u} = \hat{I} * \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \bar{\hat{\gamma}} & \hat{\delta} \end{pmatrix}. \tag{2.1}$$

If $\hat{u}_1 \in \hat{SU}(2)$ and $\hat{u}_2 \in \hat{SU}(2)$ then

$$(\hat{u}_1 * \hat{u}_2)^* = \hat{u}_2^* * \hat{u}_1^* = \hat{u}_2^{-1} * \hat{u}_1^{-1} = (\hat{u}_1 * \hat{u}_2)^{-1} \tag{2.2}$$

and $\det(\hat{u}_1 * \hat{u}_2) = 1$. Therefore, $\hat{u}_1 * \hat{u}_2 \in \hat{SU}(2)$. Also, it is easy to show that $\hat{u}_1^{-1} \in \hat{SU}(2)$. We arrive at the conclusion that $\hat{SU}(2)$ is a *group*.

Let $\hat{u} \in \hat{SU}(2)$. Since

$$\hat{u}^* = \hat{I} * \begin{pmatrix} \bar{\hat{\alpha}} & \bar{\hat{\beta}} \\ \bar{\hat{\gamma}} & \bar{\hat{\delta}} \end{pmatrix} \tag{2.3}$$

and

$$\hat{u}^{-1} = \hat{I} * \begin{pmatrix} \hat{\delta} & -\hat{\beta} \\ -\bar{\hat{\gamma}} & \hat{\alpha} \end{pmatrix}, \tag{2.4}$$

then $\hat{\delta} = \bar{\hat{\alpha}}$ and $\hat{\gamma} = -\bar{\hat{\beta}}$.

Thus, any matrix $\hat{u} \in \hat{SU}(2)$ has the form

$$\hat{u} = \hat{I} * \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ -\bar{\hat{\beta}} & \bar{\hat{\alpha}} \end{pmatrix}, \tag{2.5}$$

where $\hat{I} = \mathrm{diag}(g_{11}^{-1}, g_{22}^{-1})$, $\det I = \Delta$. Since $\det \hat{u} = 1$ then

$$|\hat{\alpha}|\Delta|\hat{\alpha}| + |\hat{\beta}|\Delta|\hat{\beta}| = 1, \tag{2.6}$$

and vice versa, if $\hat{u}$ is a matrix of the form (2.5) and Eq.(2.6) holds, then $\hat{u} \in \hat{SU}(2)$.

From the above consideration, it follows that the elements of $\hat{SU}(2)$ can be uniquely determined by two complex numbers $(\hat{\alpha}, \hat{\beta})$ obeying eq. (2.6). These complex numbers can be presented by three real parameters, for example, by $|\hat{\alpha}|$, $\mathrm{Arg}\hat{\beta}$, and $\mathrm{Arg}\hat{\alpha}$. If $\hat{\alpha} * \hat{\beta} \neq 0$, one can use another parametrization, namely Euler angles, $\hat{\varphi}$, $\hat{\theta}$, and $\hat{\psi}$, which are related to $|\hat{\alpha}|$, $\arg\hat{\beta}$, and $\arg\hat{\alpha}$ according to the following relations:

$$|\hat{\alpha}| = g_{11}^{-1/2}\cos[\theta\Delta^{1/2}/2] \equiv \mathrm{isocos}[\hat{\theta}/2],$$

$$\mathrm{Arg}\hat{\alpha} = \frac{\hat{\varphi}+\hat{\psi}}{2}, \quad \mathrm{Arg}\hat{\beta} = \frac{\hat{\varphi}-\hat{\psi}+\pi}{2}, \tag{2.7}$$

where

$$\hat{\varphi} \equiv \varphi\Delta^{1/2}, \quad \hat{\theta} \equiv \theta\Delta^{1/2}, \quad \hat{\psi} \equiv \psi\Delta^{1/2}. \tag{2.8}$$

The values of the Euler angles are not determined by (2.7) uniquely, so that we must put additionally

$$0 \leq \hat{\varphi} < 2\pi, \quad 0 < \hat{\theta} < \pi, \quad -2\pi \leq \hat{\psi} < 2\pi. \tag{2.9}$$

From (2.7) it follows that $|\hat{\beta}| = g_{22}^{-1/2}\sin(\theta\Delta^{1/2}/2)$ and that the matrix $\hat{u} = \hat{u}(\hat{\varphi}, \hat{\theta}, \hat{\psi})$ has the following form:

$$\hat{u} = \begin{pmatrix} gg_{11}^{-1/2}cos\hat{\theta}/2\Delta e^{i(\hat{\varphi}+\hat{\psi})/2} & i\Delta^2 g_{11}^{-1/2}\sin\hat{\theta}/2e^{i(\hat{\varphi}-\hat{\psi})/2} \\ i\Delta^2 g_{22}^{-1/2}\sin\hat{\theta}/2e^{-i(\hat{\psi}-\hat{\varphi})/2} & gg_{11}^{-1/2}\cos\hat{\theta}/2\Delta e^{-i(\hat{\varphi}+\hat{\psi})/2} \end{pmatrix}. \tag{2.10}$$

From (2.5) and (2.10) we have

$$g_1 1^{-1/2}\cos[\psi\Delta^{1/2}] = 2|\alpha|^2\Delta - 1,$$

$$\exp[i\Delta^{1/2}\psi/2] = -i\frac{\alpha\beta}{|\alpha||\beta|}, \tag{2.11}$$

$$\exp i\Delta^{1/2}\psi/2 = \frac{\alpha\Delta^{1/2}\exp\{-i\Delta^{1/2}\varphi/2\}}{|\alpha|}.$$

Also, from (2.10) we have the following factorization

$$\hat{u}(\hat{\varphi},\hat{\theta},\hat{\psi}) = \begin{pmatrix} \exp\{i\Delta^{1/2}\varphi/2\} & 0 \\ 0 & \exp\{-i\Delta^{1/2}\varphi/2\} \end{pmatrix} \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$$

$$\times \begin{pmatrix} g_{11}^{-1/2}\cos\hat{\theta}/2 & i\Delta gg_{22}^{-1/2}sin\hat{\theta}/2 \\ g_{22}^{-1/2}\sin\hat{\theta}/2 & i\Delta g_{11}^{-1/2}\cos\hat{\theta}/2 \end{pmatrix} \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$$

$$\times \begin{pmatrix} \exp\{i\Delta^{1/2}\psi/2\} & 0 \\ 0 & \exp\{-i\Delta^{1/2}\psi/2\} \end{pmatrix}$$

$$\equiv \hat{u}(\hat{\varphi},0,0)\Delta\hat{u}(0,\hat{\theta},0)\Delta\hat{u}(0,0,\hat{\psi}). \tag{2.12}$$

Diagonal matrices

$$\begin{pmatrix} \exp[i\Delta^{1/2}\varphi/2] & 0 \\ 0 & \exp[-i\Delta^{1/2}\varphi/2] \end{pmatrix} \tag{2.13}$$

form a one-parameter subgroup of $\hat{SU}(2)$. Thus every matrix $\hat{u} \in \hat{SU}(2)$ lies in the left and right conjugacy class in respect to ths subgroup containing the matrix of the form

$$\begin{pmatrix} g_{11}^{-1/2}\cos[\theta\Delta^{1/2}/2] & i\Delta g_{22}^{-1/2}\sin[\theta\Delta^{1/2}/2] \\ i\Delta g_{22}^{-1/2}\sin[\theta\Delta^{1/2}/2] & g_{11}^{-1/2}\cos[\theta\Delta^{1/2}/2] \end{pmatrix}. \tag{2.14}$$

Note that the matrices represented by (2.14) form a one-parameter subgroup of $\hat{SU}(2)$.

2.2 IsoEuler angles for matrix product

Let $\hat{u} = \hat{u}_1\Delta\hat{u}_2$ is a product of two matrices $\hat{u}_1, \hat{u}_2 \in \hat{SU}(2)$. Denote the corresponding isoEuler angles by ($\hat{\varphi}$, $\hat{\theta}$, $\hat{\psi}$), ($\hat{\varphi}_1$, $\hat{\theta}_1$, $\hat{\psi}_1$), and ($\hat{\varphi}_2$, $\hat{\theta}_2$, $\hat{\psi}_2$). To express the isoEuler angles of $\hat{u}$ via the isoEuler angles of $\hat{u}_1$ and $\hat{u}_2$, we consider the case when $\hat{\varphi}_1 = \hat{\psi}_1 = \hat{\psi}_2 = 0$. For this case we have

$$\hat{u} = \begin{pmatrix} g_{11}^{-1/2}\cos\hat{\theta}_1/2 & i\Delta g_{22}^{-1/2}\sin\hat{\theta}_1/2 \\ i\Delta g_{22}^{-1/2}\sin\hat{\theta}_1/2 & g_{11}^{-1/2}\cos\hat{\theta}_1/2 \end{pmatrix} \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$$

$$\times \begin{pmatrix} g_{11}^{-1/2}\Delta\cos\hat{\theta}_2/2\exp\{i\hat{\varphi}_2/2\} & i\Delta^2\sin\hat{\theta}_2/2\exp\{i\hat{\varphi}_2/2\} \\ i\Delta^2\sin\hat{\theta}_2/2\exp\{-i\hat{\varphi}_2/2\} & g_{11}^{-1/2}\Delta\cos\hat{\theta}_2/2\exp\{-i\hat{\varphi}_2/2\} \end{pmatrix}. \tag{2.15}$$

Using (2.11) we have from (2.15) in sequence

$$\cos[\theta\Delta^{1/2}] = \cos[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}]g_{11}^{-1/2}$$

$$-\sin[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]\Delta g g_{11}^{-1/2}\cos[\varphi_2\Delta^{1/2}], \tag{2.16}$$

$$\exp\{i\Delta^{1/2}\varphi\} = \frac{\sin[\theta_1\Delta^{1/2}]\Delta g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}$$

$$+\frac{(g_11)^{-1}\cos[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]\Delta g_{11}^{-1/2}\cos[\varphi_2\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}$$

$$+\frac{i\sin[\theta_2\Delta^{1/2}]\Delta\sin[\varphi_2\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}, \tag{2.17}$$

$$\exp\{i\Delta^{1/2}(\varphi+\psi)/2\} = \frac{g_{11}^{-1}\Delta\cos[\theta_2\Delta^{1/2}/2]\Delta^2\exp\{i\Delta^{1/2}\varphi_2/2\}}{\cos[\theta_1\Delta^{1/2}/2]}$$

$$-\frac{g_{22}^{-1/2}\sin[\theta_1\Delta^{1/2}/2]\Delta\sin[\theta_2\Delta^{1/2}/2]\Delta\exp\{-i\Delta^{1/2}\varphi_2/2\}}{g_{11}^{-1/2}\cos[\theta\Delta^{1/2}/2]}. \tag{2.18}$$

It is more convenient to use the following expressions:

$$\left(\frac{g_{22}}{g_{11}}\right)^{-1/2}\tan[\varphi\Delta^{1/2}]$$

$$=\frac{\sin[\theta_2\Delta^{1/2}]\Delta\sin[\psi_2\Delta^{1/2}]}{g_{11}^{-1}\cos[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]\Delta\cos[\varphi_2\Delta^{1/2}]+g_{11}^{-1/2}\sin[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}]}, \tag{2.19}$$

$$\left(\frac{g_{22}}{g_{11}}\right)^{-1/2}\tan[\psi\Delta^{1/2}]$$

$$=\frac{\sin[\theta_1\Delta^{1/2}]\Delta\sin[\varphi_2\Delta^{1/2}]}{\sin[\theta_1\Delta^{1/2}]\Delta g_{11}^{-1}\cos[\theta_2\Delta^{1/2}]\Delta\cos[\psi_2\Delta^{1/2}]+g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]}. \tag{2.20}$$

Due to the results of this particular case can easily turn to the general case. Indeed, according to (2.12) we have

$$\hat{u}(\hat{\varphi}_1,\hat{\theta}_1,\hat{\psi}_1)\Delta\hat{u}(\hat{\varphi}_2,\hat{\theta}_2,\hat{\psi}_2))$$

$$=\Delta^5\hat{u}(\hat{\varphi}_1,0,0)\hat{u}(0,\hat{\psi}_1,0)\hat{u}(0,0,\hat{\psi}_1)\hat{u}(\hat{\varphi}_2,0,0)\hat{u}(0,\hat{\theta}_2,0)\hat{u}(0,0,\hat{\varphi}_2). \tag{2.21}$$

Note that

$$\hat{u}(0,0,\hat{\psi}_1)\Delta\hat{u}(\hat{\varphi}_2,0,0)=\hat{u}(\hat{\varphi}_2+\hat{\psi}_1,0,0). \tag{2.22}$$

We observe that the result of the product $\hat{u}(\hat{\varphi},\hat{\theta},\hat{\psi})*\hat{u}(\hat{\varphi}_1,0,0)$ gives the matrix $\hat{u}(\hat{\varphi}+\hat{\varphi}_1,\hat{\theta},\hat{\psi})$. Similarly, the result of the product $\hat{u}(0,0,\hat{\psi}_1)*\hat{u}(\hat{\varphi},\hat{\theta},\hat{\psi})$ gives the matrix $\hat{u}(\hat{\varphi},\hat{\theta},\hat{\psi}+\hat{\psi}_1$. From these observations it follows that the formulas (2.16)-(2.18) are valid in general case with the replacements $\hat{\varphi}_2\to\hat{\varphi}_2+\hat{\psi}_1$, $\hat{\varphi}\to\hat{\varphi}-\hat{\varphi}_1$, and $\hat{\psi}\to\hat{\psi}-\hat{\psi}_2$.

Namely, in an explicit form

$$\cos[\theta\Delta^{1/2}]=\cos[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}]$$

$$- g g_{22}^{-1/2} \sin[\theta_1 \Delta^{1/2}] \Delta \sin[\theta_2 \Delta^{1/2}] \cos[(\varphi_2 + \psi_1) \Delta^{1/2}], \tag{2.23}$$

$$\exp\{iD\varphi\} = \frac{g_{11}^{-1/2} \sin[\theta_1 \Delta^{1/2}] \Delta \cos[\theta_2 \Delta^{1/2}]}{\sin[\theta \Delta^{1/2}]}$$

$$+ \frac{g_{11}^{-1} \cos[\theta_1 \Delta^{1/2}] \Delta \sin[\theta_2 \Delta^{1/2}] \Delta \cos[(\psi_2 + \psi_1) \Delta^{1/2}]}{\sin[\theta \Delta^{1/2}]}$$

$$+ \frac{i g_{22}^{-1/2} \sin[\theta_2 \Delta^{1/2}] \Delta \sin[(\varphi_2 + \psi_1) \Delta^{1/2}]}{\sin[\theta \Delta^{1/2}]}, \tag{2.24}$$

$$\exp\{i \Delta^{1/2} (\varphi - \varphi_1 + \psi - \psi_2)/2\}$$

$$= \frac{g_{11}^{-1} \cos[\theta_1 \Delta^{1/2}/2] \Delta \cos[\theta_2 \Delta^{1/2}/2] \Delta \exp\{i \Delta^{1/2} (\varphi_2 + \psi_1)/2\}}{g_{11}^{-1/2} \cos[\psi \Delta^{1/2}/2]}$$

$$- \frac{g_{22}^{-1/2} \sin[\psi_1 \Delta^{1/2}/2] \Delta \sin[\theta_2 \Delta^{1/2}/2] \Delta \exp\{-i \Delta^{1/2} (\varphi_2 + \psi_1)/2\}}{g_{11}^{-1/2} \cos[\psi \Delta^{1/2}/2]}. \tag{2.25}$$

2.3 Relation to the group of rotations

Let us define the relation between the groups $\hat{SU}(2)$ and $\hat{SO}(3)$. To this end, we identify the vector $\hat{x}(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ of three dimensional isoEuclidean space with the complex 2 matrix of the form

$$\hat{h}_x = \begin{pmatrix} \hat{x}_3 & \hat{x}_1 + i\hat{x}_2 \\ \hat{x}_1 - i\hat{x}_2 & -\hat{x}_3 \end{pmatrix}, \tag{2.26}$$

where $\hat{x} = x\hat{I} = x\delta^{-1}$. The set of the matrices of the form (2.26) consists of isoHermitean matrices $\hat{g}$ with $\mathrm{Tr}\hat{g} = 0$. Namely, with every matrix $\hat{u} \in \hat{SU}(2)$ we relate the transformation $\hat{T}(\hat{u})$,

$$\hat{T}(\hat{u}) \Delta \hat{h}_x = \hat{u} \Delta \hat{h}_x * \hat{u}. \tag{2.27}$$

Since for the isounitary matrices we have $\hat{u}^* = \hat{u}^{-1}$, the traces of $\hat{h}_x$ and $\hat{T}(\hat{u})\Delta\hat{h}_x$ coincide so that the trace of $\hat{T}(\hat{u})\Delta\hat{h}_x$ is zero. Also, we have

$$(\hat{T}(\hat{u}) \Delta \hat{h}(\hat{x}))^* = (\hat{u} \Delta \hat{h}_x \Delta * \hat{u})^* = \hat{u} \Delta * \hat{h}_x * \hat{u} \Delta = \hat{u} \Delta \hat{h}_x * \hat{u} = \hat{T}(\hat{u}) \Delta \hat{h}_x, \tag{2.28}$$

so that the matrix $\hat{T}(\hat{u})\Delta\hat{h}_x$ is indeed isoHermitean. On the other hand, for isoHermitean matrices we have the following representation:

$$\hat{T}(\hat{u}) \Delta \hat{h}_x = \begin{pmatrix} \hat{y}_3 & \hat{y}_1 + i\Delta\hat{y}_2 \\ \hat{y}_1 - i\Delta\hat{y}_2 & -\hat{y}_3 \end{pmatrix} \equiv \begin{pmatrix} \Delta^{-1} y_3 & \Delta y_1 + i y_2 \\ \Delta^{-1} y_1 - i y_2 & -\Delta y_3 \end{pmatrix} = \hat{h}_y, \tag{2.29}$$

where $\hat{y}(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ is a vector in three dimensional isoEuclidean space.

From (2.27) it can be seen that the components of $\hat{y}$ are linear combinations of the components of $\hat{x}$ so that $\hat{T}(\hat{u})$ is a linear transformation of the three dimensional isoEuclidean space $\hat{E}^3$. From the local isomorphism between the groups $\hat{SU}(2)$ and $\hat{SO}(3)$ it follows that rotations of $\hat{E}^3$ can be parametrized by the isoEuler angles $(\hat{\varphi}, \hat{\theta}, \hat{\psi})$. Here, the angle $\hat{\varphi}$ varies from 0 to 2π since $\hat{u}$ and $-\hat{u}$ correspond to the same rotation.

Due to (2.12) the matrices $\hat{u}(\hat{\varphi}, 0, 0)$ and $\hat{u}(0, 0, \hat{\psi})$ can be presented as

$$\begin{pmatrix} \exp\{i\Delta^{1/2}t/2\} & 0 \\ 0 & \exp\{-i\Delta^{1/2}t/2\} \end{pmatrix} \equiv \hat{\omega}_3(\hat{t}), \tag{2.30}$$

where $\hat{\omega}_3(\hat{t})$ is the rotation by the angle $\hat{t}$ around the axis $O\hat{x}_3$, and $\hat{u}(0, \hat{\theta}, 0)$ has the form

$$\begin{pmatrix} g_{11}^{-1/2}\cos[t\Delta^{1/2}/2] & i\Delta g_{22}^{-1/2}\sin[t\Delta^{1/2}/2] \\ i\Delta(g_{22})^{-1/2}\sin[t\Delta^{1/2}/2] & g_{11}^{-1/2}\cos[t\Delta^{1/2}/2] \end{pmatrix} \equiv \hat{\omega}_1(\hat{t}), \tag{2.31}$$

which is the rotation around the axis $O\hat{x}_1$. From this observation, we have the following decomposition for arbitrary rotation $\hat{g}$ of $\hat{E}^3$:

$$\hat{g}(\hat{\varphi}, \hat{\theta}, \hat{\psi}) = \hat{g}(\hat{\varphi}, 0, 0)\Delta\hat{g}(0, \hat{\theta}, 0)\Delta\hat{g}(\hat{\psi}, 0, 0)$$

$$= \begin{pmatrix} g_{11}^{-1/2}\cos\hat{\varphi} & -g_{22}^{-1/2}\sin\hat{\varphi} & 0 \\ g_{22}^{-1/2}\sin\hat{\varphi} & g_{11}^{-1/2}\cos\hat{\varphi} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}$$

$$\times \begin{pmatrix} \Delta^{-1} & 0 & 0 \\ 0 & g_{11}^{-1/2}\cos\hat{\theta} & -g_{22}^{-1/2}\sin\hat{\theta} \\ 0 & (g_{22})^{-1/2}\sin\hat{\psi} & (g_{11})^{-1/2}\cos\hat{\theta} \end{pmatrix} \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}$$

$$\times \begin{pmatrix} \Delta^{-1} & 0 & 0 \\ 0 & g_{11}^{-1/2}\cos\hat{\psi} & -g_{22}^{-1/2}\sin\hat{\psi} \\ 0 & (g_{22})^{-1/2}\sin\hat{\psi} & (g_{11})^{-1/2}\cos\hat{\psi} \end{pmatrix}. \tag{2.32}$$

3 Irreps of $\hat{SU}(2)$

Recall that with any isounimodular complex 2×2 matrix $\hat{g}$ we associate the linear transformation,

$$\hat{w}_1 = \alpha\Delta^2 z_1 + \gamma\Delta^2 z_2 = \Delta^2(\alpha z_1 + \gamma z_2), \tag{3.1}$$

$$\hat{w}_2 = \beta\Delta z_1 + \delta\delta^2 z_2 = \Delta^2(\beta z_1 + \delta z_2),$$

of two dimensional linear complex space. Such a transformation can be presented by the operator,

$$\hat{T}(\hat{g})\Delta\hat{f}(\hat{z}_1, \hat{z}_2) = \hat{f}(\hat{\alpha}\Delta\hat{z}_1 + \hat{\gamma}\Delta\hat{z}_2; \hat{\beta}\Delta\hat{z}_1 + \hat{\Delta}\hat{z}_2), \tag{3.2}$$

acting on the space of functions of two complex variables. Evidently,

$$\hat{T}(\hat{g}_1\Delta\hat{g}_2) = \hat{T}(\hat{g}_1) + \hat{T}(\hat{g}_2),$$

so that $\hat{T}(\hat{g})$ is a *representation* of the group $\hat{SL}(2, C)$. Similarly to the theorem from Ref.[4] we have the following

Proposition 1. Every irreducible isounitary representation $\hat{T}(\hat{u})$ of $\hat{SU}(2)$ is equivalent to one of the representations $\hat{T}_l(\hat{u})$, where $l = 0, 1/2, 1, \ldots$.

The prove of the Proposition 1 is analogous to that of given in Ref.[4], and we do not present it here.

From Proposition 1 it follows that in the space of subgroup $\hat{SU}(2)$ there exists the orthogonal normalized basis, $\hat{f}_{-l}, \cdots, \hat{f}_l$, such that the operators $\hat{T}(\hat{u})$ are represented in this basis by the same matrices as the operators $\hat{T}_l(\hat{u})$ in the basis $\{\hat{\psi}_k(x)\}$, where

$$\hat{\psi}_k(x) = \Delta^{-s+1/2} \frac{x^{l-k}}{\sqrt{(l-k)!(l+k)!}}, \tag{3.3}$$

$$-l \leq k \leq l, \quad s = 1, \ldots, n.$$

We call such a basis *isocanonical*. It is easy to verify that isocanonical basis is determined uniquely up to scalar factor λ, with $|\lambda| = \Delta^{-1}$. More precisely, isocanonical basis consists of normalized eigenvectors of the operator $\hat{T}(\hat{h})$, where

$$\hat{h} = \begin{pmatrix} \exp\{i\Delta^{1/2}t/2\} & 0 \\ 0 & \exp\{-i\Delta^{1/2}t/2\} \end{pmatrix}. \tag{3.4}$$

4 Matrix elements of the irreps of $\hat{SU}(2)$ and isoLegendre polynomials

In this Section, we calculate matrix elements of the irreps $\hat{T}_l(\hat{u})$ of $\hat{SU}(2)$, and express the matrix elements $\hat{t}^l_{mn}(\hat{g})$ through the isoEuler angles ($\hat{\varphi}$, $\hat{\theta}$, $\hat{\psi}$) of the matrix $\hat{g}$.

The representations $\hat{T}_l(\hat{g})$ of $\hat{SL}(2,C)$ are given by

$$\hat{T}_l(\hat{g})\Delta\hat{\varphi}(\hat{x}) = (\beta x + \delta\Delta^{-1})^{2l}\Delta\hat{\varphi}\frac{\alpha x + \gamma\Delta^{-1}}{\beta x + \delta\Delta^{-1}}, \tag{4.1}$$

where $\hat{\varphi}(\hat{x})$ is polynomial of degree $2l$ on $\hat{x}$, and $\hat{g} \in \hat{SL}(2,C)$.

Using the isocanonical basis of Sec. 3 and the formula $a_{ij} = (e_j \hat{,} e_i)$, where $\{e_i\}$ is orthonormalized basis, we write down the matrix element,

$$\hat{t}^l_{mn}(\hat{g}) = (\hat{T}_l(\hat{g})\hat{\psi}_n \hat{,} \hat{\psi}_m) = \frac{\left(\hat{T}_l(\hat{g})\hat{x}^{l-n} \hat{,} \hat{x}^{l-m}\right)}{\sqrt{(l-m)!(l+m)!(l-n)!(l+n)!}\Delta^{3/2}}, \tag{4.2}$$

where

$$\hat{\psi}_n(\hat{x}) = \frac{x^{l-n}\Delta^{3/2}}{\sqrt{(l-n)!(l+n)!}}, \tag{4.3}$$

$$-l \leq n \leq l, \quad \hat{(l-n)!} = \Delta^s(l-n)(l-n+1)\ldots, \qquad s = 1,2,\ldots$$

On the other hand,

$$\hat{T}(\hat{g})x^{l-n} = (\alpha x + \gamma\Delta^{-1})^{l-n}\Delta(\beta x + \delta\Delta^{-1})^{l+n}, \tag{4.4}$$

so that (4.2) yields

$$\hat{t}^l_{mn} = \frac{\left((\alpha x + \gamma\Delta-1)^{l-n}\Delta(\beta x + \delta\Delta^{-1})^{l+n} \hat{,} x^{l-m}\Delta^{-1}\right)}{\sqrt{(l-m)!(l+m)!(l-n)!(l+n)!}}\Delta-3/2. \tag{4.5}$$

Taking into account that $(\hat{x}^{l-k}, \hat{x}^{l-m}) = 0$ at $k \neq m$ and $(\hat{x}^{l-m}, \hat{x}^{l-m}) = (l-m)!(l+m)!\Delta^{2s+1}$, we have finally from (4.5)

$$\hat{t}^l_{mn}(\hat{g}) = \sqrt{\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!}}\Delta^{2+l}$$

$$\times \sum_{j=M}^{N} \hat{C}^{l-m-j}_{l-n}\hat{C}^j_{l+n}\alpha^{l-m-j}\beta^j\gamma^{m+j-n}\delta^{l+n-j}$$

$$= \sqrt{(l-m)!(l+m)!(l-n)!(l+n)!}\Delta^{2\alpha-4s}\alpha^{l-m}\gamma^{m-n}\delta^{l+n}$$

$$\times \sum_{j=M}^{N} \frac{\Delta^{-4s-1}}{j!(l-m-j)!(l+n-j)!(m-n+j)!}\left(\frac{\hat{\beta}\hat{\gamma}}{\hat{\alpha}\hat{\delta}}\right)^j$$

$$= \sqrt{(l-m)!(l+m)!(l-n)!(l+n)!}\alpha^{l-m}\gamma^{m-n}\delta^{l+n}$$
$$\times \sum_{j=M}^{N} \frac{(\Delta^9/\Delta^{11s})^{1/2}\Delta^j}{j!(l-m-j)!(l+n-j)!(m-n+j)!}\left(\frac{\beta\gamma}{\alpha\delta}\right)^j, \tag{4.6}$$

where $M = \max(0, n-m)$, $N = \min(l-m, l+n)$. We should to note that the matrix element (4.6) in fact does not depend on β because of isounimodularity of $\hat{g}$ implying $\beta\gamma = \alpha\delta - \Delta^{-1}$.

We are in a position to express $\hat{t}^l_{mn}(\hat{g})$ in terms of the isoEuler angles. Due to (2.32),

$$\hat{T}_l[\hat{g}(\hat{\varphi},\hat{\theta},\hat{\psi})] = \hat{T}_l[(\hat{g}(\hat{\varphi},0,0)]\Delta\hat{T}^l[\hat{g}(0,\hat{\theta},0)]\Delta\hat{T}[\hat{g}(0,0,\hat{\psi})], \tag{4.7}$$

so that finding the general matrix $\hat{T}_l(\hat{g})$ reduces to finding of the matrices $\hat{T}_l[\hat{g}(\hat{\varphi},0,0)]$, $\hat{T}_l[\hat{g}(0,\hat{\theta},0)]$, and $\hat{T}_l[\hat{g}(0,0,\hat{\psi})]$.

The matrix $\hat{g}(\hat{\varphi},0,0)$ is diagonal,

$$\hat{g}(\hat{\varphi},0,0) = \begin{pmatrix} \exp\{i\Delta^{3/2}\varphi/2\} & 0 \\ 0 & \exp\{-i\Delta^{3/2}\varphi/2\} \end{pmatrix}. \tag{4.8}$$

For this matrix, we have (see, for example, Ref.[4] for the ordinary case)

$$\hat{T}_l[\hat{g}(\hat{\varphi},0,0)]\Delta 3/2x^{l-n} = \exp -i\Delta^2 n\varphi\Delta^{1/2}\Delta^{3/2}x^{l-n}. \tag{4.9}$$

Hence, the matrix of the operator $\hat{T}_l[\hat{g}(\hat{\varphi},0,0)]$ is diagonal too, with the nonzero elements being $\exp[-i\Delta^{5/2}\phi]$, $-l \leq n \leq l$. The matrix of the operator $\hat{T}_l[\hat{g}(0,0,\hat{\psi})]$ has similar form.

Let us denote matrix element of the operator $\hat{T}_l[\hat{g}(0,\hat{\theta},0)]$ as $\hat{t}^l_{mn}(\hat{\theta})$. Then, according to diagonality of the matrices of the operators $\hat{T}_l[\hat{g}(\hat{\varphi},0,0)]$ and $\hat{T}_l[\hat{g}(0,0,\hat{\psi})]$, we obtain

$$\hat{t}^l_{mn} = \hat{t}^l_{mn}[\hat{g}(\hat{\varphi},0,0)]\Delta\hat{t}^l_{mn}(\hat{\theta})\Delta\hat{t}^l_{nn}[\hat{g}(0,0,\hat{\theta})\exp\{-i\Delta^2(m\varphi+n\psi)\}\Delta\hat{t}^l_{mn}(\theta). \tag{4.10}$$

It remains to obtain $\hat{t}^l_{mn}(\hat{\theta})$. The matrix $\hat{g}(0,\hat{\theta},0)$ has the form

$$\hat{g}(0,\hat{\theta},0) = \begin{pmatrix} g_{11}^{-1/2}\cos\hat{\theta}/2 & i\Delta g_{22}^{-1/2}\sin\hat{\theta}/2 \\ i\Delta g_{22}^{-1/2}\sin\hat{\theta}/2 & g_{11}^{-1/2}\cos\hat{\theta}/2 \end{pmatrix}, \tag{4.11}$$

where $0 \leq Re\theta < \pi$.

In the same manner as in Ref.[4] we then have

$$\hat{t}^l_{mn}(\hat{\theta}) = i^{-m-n}\Delta^{5-3s+2j}\sqrt{\frac{(l-m)!(l-n)!}{(l+m)!(l+n)!}}\left(\frac{g_{11}}{g_{22}}\right)^{1/2}\text{cotan}^{m+n}[\theta\Delta^{1/2}/2]$$

$$\times \sum_{j=max(m,n)}^{l} \frac{(l+j)!i^{2j}}{(l-j)!(j-m)!(j-n)!} g_{22}^{-1/2} \sin[\theta\Delta^{1/2}/2]. \tag{4.12}$$

Parameter $\hat{\theta}$ varies within the range $0 \leq Re\hat{\theta} < \pi$ so that, in this range, different values of $\hat{\theta}$ correspond to different values of $\hat{z} = g_{11}^{-1/2}\cos[\theta\Delta^{1/2}]$. So, $\hat{t}^l_{mn}(\hat{\theta})$ can be viewed as a function on isocos$\hat{\theta}$. In accordance to this, we put

$$\hat{t}^l_{mn}(\hat{\theta}) = \hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\theta\Delta^{1/2}]. \tag{4.13}$$

Then, (4.10) can be rewritten as

$$\hat{t}^l_{mn}(\hat{\theta}) = \exp -i\Delta^{1/2}(m\varphi + n\psi)\Delta\hat{P}^l_{mn}(\hat{z}). \tag{4.14}$$

With the use of (4.14), Eq.(4.12) leads to the following definition of the isoLegendre polynomials:

$$\hat{P}^l_{mn} = i^{-m-n}\Delta^{5-2s+3j+(m+n)/2}\sqrt{\frac{(l-m)!(l-n)!}{(l+m)!(l+n)!}}\left(\frac{\Delta^{-1}+\hat{z}}{\Delta^{-1}-\hat{z}}\right)^{(m+n)/2}$$

$$\times \sum_{j=max(m,n)}^{l} \frac{(l+j)!i^{2j}}{(l-j)!(j-m)!(j-n)!}\left(\frac{\Delta^{-1}-\hat{z}}{2}\right)^{j}. \tag{4.15}$$

The factor $((\Delta^{-1}+\hat{z})/(\Delta^{-1}-\hat{z}))^{(m+n)/2}$ is twovalued since m and n are both integer or half-integer. Single valued definition in (4.15) comes when taking into account that $0 \leq Re\hat{\theta} < \pi$ and $\hat{z}$ maps this range to the plane $\hat{z}$ cutted along the real axis, $(-\infty;-1)$ and $(1;\infty)$. In the cutted plane the factor is single valued.

5 Basic properties of the isoLegendre polynomials

In this Section, we study the basic relations obeyed by the isoLegendre polynomials.

5.1 Symmetry relations

We will show that $\hat{P}^l_{mn}(\hat{z})$ is invariant under the changing of signs of the indeces m and n. For this purpose, we use the relation

$$\hat{g}(\pi)\hat{g}(\hat{\theta}) = \hat{g}(\hat{\theta})\hat{g}(\pi), \tag{5.1}$$

where we have denoted for brevity

$$\hat{g}(\hat{\theta}) = \begin{pmatrix} g_{11}^{-1/2}\cos[t\Delta^{1/2}/2] & i\Delta g_{22}^{-1/2}\sin[t\Delta^{1/2}/2] \\ i\Delta(g_{22})^{-1/2}\sin[t\Delta^{1/2}/2] & g_{11}^{-1/2}\cos[t\Delta^{1/2}/2] \end{pmatrix}. \tag{5.2}$$

From (5.1)-(5.2) it follows that

$$\hat{T}^l(\pi)\hat{T}^l(\hat{\theta}) = \hat{T}^l(\hat{\theta})\hat{T}^l(\pi). \tag{5.3}$$

Recall that the matrix elements of $\hat{T}_l(\hat{\theta})$ are just $\hat{P}^l_{mn}(\hat{z})$. Also, it is known that $\hat{t}^l_{mn}(\pi) = 0$ at $m+n \neq 0$, and $\hat{t}^l_{m,-m}(\pi) = i^{2\Delta l}$. Replacing the operators in (5.3) by their matrix elements we obtain

$$\hat{P}^l_{m,-n}(\hat{z}) = \hat{P}^l_{-m,n}(\hat{z}), \tag{5.4}$$

from which we have

$$\hat{P}^l_{mn}(\hat{z}) = \hat{P}^l_{-m,-n}(\hat{z}). \tag{5.5}$$

According to the explicit representation (4.15), we then also obtain

$$\hat{P}^l_{mn}(\hat{z}) = \hat{P}^l_{nm}(\hat{z}). \tag{5.6}$$

The relations (5.4), (5.5) and (5.6) are the basic symmetry relations for the isoLegendre polunomials.

The relations (5.5) and (5.6) means, particularly, that $\hat{P}^l_{mn}(\hat{z})$ depends on m and n through the combinations $|m+n|$ and $|m-n|$.

Also, it is straightforward to verify that the following relation holds,

$$\hat{P}^l_{mn}(\hat{z}) = i^{2\Delta(l-m-n)}\Delta\hat{P}^l_{m,-n}(\hat{z}). \tag{5.7}$$

5.2 Counter relations

The function $\hat{P}^l_{mn}(\hat{z})$ is defined in complex plane cutted along the lines $(-\infty;-1)$ and $(1;\infty)$. On the upper and lower neighbours of these lines $\hat{P}^l_{mn}(\hat{z})$ takes different values. From (4.15) it follows that for $\hat{z} > 1$ we have

$$\hat{P}^l_{mn}(\hat{z}+i0) = -\frac{1}{\Delta^{m-n}}\Delta\hat{P}^l_{mn}(\hat{z}-i0). \tag{5.8}$$

Similarly, for $\hat{z} < -1$,

$$\hat{P}^l_{mn}(\hat{z}+i0) = -\frac{1}{\Delta^{m+n}}\Delta\hat{P}^l_{mn}(\hat{z}-i0). \tag{5.9}$$

5.3 Relation to classical orthogonal polynomials

In Sec. 4, we have defined the isoLegendre function $\hat{P}^l_{mn}(\hat{z})$, and obtained one of the representaions of it. Now, we relate $\hat{P}^l_{mn}(\hat{z})$ to some of classical orthogonal polynomials - isoJacobi, adjoint isoLegendre, and isoLegendre polynomials.

This relations allows, particularly, to establish properties of the polynomials by the use of the properties of the isoLegendre function.

5.3.1 IsoJacobi polynomials

IsoJacobi polynomials are defined by

$$\hat{P}^{\hat{\alpha},\hat{\beta}}_k(\hat{z}) = \frac{(-\Delta^{-1})^k}{2^k k!}(1-z)^{-\alpha}\Delta^{1/2}(1+z)^{-\beta}\Delta^{1/2}$$

$$\times \frac{d^k}{d\hat{z}^k}[(1-z^2)^k(1+z)^{\alpha}\Delta^{1/2}(1+z)^{\beta}\Delta^{1/2}]\Delta^{5-k-s}. \tag{5.10}$$

Comparing (5.10) with the following representation of the isoLegendre function,

$$\hat{P}^l_{mn}(\hat{z}) = \frac{\Delta^{n-m-l}}{2^l}\sqrt{\frac{(l+m)!}{(l-n)!(l+n)(l-n)!}}$$

$$\times (1+z)^{-(m+n)/2}(1-z)^{(n-m)/2}\frac{d^{l-m}}{d\hat{z}^{l-m}}[(1-z)^{l-n}(1+z)^{l+n}]\Delta^{2-s+2l}, \tag{5.11}$$

we obtain

$$\hat{P}^{\hat{\alpha},\hat{\beta}}_k(\hat{z}) = 2^{m-n}\sqrt{(l-n)!(l+n)!(l-m)!(l+m)!}$$

$$\times (1-z)^{(n-m)/2}(1-z)^{-(n+m)/2}\hat{P}^l_{mn}(\hat{z})\Delta^{2+2m-n}, \tag{5.12}$$

where

$$l = k + \frac{\hat{\alpha}+\hat{\beta}}{2}, \quad m = \frac{\hat{\alpha}+\hat{\beta}}{2}, \quad n = \frac{\hat{\beta}-\hat{\alpha}}{2}. \tag{5.13}$$

From (2.30) we see that $\hat{\alpha} = m - n$ and $\hat{\beta} = m + n$ are integer numbers. Thus, $\hat{P}^l_{mn}(\hat{z})$ is reduced to isoJacobi polymomials, for which $\hat{\alpha}$ and $\hat{\beta}$ are integer numbers.

5.3.2 isoLegendre polynomials

IsoLegendre polynomials are defined by

$$\hat{P}_l(\hat{z}) = -\frac{\Delta^{-l}}{2^l l!}\frac{d^l}{d\hat{z}^l}[(1-z^2)^l]\Delta^{-(s+l)}. \tag{5.14}$$

This implies that $\hat{P}_l(\hat{z}) = \hat{P}_l^{00}(\hat{z})$. Comparing (5.14) and (5.11) we obtain

$$\hat{P}_l(\hat{z}) = \hat{P}^l_{00}(\hat{z}). \tag{5.15}$$

5.3.3 Adjoint isoLegendre functions

The adjoint isoLegendre function $\hat{P}_l^m(\hat{z})$, where $m \geq 0$ (l, m are integer), is defined by

$$\hat{P}_l^m(\hat{z}) = \frac{(-\Delta^{-1})^l}{2^{l+m} l!}(1-z^2)^{m/2}\frac{d^l}{d\hat{z}^l}[(1-z^2)^l]\Delta^{-(s+2l+m/2)}, \tag{5.16}$$

that is

$$\hat{P}_l^m(\hat{z}) = \frac{2^m(l+m)!}{l!}(1-z^2)^{m/2}\hat{P}_{l+m}^{-m,-m}(\hat{z})\Delta^{m/2}. \tag{5.17}$$

Comparing (5.17) with (5.11) leads to the following relation:

$$\hat{P}_l^m(\hat{z}) = i^m\sqrt{\frac{(l+m)!}{(l-m)!}}\hat{P}^l_{-m0}(\hat{z})\Delta^{2-m}. \tag{5.18}$$

Let us rewrite (5.18) by taking into account (5.5),

$$\hat{P}_l^m(\hat{z}) = i^m\sqrt{\frac{(l+m)!}{(l-m)!}}\hat{P}^l_{m0}(\hat{z})\Delta^{2-m}, \quad m \geq 0. \tag{5.19}$$

6 Functional relations for isoLegendre functions

In this Section, we derive basic theorems of composition and multiplication of $\hat{P}^l_{mn}(\hat{z})$, and the condition of its orthogonality.

6.1 Theorem of composition

Many important properties of $\hat{P}^l_{mn}(\hat{z})$ are related to the theorem of composition. To derive the rule, let us use the relation

$$\hat{T}^l(\hat{g}_1\Delta\hat{g}_2) = \hat{T}^l(\hat{g}_1)\Delta\hat{T}^l(\hat{g}_2), \tag{6.1}$$

from which it follows that

$$\hat{t}^l(\hat{g}_1\Delta\hat{g}_2) = \sum_{k=-l}^{l} \hat{t}^l(\hat{g}_1)\Delta\hat{t}^l(\hat{g}_2), \tag{6.2}$$

and it can be rewritten as

$$\hat{t}^l_{mn}(\hat{g}_1\Delta\hat{g}_2) = \exp\{-i\Delta^{1/2}(m\varphi + n\psi)\}\hat{P}^l_{mn}(\hat{z}). \tag{6.3}$$

For

$$\hat{t}^l_{mn}(\hat{g}_1 = \hat{P}^l_{mk}(\hat{z}), \quad \hat{t}^l_{kn}(\hat{g}_2 = \exp\{-i\Delta^{3/2}\varphi_2\}\Delta\hat{P}^l_{kn}(\hat{z}), \tag{6.4}$$

where $\hat{\varphi}$, $\hat{\theta}$, $\hat{\psi}$ are isoEuler angles of the matrix $\hat{g}_1\Delta\hat{g}_2$. These angles are expressed through the angles $\hat{\theta}_{1,2}$, $\hat{\varphi}_2$ due to the following formulas:

$$\cos[\theta\Delta^{1/2}] = \cos[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}]$$

$$- g_{22}^{-1/2}\sin[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]\cos[\varphi_2\Delta^{1/2}], \tag{6.5}$$

$$\exp\{i\Delta^{1/2}\varphi\} = \frac{\sin[\theta_1\Delta^{1/2}]\Delta g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}$$

$$+\frac{g_{11}^{-1}\cos[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]\Delta\cos[\psi_2]\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}$$

$$+\frac{ig_{22}^{-1/2}\ sin[\theta_2\Delta^{1/2}]\Delta\sin[\varphi_2+\psi_1]\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}, \tag{6.6}$$

$$\exp\{i(\hat{\varphi}+\hat{\psi})/2\} = \frac{g_{11}^{-1}\cos[\hat{\theta}_1/2]\cos[\hat{\theta}_2/2]\Delta\exp\{i\hat{\varphi}_2/2\}\Delta\exp\{\hat{\varphi}_2/2\}}{g_{11}^{-1/2}\cos[\hat{\psi}/2]},$$

$$-\frac{g_{22}^{-1/2}\sin[\hat{\psi}_1/2]\Delta\sin[\hat{\theta}_2/2]\Delta\exp\{-i\hat{\varphi}_2/2\}}{g_{11}^{-1/2}\cos[\hat{\psi}/2]}, \tag{6.7}$$

where $0 \leq Re\,\hat{\theta} < \pi$, $0 \leq Re\,\hat{\varphi} < 2\pi$, and $-2\pi \leq Re\,\hat{\psi} < 2\pi$.

Inserting equations (6.3) and (6.4) into (6.2), we obtain

$$\exp\{-i\Delta^{1/2}(m\varphi+n\psi)\}\hat{P}^l_{mn}(\hat{z})$$

$$= \sum_{k=-l}^{l}\exp\{-i\Delta^{3/2}\varphi_2\}\hat{P}^l_{mk}(\hat{z}_1)\hat{P}^l_{kn}(\hat{z}_2)\Delta^2. \tag{6.8}$$

(a) Let $\hat{\varphi}_2 = 0$, then if $Re(\hat{\theta}_1 + \hat{\theta}_2) < \pi \Rightarrow \hat{\theta} = \hat{\theta}_1 + \hat{\theta}_2$ and $\hat{\varphi} = \hat{\psi} = 0$. Accordingly, (6.8) takes the form

$$\hat{P}^l_{mn}(\hat{z}_1 + \hat{z}_2) = \sum_{k=-l}^{l} \hat{P}^l_{mk}(\hat{z}_1)\Delta\hat{P}^l_{kn}(\hat{z}_2) \Leftrightarrow$$

$$\hat{P}^l_{mn}[g_{11}^{-1/2}\cos[(\hat{\theta}_1 + \hat{\theta}_2)\Delta^{1/2}] \tag{6.9}$$

$$= \sum_{k=-l}^{l} \hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}2]\Delta\hat{P}^l_{kn}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}2]).$$

(b) Let $\varphi_2 = 0$, then if $Re(\hat{\theta}_1 + \hat{\theta}_2) > \pi \Rightarrow \hat{\theta} = 2\pi - \hat{\theta}_1 - \hat{\theta}_2$, $\hat{\varphi} = \hat{\psi} = \pi$. Therefore,

$$\hat{P}^l_{mn}(\hat{z}_1 + \hat{z}_2) = -\Delta^{2-m-n}$$

$$\sum_{k=-l}^{l} \hat{P}^l_{mk}(g_{11}^{-1/2}cos[\theta_1\Delta^{1/2}2])\Delta\hat{P}^l_{kn}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}2]). \tag{6.10}$$

(c) Let $\varphi_2 = \pi$, then if $Re\,\hat{\theta}_1 \geq Re\,\hat{\theta}_2$, $\Rightarrow \hat{\theta} = \hat{\theta}_1 - \hat{\theta}_2$, $\hat{\varphi} = 0$, $\hat{\psi} = \pi$. Therefore,

$$\hat{P}^l_{mn}(\hat{z}_1 + \hat{z}_2)$$

$$= \sum_{k=-l}^{l}(-\Delta^{2-n-k})\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}2])\Delta\hat{P}^l_{kn}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}2]). \tag{6.11}$$

(d) In particular, at $\hat{\theta}_1 = \hat{\theta}_2 = \hat{\theta}$, we have

$$\sum_{k=-l}^{l}(-\Delta^{1-n-k})\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}2])$$

$$\times\, \Delta\hat{P}^l kn(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}2]) = \delta^{mn}. \tag{6.12}$$

(e) At $\hat{\varphi} = \frac{\pi}{2}$, the formulas (6.5)-(6.7) take the following forms:

$$\cos[\theta\Delta^{1/2}] = \cos[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}], \tag{6.13}$$

$$\exp\{i\Delta^{1/2}\varphi\} = \frac{\sin[\theta_1\Delta^{1/2}]\Delta g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}] + i\Delta\sin[\theta_2\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}, \tag{6.14}$$

$$\exp\{i\Delta^{1/2}(\varphi + \psi)/2\} = \frac{\cos[(\theta_1 + \theta_2)\Delta^{1/2}/2] + i\Delta\cos[(\theta_1 - \theta_2)\Delta^{1/2}/2]}{\cos[\theta_2\Delta^{1/2}]}, \tag{6.15}$$

respectively.

Instead of (6.14) and (6.15), it is more convenient to define

$$gg_{22}^{-1/2} tan[\varphi\Delta^{1/2}] = \frac{\sin[\{\theta_2\Delta^{1/2}]}{\sin}[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}], \tag{6.16}$$

$$g_{22}^{-1/2}\tan[\psi\Delta^{1/2}] = \frac{\sin[\{\theta_1\Delta^{1/2}]}{\cos[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]}. \tag{6.17}$$

Then

$$\exp\{-i\Delta^{1/2}(m\varphi + n\psi)\}\Delta\hat{P}^l_{mn}(g_{11}^{-1/2} cos[\theta_2\Delta^{1/2}])$$
$$= i^{-k}\sum_{k=-l}^{l}\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\Delta\hat{P}^l_{kn}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}]). \tag{6.18}$$

6.1.1 Theorem of composition for isoLegendre polynomials

Consider particular cases of the function $\hat{P}^l_{mn}(\hat{z})$, namely, the IsoLegendre polynomials and adjoint isoLegendre polynomials. The polynomials are defined due to

$$\hat{P}_l(\hat{z}) = \hat{P}^l_{00}(\hat{z}), \quad \hat{P}^m_l = i^m\sqrt{\frac{(l+m)!}{(l-m)!}}\hat{P}^l_{-m0}(\hat{z})\Delta^{2-m}. \tag{6.19}$$

Taking into account the formulas from Sec 6.1 and using (6.19) we get

$$\exp\{i\Delta^{3/2}m\varphi\}\hat{P}^l_m(g_{11}^{-1/2}\cos[\theta^{1/2}_{\Delta}])$$
$$= i^m\sqrt{\frac{(l+m)!}{(l-m)!}}\sum_{k=-l}^{l} i^{-k}\sqrt{\frac{(l-k)!}{(l+k)!}} \tag{6.20}$$
$$\times\exp\{-i\Delta^{3/2}m\varphi_2\}\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}^k_l(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])\Delta^{5+m-k},$$

where $\hat{\varphi}$, $\hat{\theta}$, $\hat{\varphi}_2$, and $\hat{\theta}_1$, $\hat{\theta}_2$ are related to each other as in Sec. 6.1.

If we put $m = n = 0$, we obtain, particularly,

$$\hat{P}^l(g_{11}^{-1}\cos[\theta_1\Delta^{1/2}])\Delta\cos[\theta_2\Delta^{1/2}]$$
$$- g_{22}^{-1}\sin([\theta_1\Delta^{1/2}]\sin([\theta_2\Delta^{1/2}]g_{11}^{-1/2}\cos[\varphi_2\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]) \tag{6.21}$$
$$= (-1)\sum_{k=-l}^{l} i^{-k}\sqrt{\frac{(l-k)!}{(l+k)!}}\exp\{-i\Delta^{3/2}m\varphi_2\}$$

$$\times \hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])\Delta^4.$$

Due to the symmetry $\hat{P}^l_{m0}(\hat{z}) = \hat{P}^l_{-m0}(\hat{z})$ and (6.19), the relation (6.21) can be reduced:

$$\hat{P}_l^{-m}(\hat{z}) = -1\sqrt{\frac{(l+m)!}{(l-m)!}}\hat{P}_l^m(\hat{z})\Delta^{2-m}. \tag{6.22}$$

Thus, from (6.21) it follows that the polynomials - $\hat{P}_l(\hat{z})$ obey the following *theorem of composition:*

$$\begin{aligned}&\hat{P}^l(g_{11}^{-1}\cos[\theta_1\Delta^{1/2}])\Delta\cos[\theta_2\Delta^{1/2}]\\ &- g_{22}^{-1}\sin([\theta_1\Delta^{1/2}]\sin[\varphi_2\Delta^{1/2}]g_{11}^{-1/2}\cos[\varphi_2\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}])\\ &= (-1)\sum_{k=-l}^{l} i^{-k}\sqrt{\frac{(l-k)!}{(l+k)!}}\exp\{-i\Delta^{3/2}m\varphi_2\}\\ &\times\hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])\Delta^3.\end{aligned} \tag{6.23}$$

6.2 Multiplication rules

Let in the composition rule

$$\begin{aligned}&\exp\{-i\Delta^{3/2}(m\varphi+n\psi)\}\hat{P}^l(g_{11}^{-1/2}\cos[\theta\Delta^{1/2}])\\ &= \exp\{-i\Delta^{3/2}k\varphi_2\}\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}^l_{kn}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])\Delta^2.\end{aligned} \tag{6.24}$$

If φ_2 is a real angle, then this formula can be viewed as a Fourier expansion of the function

$$\exp\{-i\Delta^{3/2}(m\varphi+n\psi)\}\Delta\hat{P}^l(g_{11}^{-1/2}\cos[\theta\Delta^{1/2}]).$$

Therefore,

$$\begin{aligned}&\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\Delta\hat{P}^l_{kn}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])\\ &= \frac{\Delta^3}{2\pi}\int_{-\pi}^{\pi}\exp\{-i\Delta^{3/2}(k\varphi_2-m\varphi-n\psi)\}\hat{P}^l_{mn}(g_{11}^{-1/2})d(\varphi_2\Delta^{1/2}).\end{aligned} \tag{6.25}$$

Putting $m = n = 0$ in this formula, we get

$$\frac{\Delta^3}{2\pi}\int_{-\pi}^{\pi}\exp\{-i\Delta^{3/2}k\varphi_2\}\hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\theta\Delta^{1/2})d(\varphi_2\Delta^{1/2})$$

$$= \sqrt{\frac{(l-k)!}{(l+k)!}}\hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])\Delta^{3-k}. \qquad (6.26)$$

Since $\hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\theta\Delta^{1/2}])$ is an even function in respect to φ_2, the above equality can be rewritten

$$\frac{\Delta^2}{\pi}\int_{-\pi}^{\pi}\hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\theta\Delta^{1/2}])g_{11}^{-1/2}\cos[k\varphi_2\Delta^{1/2}]d(\varphi_2\Delta^{1/2})$$

$$= \hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}]). \qquad (6.27)$$

If we now let additionally $k=0$, we obtin the further reduction

$$\frac{\Delta}{\pi}\int_{-\pi}^{\pi}\hat{P}^l_{mn}(g_{11}^{-1/2}cos[\theta\Delta^{1/2})d(\varphi_2\Delta^{1/2})$$

$$= \hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}]). \qquad (6.28)$$

Let us rewrite eq. (6.28) in a more convenient form. Assuming θ_1, θ_2, φ_2 to be real numbers such that $0 \le \theta_1 < \pi$ and $0 \le \theta_1+\theta_2 < \pi$, we redefine the variable

$$\cos[\theta\Delta^{1/2}] = \cos[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}]$$
$$- g_{22}^{-1}\sin[\theta_1\Delta^{1/2}]\sin[\theta_2\Delta^{1/2}]\cos[\varphi_2\Delta^{1/2}]\Delta^2. \qquad (6.29)$$

Introduce the notation

$$\hat{T}_n(\hat{x}) = g_{11}^{-1/2}\cos[n\Delta g_{11}^{-1/2}\mathrm{arccos}\hat{x}].$$

This function defines *Chebyshev-I polynomial.* From the last equation it follows that

$$g_{11}^{-1/2}\cos[k\Delta^{3/2}\varphi_2]$$
$$= \hat{T}_k\frac{g_{11}^{-1}(\cos[\theta_1\Delta^{1/2}]\Delta g_{11}^{-1}\cos[\theta_2\Delta^{1/2}] - \cos[\theta\Delta^{1/2}])}{g_{22}^{-1}\sin[\theta_1\Delta^{1/2}]\Delta g_{22}^{-1}\sin[\theta_2\Delta^{1/2}]}. \qquad (6.30)$$

In turn, from the condition (6.30) it follows

$$d\hat{\varphi}_2 =$$

$$\frac{g_{22}^{-1/2}\sin[\theta\Delta^{1/2}]\Delta d\hat{\theta}}{\sqrt{g_{11}^{-1/2}(\cos[\theta\Delta^{1/2}] - \cos[(\theta_1+\theta_2)\Delta^{1/2}])\Delta g_{11}^{-1/2}(\cos[(\theta_1-\theta_2) - \cos[\theta\Delta^{1/2}]}}. \qquad (6.31)$$

Since when varying $\hat{\varphi}_2$ from $\hat{\theta}$ to π the variable $\hat{\theta}$ varies in the range from $|\hat{\theta}_1 + \hat{\theta}_2|$ to $|\hat{\theta}_1 - \hat{\theta}_2|$, the above made redefinition transfroms the integral to the form

$$\frac{\Delta^2}{2} \int\limits_{|\theta_1+\theta_2|}^{\theta_1+\theta_2} \hat{P}_l^k(g_{11}^{-1/2}\cos[\theta\Delta^{1/2}])$$

$$\times \hat{T}_k \frac{g_{11}^{-1}\cos[\theta_1\Delta^{1/2}]\Delta g_{11}^{-1}\cos[\theta_2\Delta^{1/2}] - g_{11}^{-1/2}\cos[\theta\Delta^{1/2}]}{g_{22}^{-1}\sin[\theta_1\Delta^{1/2}]\Delta g_{22}^{-1}\sin[\theta_2\Delta^{1/2}]}$$

$$\times \frac{g_{22}^{-1/2}\sin[\theta\Delta^{1/2}]\Delta d(\hat{\theta}\Delta^{1/2})}{g_{11}^{-1/2}(\cos[\theta\Delta^{1/2}] - \cos[(\theta_1+\theta_2)\Delta^{1/2}])\Delta g_{11}^{-1/2}(\cos[(\theta_1-\theta_2)\Delta^{1/2}]\cos[\theta\Delta^{1/2}])}$$

$$= \hat{P}_l^k(g_{11}^{-1/2}cos[\theta_1\Delta^{1/2}])\Delta\hat{P}_l^{-k}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}]). \tag{6.32}$$

The expression in the denominator has a simple geometrical meaning: it is equal to the square of the spherical triangle with the sides $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}$, divided to $4\pi^2$.

6.3 Orthogonality relations

In this Section, we apply theorems of orthogonality and completeness of the system of matrix elements of pairwise nonequivalent irreducible isounitary representations of compact group to the group $\hat{SU}(2)$. Since dimension of the representation $\hat{T}_l(\hat{u})$ of the group $\hat{SU}(2)$ is $2l+1$, the functions $\sqrt{2l+1}\Delta\hat{t}^l_{mn}(\hat{u})$ form complete orthogonal normalized system in respect to invariant measure $d\hat{u}$ on this group. In other words, the functions $\hat{t}^l_{mn}(\hat{u})$ fulfill the relations

$$\int\limits_{\hat{SU}(2)} \hat{t}^l_{mn}(\hat{u})\Delta^2\hat{t}^{*k}_{pq}(\hat{u})d\hat{u} = \frac{\Delta^2}{(2l+1)}\delta_{lk}\delta_{mp}\delta_{nq}. \tag{6.33}$$

Inserting expression for the matrix elements

$$\hat{t}^l_{mn}(\hat{\varphi},\hat{\theta},\hat{\psi}) = \exp\{-i\Delta^{3/2}(m\varphi + n\psi)\}\hat{P}^l(g_{11}^{-1/2}\cos[\theta\Delta^{1/2}]) \tag{6.34}$$

into (6.33) and using the fact that the measure $d\hat{u}$ on the group $\hat{SU}(2)$ is given by

$$d\hat{u} = \frac{\Delta^4}{16\pi} g g_{22}^{-1/2} sin[\theta\Delta^{1/2}]d\hat{\varphi}d\hat{\theta}d\hat{\psi}, \tag{6.35}$$

we turn to the following specific cases.

(a) If $l \neq k$ or $m \neq p$ or $n \neq q$, then

$$\int\limits_{-2\pi}^{2\pi}\int\limits_{0}^{2\pi}\int\limits_{0}^{\pi} \hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}^{*k}_{mn}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])g_{22}^{-1/2}\sin[\theta\Delta^{1/2}]$$

$$\times\,\Delta^6\exp\{i\Delta^{3/2}(p-m)\varphi\}\exp\{i\Delta^{3/2}(q-n)\psi\}\psi d(\theta\Delta^{1/2})d(\varphi\Delta^{1/2})d(\psi\Delta^{1/2}). \tag{6.36}$$

(b) Let $p = m$ and $q = n$, then, at $l \neq k$,

$$\int\limits_{0}^{\pi} \hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}^{*k}_{mn}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])g_{11}^{-1/2}\cos[\hat{\theta}]\Delta^3 d(\theta\Delta^{1/2}) = 0. \tag{6.37}$$

Analogously, from (6.33) it follows

$$\int\limits_{0}^{\pi} |\hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])|^2 g_{11}^{-1/2}\sin[\theta\Delta^{1/2}]d(\theta\Delta^{1/2}) = \frac{2}{2l+1}. \tag{6.38}$$

Further, putting $\hat{x} = g_{11}^{-1/2}\cos[\theta\Delta^{1/2}]$ we get the orthogonality relations for $\hat{P}^l_{mn}(\hat{x})$:

$$\int\limits_{-1}^{1} \hat{P}^l_{mn}(\hat{x})\hat{P}^{*k}_{mn}(\hat{x})d(\hat{x}) = \frac{2}{2l+1}\delta_{lk}. \tag{6.39}$$

7 Recurrency relations for isoLegendre functions

In this Section, we derive the formulas relating the functions $\hat{P}^l_{mn}(\hat{z})$, indeces of which are differ from each other by one, that is, recurrency relations, which can be viewed as an infinitesimal form of the theorem of composition. These relations are then follow from the composition rules at infinitesimal $\hat{\theta}_2$.

To obtain the reccurency rules, we diffirentiate the equation below on $\hat{\theta}_2$ and put $\hat{\theta}_2 = 0$:

$$\hat{P}^l_{mn}(\hat{z}) = \sqrt{\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!}}\times$$

$$\times\frac{\Delta^7}{2\pi}\int\limits_{0}^{2\pi} d\varphi(g_{11}^{-1/2}\cos[\frac{\theta\Delta^{1/2}}{2}]\exp\frac{i\Delta^{1/2}}{2}$$

$$+ig_{22}^{-1/2}\sin[\frac{\theta\Delta^{1/2}}{2}]\exp\frac{-i\Delta^{1/2}}{2})^{l-n}(ig_{22}^{-1/2}\sin[\frac{\theta\Delta^{1/2}}{2}]\exp\frac{i\Delta^{1/2}}{2}$$

$$\times g_{11}^{-1/2}\cos[\frac{\theta\Delta^{1/2}}{2}]\exp\frac{-i\Delta^{1/2}\varphi}{2})^{l-n}\exp i\Delta^{1/2}\varphi. \tag{7.1}$$

First, we find

$$\frac{d}{d\hat{\theta}}[\hat{P}^l_{mn}(g-11^{-1/2}\cos[\theta\Delta^{1/2}])]_{|\hat{\theta}=0}$$

$$=\frac{\Delta^5}{4\pi}\sqrt{\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!}} \tag{7.2}$$

$$\times\int\limits_0^{2\pi}d\varphi(l-n)\exp-i\Delta^{3/2}(n+1)\varphi+(l+n)\exp-i\Delta^{3/2}(n-1)\varphi\exp\Delta^{3/2}m\varphi.$$

It is obvious that the r.h.s. of this equation is zero unless $m=n\pm1$. At $m=n+1$, from (7.2) we get

$$\frac{d}{d\hat{\theta}}[\hat{P}^{n+1}_n\cos[\frac{\theta\Delta^{1/2}}{2}]_{|\hat{\theta}=0}=\frac{i}{2}\Delta^{3/2}\sqrt{(l-n)(l+n+1)}. \tag{7.3}$$

Similarly,

$$\frac{d}{d\hat{\theta}}[\hat{P}^{n-1}_n(g_{11}^{-1/2}\cos[\frac{\theta\Delta^{1/2}}{2}]_{\hat{\theta}=0}=\frac{i}{2}\Delta^{3/2}\sqrt{(l+n)(l-n+1)}. \tag{7.4}$$

Now, we are ready to derive the recurrency relations.
Using the factorization

$$\hat{P}^l_{mn}[g_{11}^{-1/2}\cos[(\theta_1+\theta_2)\Delta^{1/2}]]=\sum_{k=-l}^{l}\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])$$
$$\Delta\hat{P}^l_{kn}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])$$

obtained in Sec. GS6, putting $\hat{\theta}_2=0$ and replacing $g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])$ by $\hat{z}$, we obtain the recurrency relation in the form

$$\sqrt{1-z^2}\frac{d\hat{P}^l_{mn}(\hat{z})}{dz}=-\frac{i}{2}\Delta^{3/2}[\sqrt{(l-n)(l+n+1)} \tag{7.5}$$

$$\times\hat{P}^l_{m,n+1}(\hat{z})+\sqrt{(l+n)(l-n+1)}\hat{P}^l_{m,n-1}(\hat{z})].$$

To derive the second recurrency relation, we use the particular case of the theorem of composition which corresponds to $\hat{\varphi}_2 = \frac{\pi}{2}$. Namely, we differentiate the formula

$$\exp -i\Delta^{3/2}(m\varphi + n\psi)\hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}]) =$$

$$\sum_{k=-l}^{l} i^{-k}\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}]$$

$$\Delta\hat{P}^l kn(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])$$

and put $\theta_2 = 0$. After strightforward computations, we have

$$i\Delta^3\left[m\frac{d\hat{\varphi}}{d\hat{\theta}_2} + n\frac{d\hat{\psi}}{d\hat{\theta}_2}\right]|_{\hat{\theta}_2=0}$$

$$\times\, \hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}] - \frac{d\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])}{d\hat{\theta}_1}\frac{d\hat{\theta}}{\hat{\theta}_2}|_{\hat{\theta}_2=0} \tag{7.6}$$

$$-\frac{1}{2}\Delta^{3/2}\sqrt{(l+n)(l-n+1)}\hat{P}^l_{m,n-1}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}] -$$

$$\sqrt{(l-n)(l+n+1)}\hat{P}^l_{m,n+1}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])$$

It remains to find $d\hat{\varphi}/d\hat{\theta}_2$ and $d\hat{\psi}/d\hat{\theta}_2$. To this end, we differentiate the equality $\cos[\theta\Delta^{1/2}] = \cos[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}]$. Since, at $\hat{\theta}_2 = 0$, we have $\hat{\theta} = \hat{\theta}_1$; $\hat{\varphi} = 0$, and $\hat{\psi} = \hat{\varphi}_2 = \frac{\pi}{2}$, it follows that $d\hat{\theta}/d\hat{\theta}_2|_{\hat{\theta}_2=0} = 0$. Similarly,

$$\frac{d\hat{\varphi}}{d\hat{\theta}_2}|_{\hat{\theta}_2=0} = \frac{g_{11}^{1/2}}{\cos[\theta_1\Delta^{1/2}]} \tag{7.7}$$

and

$$\frac{d\hat{\psi}}{d\hat{\theta}_2}|_{\hat{\theta}_2=0} = -(\frac{g_{11}}{g_{22}})^{1/2} ctan[\theta_1\Delta^{1/2}]$$

$$i\Delta\left[\frac{m-nz}{1-z^2}\right]\hat{P}^l_{mn}(\hat{z})$$

$$= \frac{1}{2}\left[\sqrt{(l+n)(l-n+1)}\hat{P}^l_{m,n-1}(\hat{z}) - \sqrt{(l-n)(l+n+1)}\hat{P}^l_{m,n+1}(\hat{z})\right]. \tag{7.8}$$

From the reccurency relations obtained above it is straightforward to write down the following recurrency relations:

$$\sqrt{1-z^2}\frac{d\hat{P}^l_{mn}(z)}{d\hat{z}} + \frac{nz-m}{\sqrt{1-z^2}}\hat{P}^l_{mn}(\hat{z}) = -i\Delta^{5/2}\sqrt{(l-n)(l+n+1)}\hat{P}^l_{m,n+1}(\hat{z}) \tag{7.9}$$

and, analogously,

$$\sqrt{1-z^2}\frac{d\hat{P}^l_{mn}(z)}{d\hat{z}} - \frac{nz-m}{\sqrt{1-z^2}}\hat{P}^l_{mn}(\hat{z}) = -i\Delta^{5/2}\sqrt{(l+n)(l-n+1)}\hat{P}^l_{m,n+1}(\hat{z}). \tag{7.10}$$

Due to the symmetry, we have from (7.9) and (7.10)

$$\sqrt{1-z^2}\frac{d\hat{P}^l_{mn}(z)}{d\hat{z}} + \frac{mz-n}{\sqrt{1-z^2}}\hat{P}^l_{mn}(\hat{z}) = -i\Delta^{5/2}\sqrt{(l-m)(l+m+1)}\hat{P}^l_{m,n+1}(\hat{z}) \tag{7.11}$$

and

$$\sqrt{1-z^2}\frac{d\hat{P}^l_{mn}(z)}{d\hat{z}} - \frac{mz-n}{\sqrt{1-z^2}}\hat{P}^l_{mn}(\hat{z}) = -i\Delta^{5/2}\sqrt{(l+m)(l-m+1)}\hat{P}^l_{m,n+1}(\hat{z}). \tag{7.12}$$

Adding (7.9) to (7.10), we obtain the recurrency relations for three $\hat{P}$'s:

$$\begin{aligned} 2\left[\frac{n-mz}{1-z^2}\right]\hat{P}^l_{mn}(\hat{z}) &= i\Delta^{3/2}\Bigg[\sqrt{(l+n)(l-n+1)}\hat{P}^l_{m,n-1}(\hat{z}) \\ &\quad - \sqrt{(l-n)(l+n+1)}\hat{P}^l_{m,n+1}(\hat{z})\Bigg], \end{aligned} \tag{7.13}$$

$$\begin{aligned} \sqrt{1-z^2}\hat{P}^l_{mn}(\hat{z}) &= -i\Delta^{5/2}\Bigg[\sqrt{(l+n)(l-n+1)}\hat{P}^l_{m,n-1}(\hat{z}) \\ &\quad + \sqrt{(l-n)(l+n+1)}\hat{P}^l_{m,n+1}(\hat{z})\Bigg]. \end{aligned} \tag{7.14}$$

Putting $m = 0$ in (7.9) and (7.10), and using

$$\hat{P}^l_{0n}(\hat{z}) = i^{-n}\Delta^2\sqrt{\frac{(l-n)}{(l+n)}}\hat{P}^n_l(\hat{z}) \tag{7.15}$$

we obtain, finally, the recurrency rules for the adjoint isoLegendre polynomials,

$$\sqrt{1-z^2}\frac{d\hat{P}^n_l(\hat{z})}{d\hat{z}} + \Delta^2\frac{nz}{1-z^2}\hat{P}^n_l(\hat{z}) = -\hat{P}^{n+1}_l(\hat{z}) \tag{7.16}$$

and

$$\sqrt{1-z^2}\frac{d\hat{P}^n_l(\hat{z})}{d\hat{z}} - \Delta^2\frac{n}{1-z^2}\hat{P}^n_l(\hat{z}) = -\Delta^3(l+n)(l-n+1)\hat{P}^{n-1}_l(\hat{z}). \tag{7.17}$$

8 The group $\hat{QU}(2)$

In this Section, we consider the group $\hat{QU}(2)$ consisting of isounimodular isoquasiunitary matrices representations of which lead to isoJacobi and isoLegendre functions.

8.1 Definitions

The representations of $\hat{QU}(2)$ are in many ways similar to that of the group $\hat{SU}(2)$. However, in contrast to $\hat{SU}(2)$, the group $\hat{QU}(2)$ is not compact, thus having continuous series of isounitary representations.

Similarly to the description of the group $\hat{SU}(2)$, we describe the group $\hat{QU}(2)$ as a set of isounimodular isoquasiunitary 2×2 matrices

$$\hat{g}_0 = \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \bar{\hat{\beta}} & \bar{\hat{\alpha}} \end{pmatrix}, \tag{8.1}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are given by (2.7), satisfying

$$\hat{g}_0 \Delta \hat{s} \Delta \hat{g}_0^* = \hat{s}, \tag{8.2}$$

where

$$\hat{s} = \begin{pmatrix} \Delta^{-1} & 0 \\ 0 & \Delta^{-1} \end{pmatrix}, \quad \hat{g}_0^* = \begin{pmatrix} \bar{\hat{\alpha}} & \hat{\beta} \\ \bar{\hat{\beta}} & \hat{\alpha} \end{pmatrix} \quad \det \hat{g}_0 = 1, \quad |\hat{\alpha}|^{\hat{2}} - |\hat{\beta}|^{\hat{2}} = \Delta^{-1}. \tag{8.3}$$

8.2 Parametrizations

The matrices $\hat{g}_0$ above have been defined by the complex numbers $\hat{\alpha}$ and $\hat{\beta}$. However, in various aspects it is suitable to define them by the isoEuler angles. Constraints on the isoEuler angles following from the requirement that $\hat{g}_0 \in \hat{QU}(2)$ are

$$\begin{aligned} & g_{11}^{-1/2} \cos[\frac{\theta\Delta^{1/2}}{2}]\Delta \exp\{-i\Delta^{3/2}(\varphi + \psi)/2\} \\ &= g_{11}^{-1/2} \overline{\cos[\theta\Delta^{1/2}/2]\Delta \exp\{-i\Delta^{3/2}(\bar{\varphi} + \bar{\psi})/2\}} \end{aligned} \tag{8.4}$$

and

$$g_{11}^{-1/2} \sin[\frac{\theta\Delta^{1/2}}{2}]\Delta \exp\{i\Delta^{3/2}(\varphi - \psi)/2\}$$

$$= -g_{22}^{-1/2}\sin[\frac{\bar{\theta}\Delta^{1/2}}{2}]\Delta\exp\{i\Delta^{3/2}(\bar{\varphi}-\bar{\psi})/2\}, \qquad (8.5)$$

which we rewrite in the following form:

$$\cos[\frac{i\theta\Delta^{1/2}}{2}] = \cos[\frac{\bar{\theta}\Delta^{1/2}}{2}]\exp\{i\Delta^{3/2}(\varphi-\bar{\varphi}+\psi-\bar{\psi})/2\} \qquad (8.6)$$

and

$$\sin[\frac{i\theta\Delta^{1/2}}{2}] = -\sin[\frac{i\bar{\theta}\Delta^{1/2}}{2}]\exp\{i\Delta^{3/2}(\bar{\psi}-\varphi+\psi-\bar{\varphi})/2\}. \qquad (8.7)$$

The angles $\varphi-\bar{\varphi}+\psi-\bar{\psi}$ and $\bar{\psi}-\varphi+\psi-\bar{\varphi}$ are real. So, if $\hat{g}_0 = \hat{g}_0(\hat{\varphi},\hat{\theta},\hat{\psi}) \in \hat{QU}(2)$ then $\cos[\theta\Delta^{1/2}/2]$ is an imaginary number, *i.e.* $\hat{\tau} = i\hat{\theta}$ is real.

Taking into account the constraints (8.6) and (8.7), we obtain the following ranges for the parameters:

$$0 \le \hat{\varphi} < 2\pi, \qquad 0 \le \hat{\tau} < \infty, \qquad -2\pi \le \hat{\psi} < 2\pi. \qquad (8.8)$$

In terms of these parameters, the matrix $\hat{g}_0$ is

$$\hat{g}_0 = \begin{pmatrix} g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}]\Delta e^{\frac{i\Delta^{3/2}(\varphi+\psi)}{2}} & -i\Delta^2 g_{22}^{-1/2}\sin[i\tau\Delta^{1/2}]e^{\frac{i\Delta^{3/2}(\varphi-\psi)}{2}} \\ -i\Delta^2 g_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}]e^{\frac{i\Delta^{3/2}(\psi-\varphi)}{2}} & g_{11}^{-1/2}\cos[\frac{\psi\Delta^{1/2}}{2}]\Delta e^{\frac{-i\Delta^{3/2}(\varphi+\psi)}{2}} \end{pmatrix}. \qquad (8.9)$$

Thus, we see that the group $\hat{QU}(2)$ is one of the real types of subgroups of $\hat{SL}(2,C)$. In the following, we use the parameters (8.8) instead of the isoEuler angles $(\hat{\varphi},\hat{\theta},\hat{\psi})$.

Let us find the transformation laws for these parameters under the multiplying of two elements of $\hat{QU}(2)$. We introduce the notation $\hat{g}_{01} = (0,\hat{\tau}_1,0)$ and $\hat{g}_{02} = (\hat{\varphi}_2,\hat{\tau}_2,0)$ so that $\hat{g}_{01}\Delta\hat{g}_{02} = (\hat{\varphi},\hat{\tau},\hat{\psi})$. Using the formulas (2.16)-(2.18) we find

$$\cos[i\tau\Delta^{1/2}] == \cos[i\tau_1\Delta^{1/2}]\Delta\cos[i\tau_2\Delta^{1/2}]g_{11}^{-1/2} \qquad (8.10)$$

$$-\sin[i\tau_1\Delta^{1/2}]\Delta\sin[i\tau_2\Delta^{1/2}]\Delta g_{11}^{-1/2}\cos[\varphi_2\Delta^{1/2}],$$

$$\exp\{i\Delta^{1/2}(\varphi+\psi)/2\} = \Delta^2(\frac{g_{11}^{-1}\cos[i\tau_1\Delta^{1/2}]\Delta\cos[i\tau_2\Delta^{1/2}]\exp\{i\Delta^{1/2}\varphi_2/2\}}{\cos[i\tau\Delta^{1/2}]}$$

$$+\frac{g_{22}^{-1}\sin[i\tau_1\Delta^{1/2}]\Delta\sin[i\tau_2\Delta^{1/2}]\exp\{-i\Delta^{1/2}\varphi_2/2\})}{\cos[i\tau\Delta^{1/2}]} \qquad (8.11)$$

and

$$\exp\{i\Delta^{1/2}\varphi\} = \Delta^{-1/2}(\frac{\sin[i\tau_1\Delta^{1/2}]\Delta\cos[i\tau_2\Delta^{1/2}]}{g_{22}^{-1/2}\sin[i\tau\Delta^{1/2}]} \tag{8.12}$$

$$+\frac{\Delta^{-1/2}\cos[i\tau_1\Delta^{1/2}]\Delta\sin[i\tau_2\Delta^{1/2}]\Delta g_{11}^{-1/2}\cos[\varphi_2\Delta^{1/2}]}{g_{22}^{-1/2}\sin[i\tau\Delta^{1/2}]}$$

$$-\frac{ig_{22}^{-1}\sin[i\tau_2\Delta^{1/2}]\Delta\sin[\varphi_2\Delta^{1/2}]}{g_{22}^{-1/2}\sin[i\tau\Delta^{1/2}]}).$$

It is easy to check that the element $\hat{g}_0(\hat{\varphi},\hat{\tau},\hat{\psi})$ is an inverse of $\hat{g}_0(\pi - \hat{\varphi},\hat{\tau},-\pi-\hat{\psi})$.

8.3 Relation to the group $\hat{SH}(3)$

Let us define the group $\hat{SH}(3)$ as the group of isolinear transformation of three dimensional isoEuclidean space $\hat{E}_3$ acting transitively on (iso)hyperboloids and (iso)conics. This transformation is an isohyperbolic one.

The relation between the groups $\hat{QU}(2)$ and $\hat{SH}(3)$ is similar to that between $\hat{SU}(2)$ and $\hat{S0}(3)$. Namely, to every point $\hat{x}(\hat{x}_1,\hat{x}_2,\hat{x}_3) \in \hat{E}_3$ we associate the quasiuntary matrix

$$\hat{h}_x = \begin{pmatrix} \hat{x}_1 & \hat{x}_2 + i\hat{x}_3 \\ \hat{x}_2 - i\hat{x}_3 & -\hat{x}_1 \end{pmatrix}. \tag{8.13}$$

Then,

$$\hat{T}(\hat{g}_0)\Delta\hat{h}_x = \hat{g}_0\Delta\hat{h}_x^*\hat{g}_0. \tag{8.14}$$

Accordingly,

$$\hat{T}(\hat{g}_0)\Delta\hat{h}_x = \begin{pmatrix} \Delta^{-1}y_1 & \Delta y_2 + iy_3 \\ \Delta y_2 - iy_3 & \Delta y_1 \end{pmatrix}, \tag{8.15}$$

where $\hat{x} = g_{11}^{1/2}x$, $\hat{y} = g_{22}^{1/2}y$, and $\hat{y}(\hat{y}_1,\hat{y}_2,\hat{y}_3) in \hat{E}_3$.

9 Irreps of $\hat{QU}(2)$

9.1 Description of the irreps

Denote $\hat{\chi} = (l,\varepsilon)$, where l is complex number and $\varepsilon = 0, 1/2$. With every $\hat{\chi}$ we associate the space $D_{\hat{\chi}}$ of functions $\hat{\varphi}(\hat{z})$ of complex variable $\hat{z} = \hat{x} + i\hat{y}$ such that:

(1) $\hat{\varphi}(\hat{z})$ is of C^∞ class on $\hat{x}$ and $\hat{y}$ at every point $\hat{z} = \hat{x} + i\hat{y}$ except for $\hat{z} = 0$;

(2) for any $a > 0$ the following equation is satisfied:

$$\hat{\varphi}(a\Delta\hat{z}) = a^{2\Delta}\Delta\hat{\varphi}(\hat{z}). \tag{9.1}$$

(3) $\hat{\varphi}(\hat{z})$ is an even (odd) function at $\varepsilon = 0(1/2)$,

$$\hat{\varphi}(-\hat{z}) = (-\Delta^{-1})^{2\varepsilon}\Delta\hat{\varphi}(\hat{z}). \tag{9.2}$$

For subsequent purposes, we realize the space $D_{\hat{\chi}}$ on a circle. Namely, with every function $\hat{\varphi}(\hat{z})$ we associate the function $\hat{f}$ such that, at $\varepsilon = 0$,

$$\hat{f}(\exp\{i\theta\Delta^{1/2}\}) = \hat{\varphi}(\exp\{i\theta\Delta^{1/2}\}) \tag{9.3}$$

and, at $\varepsilon = 1/2$,

$$\hat{f}(\exp\{i\theta\Delta^{1/2}\}) = \exp\{i\theta\Delta^{1/2}\}\Delta\hat{\varphi}(\exp\{i\theta\Delta^{1/2}\}). \tag{9.4}$$

Thus, the space $D_{\hat{\chi}}$ can be represented as the space D of functions on circle.

9.2 Representations $\hat{T}_{\hat{\chi}}(\hat{g}_0)$

To every element

$$\hat{g}_0 = \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \bar{\hat{\beta}} & \bar{\hat{\alpha}} \end{pmatrix}$$

of the group $\hat{Q}U(2)$ we associate the operator in the space $D_{\hat{\chi}}$,

$$\hat{T}_{\hat{\chi}}(\hat{g}_0)\Delta\hat{\varphi}(\hat{z}) = \hat{\varphi}(\hat{\alpha}\hat{z} + \bar{\hat{\beta}}\hat{z}). \tag{9.5}$$

Clearly, function $\hat{T}_{\chi}(\hat{g}_0)\Delta\hat{\varphi}(\hat{z})$ has the same homogeneity degree as the function $\hat{\varphi}(\hat{z})$, and so the operator $\hat{T}_{\hat{\chi}}(\hat{g}_0)$ is an automorphism of the space $D_{\hat{\chi}}$. Also, it is easy to verify that

$$\hat{T}_{\chi}(\hat{g}_{01})\Delta\hat{T}_{\chi}(\hat{g}_{01}) = \hat{T}_{\chi}(\hat{g}_{01}\Delta\hat{g}_{02}). \tag{9.6}$$

Action of the operator $\hat{T}_{\hat{\chi}}(\hat{g}_0)$ can then be strightforwardly derived. Namely, for $\hat{\chi} = (l, 0)$ we have

$$\hat{T}_{\hat{\chi}}(\hat{g})\Delta\hat{f}(\exp\{i\theta\Delta^{1/2}\})$$

$$= |\hat{\beta}\Delta\exp\{i\theta\Delta^{1/2}\} + \bar{\hat{\alpha}}|^{2l}\Delta^{2l+1}\hat{f}\left(\frac{\hat{\alpha}\exp\{i\theta\Delta^{1/2}\} + \bar{\hat{\beta}}}{\hat{\beta}\Delta\exp\{i\theta\Delta^{1/2}\} + \bar{\hat{\alpha}}}\right), \qquad (9.7)$$

and for $\hat{\chi} = (l, 1/2)$

$$\hat{T}_{\hat{\chi}}(\hat{g})\Delta\hat{f}(\exp\{i\theta\Delta^{1/2}\})$$

$$= |\hat{\beta}\Delta\exp\{i\theta\Delta^{1/2}\}+\bar{\hat{\alpha}}|^{2l-1}\Delta^{2l+1}(\hat{\beta}\Delta\exp\{i\theta\Delta^{1/2}\}+\bar{\hat{\alpha}})\hat{f}\left(\frac{\hat{\alpha}\exp\{i\theta\Delta^{1/2}\} + \bar{\hat{\beta}}}{\hat{\beta}\Delta\exp\{i\theta\Delta^{1/2}\} + \bar{\hat{\alpha}}}\right). \qquad (9.8)$$

10 Matrix elements of the irreps of $\hat{QU}(2)$ and iso-Jacobi functions

10.1 The matrix elements

Let us choose the basis $\exp\{-im\theta\Delta^{3/2}\}$ in space $\hat{D}_{\hat{\chi}}$, and define the matrix elemets of $\hat{T}_{\hat{\chi}}(\hat{h})$, where

$$\hat{h} = \begin{pmatrix} \exp\{it\Delta^{1/2}/2\} & 0 \\ 0 & \exp\{-it\Delta^{1/2}/2\} \end{pmatrix}. \qquad (10.1)$$

In the same manner as for $\hat{g}$ of $\hat{QU}(2)$ we can represent

$$\hat{h} = \begin{pmatrix} e^{i\varphi\Delta^{1/2}/2} & 0 \\ 0 & e^{-i\varphi\Delta^{1/2}/2} \end{pmatrix}\Delta\begin{pmatrix} g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] & -ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}] \\ -ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}] & g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] \end{pmatrix} \qquad (10.2)$$

$$\times\Delta\begin{pmatrix} e^{\frac{i\psi\Delta^{1/2}}{2}} & 0 \\ 0 & e^{\frac{-i\psi\Delta^{1/2}}{2}} \end{pmatrix},$$

where φ, τ, and $\hat{\psi}$ are isoEuler angles of $\hat{g}_0$. From (10.4) we define $\hat{T}_{\hat{\chi}}(\hat{g}_\tau)$, namely,

$$\begin{pmatrix} g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] & -ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}] \\ -ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}] & g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] \end{pmatrix}. \qquad (10.3)$$

Then, straighforward calculations yield [4]

$$\hat{t}^{\hat{\chi}}_{mn} = \frac{\Delta^{4+l}}{2\pi}\int_0^{2\pi} d\hat{\theta}\,(g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] - ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}]\exp\{i\theta\Delta^{1/2}\})^{l+n+\varepsilon}$$

$$\times\,(g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] \tag{10.4}$$

$$-ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}]\exp\{-i\theta\Delta^{1/2}\})^{l-n-\varepsilon}\exp\{i\theta(m-n)\Delta^{3/2}\}.$$

Introduce the function $B^l_{mn}(isocosh\hat{\tau})$ defining

$$B^l_{mn}(g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}]) = \frac{\Delta^{4+l}}{2\pi}\int_0^{2\pi} d\hat{\theta}\,(g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] - ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}]$$

$$\times\exp\{i\theta\Delta^{1/2}\})^{l+n}(g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] \tag{10.5}$$

$$-ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}]\exp\{-i\theta\Delta^{1/2}\})^{l-n}\exp\{i\theta(m-n)\Delta^{3/2}\}.$$

Comparing (10.4) and (10.5) we have

$$\hat{t}^{\hat{\chi}}_{mn} = B^l_{mn}(g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}]) \quad \hat{\chi} = (l,\varepsilon), \tag{10.6}$$

where

$$m' = m+\varepsilon, \quad n' = n+\varepsilon, \quad 0 \le \tau < \infty, \tag{10.7}$$

l is a complex number, m and n are simultaneously integer or half-integer numbers. From the expansion (10.4) it follows that

$$\hat{T}_{\hat{\chi}}(\hat{g}_0) = \hat{T}_{\hat{\chi}}(\hat{h}\varphi)\Delta\hat{T}_{\hat{\chi}}(\hat{g}_{\hat{\tau}})\Delta\hat{T}_{\hat{\chi}}(\hat{h}\psi). \tag{10.8}$$

So we can write

$$\hat{t}^{\hat{\chi}}_{mn}(\hat{\varphi},\hat{\tau},\hat{\psi}) = \exp\{-i\Delta^{3/2}(m'\varphi+n'\psi)\}B^l_{mn}(g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}]), \tag{10.9}$$

where m',n', and τ' are defined according to (10.7). Since $B^l_{mn}(\hat{z})$ plays the same role for $\hat{QU}(2)$ as the function $P^l_{mn}(\hat{z})$ for $\hat{SU}(2)$, we call $B^l_{mn}(\hat{z})$ *isoJacobi function* of the variable $\hat{z} = g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}]$.

11 IsoJacobi function $\hat{B}^l_{mn}(\hat{z})$

Integral representation of the isoJacobi function $B^l_{mn}(\hat{z})$ can be readily derived (see [4] for the usual case),

$$B^l_{mn}(g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}]) =$$

$$\frac{\Delta^{4+l}}{2\pi}\int_{-\tau}^{\tau}\frac{\exp\{(l-n+1/2)\Delta^{3/2}t\}}{\sqrt{2\Delta g_{11}^{-1/2}(\cos[i\tau\Delta^{1/2}]-\cos[it\Delta^{1/2}])}}\Big[\hat{z}_{+}^{m-n}\Delta(g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}/2]$$
$$-\hat{z}_{+}ig_{22}^{-1/2}\sin[i\tau\Delta^{1/2}/2])^{2n}+\hat{z}_{-}^{m-n}\Delta(g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}/2]$$
$$-\hat{z}_{-}igg_{22}^{-1/2}\sin[i\tau\Delta^{1/2}/2])^{2n}\Big]d\hat{t}, \tag{11.1}$$

where

$$\hat{z}_{\pm}=\frac{\exp\{\tau\Delta^{1/2}\}-g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}]}{g_{22}^{-1/2}\sin[i\tau\Delta^{1/2}]}$$
$$\frac{\pm i\exp\{it\Delta^{1/2}/2\}\Delta\sqrt{2\Delta g_{11}^{-1/2}(\cos[i\tau\Delta^{1/2}]-\cos[it\Delta^{1/2}])}}{g_{22}^{-1/2}\sin[i\tau\Delta^{1/2}]}. \tag{11.2}$$

As one can see, the representation (11.1) is simplified when $n=m$ and also when $n=0$.

When $n=m$ we have directly from (11.1)

$$\hat{B}^{l}_{nn}(\hat{z})=\frac{\Delta^{5/2}}{\pi}\int_{0}^{\tau}\frac{g_{11}^{-1/2}\cos[(l-n+\frac{1}{2}\Delta^{1/2}]\Delta g_{11}^{-1/2}\cos[(2n\Delta^{3/2}\alpha)\Delta^{1/2}]d\hat{t}}{\sqrt{g_{11}^{-1}\cos[\frac{i\tau\Delta^{1/2}}{2}]+g_{22}^{-1}\sin[\frac{i\tau\Delta^{1/2}}{2}]}}. \tag{11.3}$$

When $n=0$ we have

$$\hat{B}^{l}_{m0}(\hat{z})=\frac{\Delta^{\frac{5}{2}+m}}{2\pi}\int_{-\tau}^{\tau}\frac{\exp\{(l+\frac{1}{2})\Delta^{3/2}t\}(\hat{z}_{+}^{\hat{m}}+\hat{z}_{-}^{\hat{m}})d\hat{t}}{2\Delta(g_{11}^{-1/2}(\cos[i\tau\Delta^{1/2}]-cos[it\Delta^{1/2}]))}. \tag{11.4}$$

Particularly, when in addition $m=0$ we have

$$\hat{B}^{l}_{00}(\hat{z})=\frac{\Delta^{3/2}}{\pi}\int_{0}^{\tau}\frac{\cos[i(l+1/2)t]dt}{\sqrt{\cos^{\hat{2}}[\frac{i\tau\Delta^{1/2}}{2}]-\cos^{\hat{2}}[\frac{it\Delta^{1/2}}{2}]}}. \tag{11.5}$$

12 IsoJacobi function $\hat{B}^{l}_{00}$

Let us put $m=n=0$ in (10.6). Then

$$\hat{t}^{\chi}_{00}(\hat{g}_0)=\hat{B}^{l}_{00}(\hat{z}). \tag{12.1}$$

We call $\hat{B}^{l}_{00}(\hat{z})$ isoJacobi function with index l and denote it simply $\hat{B}_{l}(\hat{z})$, namely,

$$\hat{B}_{l}(\hat{z})=\hat{t}^{\chi}_{00}(0,\hat{\tau},0,)=\hat{B}^{l}_{00}(\hat{z}), \tag{12.2}$$

where $\hat{\chi} = (l, 0)$ and $\hat{z} = g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}]$.

The following integral representations for the isoJacobi function $\hat{B}^l(\hat{z})$ can be written:

$$\hat{B}_l(\hat{z}) = \frac{\Delta^{l+1}}{2\pi}\int_0^\pi (g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}] - g_{22}^{-1/2}\sin[i\tau\Delta^{1/2}]g_{11}^{-1/2}\cos[i\theta\Delta^{1/2}])^l d\hat{\theta}, \tag{12.3}$$

$$\hat{B}_l(\hat{z}) = \frac{\Delta^{l+1}}{2\pi i}\int (g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}] - i\frac{\hat{z}^2+1}{2\Delta^2\hat{z}}g_{22}^{-1/2}\sin[i\tau\Delta^{1/2}])^l\frac{d\hat{z}}{\hat{z}}, \tag{12.4}$$

$$\hat{B}_l(\hat{z}) = \frac{\Delta^l\sqrt{2}g_{22}^{-1/2}\sin[\pi\Delta^{1/2}l]}{\pi}\int_0^\infty \frac{(g_{11}^{-1/2}\cos[i(l+\frac{1}{2})t\Delta^{1/2}]d\hat{t}}{\sqrt{\cos[it\Delta^{1/2}])^l + \cos[i\tau\Delta^{1/2}]}}. \tag{12.5}$$

From (12.3) it can be seen that when l is integer the isoJacobi function $\hat{B}_l(\hat{z})$ coincides with the isoLegendre polynomial,

$$\hat{B}_l(\hat{z}) = \hat{P}_l(\hat{z}), \tag{12.6}$$

which has been considered in Secs. 2-7.

12.1 Symmetry relations for $\hat{B}^l_{mn}(\hat{z})$ and $\hat{B}_l(\hat{z})$

Similarly to the isoLegendre polynomials $\hat{P}^l_{mn}(\hat{z})$, the isoJacobi functions $\hat{B}^l_{mn}(\hat{z})$ satisfy the following symmetry relations:

$$\hat{B}^l_{mn}(\hat{z}) = \hat{B}^l_{-m-n}(\hat{z}) \tag{12.7}$$

and

$$\hat{B}_l(\hat{z}) = \hat{B}_{-l-1}(\hat{z}). \tag{12.8}$$

13 Functional relations for $\hat{B}^l_{mn}(\hat{z})$

Functional relations for isoJacobi functions $\hat{B}^l_{mn}(\hat{z})$ can be derived in a similar fashion as it for isoLegendre functions $\hat{P}^l_{mn}(\hat{z})$. Particularly, we have

$$\exp\{-i\Delta^{3/2}(m\varphi + n\psi)\}\hat{B}^l_{mn}(\hat{z}) = \sum_{k=-\infty}^{\infty} \exp -i\Delta^{3/2}k\varphi_2\hat{B}^l_{mk}(\hat{z}_1)\Delta\hat{B}^l_{kn}(\hat{z}_2), \tag{13.1}$$

where $\hat{z} = g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}]$, $\hat{z}_1 = g_{11}^{-1/2}\cos[i\tau_1\Delta^{1/2}]$, $\hat{z}_2 = g_{11}^{-1/2}\cos[i\tau_2\Delta^{1/2}]$, and τ, τ_1, τ_2, φ, and ψ are defined due to eqs. (8.10)-(8.12).

So, as a consequence of (13.1) we have the following particular cases.

(a) Let $\hat{\varphi}_2 = 0$, then $\hat{\tau} = \hat{\tau}_1 + \hat{\tau}_2$, $\hat{\varphi} = \hat{\psi} = 0$, and we have

$$\hat{B}^l_{mn}(g_{11}^{-1/2}\cos[i(\tau_1+\tau_2)\Delta^{1/2}]) = \sum_{k=-\infty}^{\infty} \hat{B}^l_{mk}(g_{11}^{-1/2}\cos[i\tau_1\Delta^{1/2}])\Delta \tag{13.2}$$

$$\times \hat{B}^l_{kn}(g_{11}^{-1/2}\cos[i\tau_2\Delta^{1/2}]).$$

(b) Let $\hat{\varphi}_2 = \pi$, then $\hat{\tau}_1 \geq \hat{\tau}_2$, $\hat{\tau} = \hat{\tau}_1 - \hat{\tau}_2$, $\hat{\varphi} = 0$, $\psi = \pi$, and we have

$$\hat{B}^l_{mn}(g_{11}^{-1/2}\cos[i(\tau_1-\tau_2)\Delta^{1/2}]) = \sum_{k=-\infty}^{\infty} \hat{B}^l_{mk}(g_{11}^{-1/2}\cos[i\tau_1\Delta^{1/2}])\Delta^2 \tag{13.3}$$

$$\times \hat{B}^l_{kn}(g_{11}^{-1/2}\cos[i\tau_2\Delta^{1/2}]).$$

(c) Particularly, when in addition $\hat{\tau}_1 = \hat{\tau}_2$, we have

$$\sum_{k=-\infty}^{\infty} \hat{B}^l_{mk}(g_{11}^{-1/2}\cos[i\tau_1\Delta^{1/2}])\Delta^2 \hat{B}^l_{kn}(g_{11}^{-1/2}\cos[i\tau_2\Delta^{1/2}]) = \hat{B}^l_{mn}(1) \tag{13.4}$$

$$= \hat{\delta}_{mn} \equiv \delta_{mn}\Delta^{-1}.$$

Theorem of composition for isoLegendre function.

Let us define isoLegendre function and adjoint isoLegendre function as follows

$$\hat{B}_l(\hat{z}) = \hat{B}^l_{00}(\hat{z}) \tag{13.5}$$

and

$$\hat{B}^m_l(\hat{z}) = \frac{\hat{\Gamma}(l+m+1)}{\hat{\Gamma}(l+1)}\Delta \hat{B}^{m0}_l(\hat{z}), \quad \hat{B}^l_{m0}(\hat{z}) = \frac{\hat{\Gamma}(l+1)}{\hat{\Gamma}(l-m+1)}\Delta \hat{B}^{0m}_l(\hat{z}). \tag{13.6}$$

Putting $m = n = 0$ in (13.1) and using (13.5) and (13.6) we obtain

$$\hat{B}_l(\hat{z}) = \frac{\hat{\Gamma}(l-k+1)}{\hat{\Gamma}(l+k+1)}\Delta^3 e^{-i\Delta^{3/2}k\varphi_2}\hat{B}^k_l(\hat{z}_1)\hat{B}^k_l(\hat{z}_2), \tag{13.7}$$

where

$$g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}] = g_{11}^{-1}\cos[i\tau_1\Delta^{1/2}]\Delta g_{11}^{-1}\cos[i\tau_2\Delta^{1/2}]+ \tag{13.8}$$

$$g_{22}^{-1}\sin[i\tau_1\Delta^{1/2}]\Delta^2\sin[i\tau_2\Delta^{1/2}]g_{11}^{-1}\cos[i\varphi_2\Delta^{1/2}].$$

The composition formula for the adjoint isoLegendre function follows from (13.1) with $n = 0$, namely, we have

$$\hat{B}_l(\hat{z}) = \frac{\hat{\Gamma}(l+m+1)}{\hat{\Gamma}(l+k+1)}\Delta^3 e^{-i\Delta^{3/2}k\varphi_2}\hat{B}_l^k(\hat{z}_1)\hat{B}_l^k(\hat{z}_2), \tag{13.9}$$

where

$$\hat{\Gamma}(l+m+1) = \hat{\Gamma}(l+m)\Delta(l+m) and \hat{\Gamma}(l+m+1) = \int_0^\infty e^{-l-m}\Delta^2\hat{x}^{l+m-1}dx. \tag{13.10}$$

Multiplication formula.

Multiplying both sides of the equation (13.1) by $\exp\{i\Delta^{3/2}k\varphi_2\}$ we obtain

$$\hat{B}^l_{mk}(\hat{z}_1)\hat{B}^l_{mk}(\hat{z}_2) = \frac{\Delta^2}{2\pi}\int_0^{2\pi} d\varphi_2\, e^{-i\Delta^{3/2}(k\varphi_2 - m\varphi - n\psi)}\hat{B}^l_{mn}(\hat{z}). \tag{13.11}$$

Putting $m = n = 0$ in (13.11) and using the symmetry relations we get

$$\hat{B}_l^k(\hat{z}_1)\hat{B}_l^{-k}(\hat{z}_2) = \frac{\Delta^2}{2\pi}\int_0^{2\pi} e^{-i\Delta^{3/2}k\varphi_2} \tag{13.12}$$

$$\hat{B}^l_{mn}(\hat{z}_1\Delta\hat{z}_2 + \hat{z}_3\Delta\hat{z}_4\Delta\hat{z}_5)d\varphi_2,$$

where $\hat{z}_1 = g_{11}^{-1/2}\cos[i\tau_1\Delta^{1/2}]$, $\hat{z}_2 = g_{11}^{-1/2}\cos[i\tau_2\Delta^{1/2}]$, $\hat{z}_3 = g_{11}^{-1/2}\sin[i\tau_1\Delta^{1/2}]$, $\hat{z}_4 = g_{11}^{-1/2}\sin[i\tau_2\Delta^{1/2}]$, and $\hat{z}_5 = g_{11}^{-1/2}\cos[i\varphi_2\Delta^{1/2}]$. Particularly,

$$\hat{B}_l(\hat{z}_1)\hat{B}_l(\hat{z}_2) = \frac{\Delta}{2\pi}\int_0^{2\pi} \hat{B}^l_{mn}(\hat{z}_1\Delta\hat{z}_2 + \hat{z}_3\Delta\hat{z}_4\Delta\hat{z}_5)d\varphi_2. \tag{13.13}$$

14 Recurrency relations for $\hat{B}^l_{mn}$

Recurrency relations for $\hat{B}^l_{mn}$ can be derived in the same manner as it for $\hat{P}^l_{mn}$. So, we do not represent the calculations here, and write down the final results.

$$\sqrt{\hat{z}^2-1}\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}} = \frac{(l+n)}{2}\Delta\hat{B}^l_{m,n-1}(\hat{z}) + \frac{(l-n)}{2}\Delta\hat{B}^l_{m,n+1}(\hat{z}), \tag{14.1}$$

$$\frac{m-n\Delta\hat{z}}{\sqrt{\hat{z}^2-1}}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}} = -\frac{(l+n)}{2}\Delta\hat{B}^l_{m,n+l}(\hat{z}) + \frac{(l-n)}{2}\Delta\hat{B}^l_{m,n+1}(\hat{z}). \tag{14.2}$$

From (14.1) and (14.2) we have

$$\sqrt{\hat{z}^2-1}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}}-\frac{m-n\Delta\hat{z}}{\sqrt{\hat{z}^2-1}}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}}=(l-n)\Delta\hat{B}^l_{m,n+1}(\hat{z}) \quad (14.3)$$

and

$$\sqrt{\hat{z}^2-1}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}}+\frac{m-n\Delta\hat{z}}{\sqrt{\hat{z}^2-1}}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}}=(l+n)\Delta\hat{B}^l_{m,n+1}(\hat{z}). \quad (14.4)$$

Using the symmetry relations we have

$$\sqrt{\hat{z}^2-1}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}}+\frac{n-m\Delta\hat{z}}{\sqrt{\hat{z}^2-1}}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}}=(l+m+1)\Delta\hat{B}^l_{m,n+1}(\hat{z}) \quad (14.5)$$

and

$$\sqrt{\hat{z}^2-1}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}}+\frac{m-n\Delta\hat{z}}{\sqrt{\hat{z}^2-1}}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}}=(l-m+1)\Delta\hat{B}^l_{m,n+1}(\hat{z}). \quad (14.6)$$

Also,

$$(l-n)\hat{B}^l_{m,n+1}(\hat{z})-(l+n)\hat{B}^l_{m,n-1}(\hat{z})=\frac{2(m-n\Delta\hat{z})}{\sqrt{\hat{z}^2-1}}\Delta^2B^l_{mn}(\hat{z}), \quad (14.7)$$

$$(l+m+1)\hat{B}^l_{m+1,n}(\hat{z})-(l-m+1)\hat{B}^l_{m-1,n}(\hat{z})=\frac{2(n-m\Delta\hat{z})}{\sqrt{\hat{z}^2-1}}\Delta^2B^l_{mn}(\hat{z}). \quad (14.8)$$

The differential equation satisfied by isoJacobi function is

$$\sqrt{\hat{z}^2-1}\frac{d^2\hat{B}^l_{mn}(\hat{z})}{d\hat{z}^2}-2z\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}}-\frac{m^2+n^2-2mn\Delta}{z^2-1}\Delta\hat{B}^l_{mn}(\hat{z})=l(l+1)\hat{B}^l_{mn}(\hat{z}). \quad (14.9)$$

The differential equation satisfied by adjoint isoLegendre function is

$$\sqrt{\hat{z}^2-1}\frac{d^2\hat{B}^m_l(\hat{z})}{d\hat{z}^2}-2z\Delta\frac{d\hat{B}^m_l(\hat{z})}{d\hat{z}}-\frac{m^2\Delta^2}{z^2-1}\Delta\hat{B}^m_l(\hat{z})=l(l+1)\Delta\hat{B}^m_l(\hat{z}), \quad (14.10)$$

and the equation satisfied by isoLegendre function is

$$\sqrt{\hat{z}^2-1}\frac{d^2\hat{B}_l(\hat{z})}{d\hat{z}^2}-2z\Delta\frac{d\hat{B}_l(\hat{z})}{d\hat{z}}=l(l+1)\Delta\hat{B}_l(\hat{z}). \quad (14.11)$$

15 The group $\hat{M}(2)$

In this Section, we consider linear transformations of isoEuclidean plane.

15.1 Definitions

The motion of isoEuclidean plane $\hat{E}^2$ is similar to that of the ordinary Euclidean plane E^2 so the definition of the group $\hat{M}(2)$ is similat to that of $\hat{M}(2)$.

Choosing local coordinates $(\hat{x}, \hat{y})$ on $\hat{E}^2$, we write the motion $\hat{g} : (\hat{x}, \hat{y}) \to (\hat{x}', \hat{y}')$ in th following form:

$$\begin{aligned} \hat{x}' &= \hat{x}\Delta g_{11}^{-1/2} \cos[\alpha\Delta^{1/2}] - \hat{y}\Delta g_{22}^{-1/2} \sin[\alpha\Delta^{1/2}] + a, \\ \hat{y}' &= \hat{x}\Delta g_{22}^{-1/2} \sin[\alpha\Delta^{1/2}] - \hat{y}\Delta g_{11}^{-1/2} \cos[\alpha\Delta^{1/2}] + b, \end{aligned} \tag{15.1}$$

where

$$\hat{x} = g_{11}^{1/2} x, \quad \hat{y} = g_{22}^{1/2} y, \tag{15.2}$$

so that, in an explicit form,

$$\begin{aligned} \hat{x}' &= \hat{x}(g_{11}^{1/2} g_{22}) \cos[\alpha\Delta^{1/2}] - \hat{y}(g_{11}^{1/2} g_{22}) \sin[\alpha\Delta^{1/2}] + a, \\ \hat{y}' &= \hat{x}\Delta g_{11}^{3/2} \sin[\alpha\Delta^{1/2}] - \hat{y}\Delta(g_{11}^{1/2} g_{22}) \cos[\alpha\Delta^{1/2}] + b. \end{aligned} \tag{15.3}$$

Here, a, b, and α parametrize the motion $\hat{g}$ so that every element $\hat{g} \in \hat{M}(2)$ can be defined by the three parameters having the following ranges:

$$-\infty < a < \infty, \quad -\infty < b < \infty, \quad 0 \leq \alpha < 2\pi. \tag{15.4}$$

Another realization of $\hat{M}(2)$ comes with the identification of $\hat{g}(a, b, \alpha)$ with the matrix

$$\hat{T}(\hat{g}) = \begin{pmatrix} g_{11}^{-1/2} \cos[\alpha\Delta^{1/2}] & -g_{22}^{-1/2} \sin[\alpha\Delta^{1/2}] & a \\ g_{22}^{-1/2} \sin[\alpha\Delta^{1/2}] & g_{11}^{-1/2} \cos[\alpha\Delta^{1/2}] & b \\ 0 & 0 & \Delta^{-1} \end{pmatrix}. \tag{15.5}$$

It can be easily verified that

$$\hat{T}(\hat{g}_1)\Delta\hat{T}(\hat{g}_2) = \hat{T}(\hat{g}_1\Delta\hat{g}_1),$$

so that $\hat{T}(\hat{g})$ is a representation of $\hat{M}(2)$. This representation is an exact one, i.e. $\hat{T}(\hat{g}_1) \neq \hat{T}(\hat{g}_2)$ if $\hat{g}_1 \neq \hat{g}_2$. Thus, we conclude that the group $\hat{M}(2)$ is realized as group of 3×3 real matrices (15.6).

The group $\hat{M}(2)$ can be realized also as the group of 2×2 complex matrices. Namely, by the identification of $\hat{g}(a, b, \alpha)$ with the matrix

$$\hat{Q}(\hat{g}) = \left(\begin{array}{cc} \exp\{i\Delta^{3/2}\alpha\} & \hat{z} \\ 0 & \Delta^{-1} \end{array} \right), \tag{15.6}$$

where

$$\hat{z} = a + i\Delta b. \tag{15.7}$$

It is easy to verify that $\hat{Q}(\hat{g}_1)\Delta\hat{Q}(\hat{g}_2) = \hat{Q}(\hat{g}_1\Delta\hat{g}_1)$ and $\hat{Q}(\hat{g}_1) \neq \hat{Q}(\hat{g}_2)$ if $\hat{g}_1 \neq \hat{g}_2$.

15.2 Parametrizations

For the parametrization above, let us find the composition law. Let $\hat{g}_1 = \hat{g}(a_1, b_1, \alpha_1)$ and $\hat{g}_2 = \hat{g}(a_2, b_2, \alpha_2)$. Then

$$\hat{T}(\hat{g}_1\Delta\hat{g}_2) = \tag{15.8}$$

$$\left(\begin{array}{ccc} g_{11}^{-1/2}\cos[\hat{\alpha}_1 + \hat{\alpha}_2] & -g_{22}^{-1/2}\sin[\hat{\alpha}_1 + \hat{\alpha}_2] & a_1 + a_2\Delta g_{11}^{-1/2}\cos[\hat{\alpha}_1] - b_2\Delta g_{22}^{-1/2}\sin[\hat{\alpha}_1 \\ g_{22}^{-1/2}\sin[\hat{\alpha}_1 + \hat{\alpha}_2] & g_{11}^{-1/2}\cos[\hat{\alpha}_1 + \hat{\alpha}_2] & b_1 + a_2\Delta g_{22}^{-1/2}\sin[\hat{\alpha}_1] + b_2\Delta g_{11}^{-1/2}\cos[\hat{\alpha}_1 \\ 0 & 0 & \Delta^{-1} \end{array} \right.$$

so that the law is

$$a = a_1 + a_2\Delta g_{11}^{-1/2}\cos[\alpha_1\Delta^{1/2}] - b_2\Delta g_{22}^{-1/2}\sin[\alpha_1\Delta^{1/2}], \tag{15.9}$$

$$b = b_1 + a_2\Delta g_{22}^{-1/2}\sin[\alpha_1\Delta^{1/2}] + b_2\Delta g_{11}^{-1/2}\cos[\alpha_1\Delta^{1/2}], \tag{15.10}$$

$$\hat{\alpha} = \hat{\alpha}_1 + \hat{\alpha}_2. \tag{15.11}$$

Denoting $\hat{x} = (a_1, b_1)$ and $\hat{y} = (a_1, b_1)$ we rewrite the formulas (15.9)-(15.11) as follows:

$$\hat{g}(\hat{x}, \hat{\alpha})\Delta\hat{g}(\hat{y}, \hat{\beta}) = \hat{g}(\hat{x} + \hat{y}_\alpha, \hat{\alpha} + \hat{\beta}). \tag{15.12}$$

From this equation it follows that if $\hat{g} = \hat{g}(\hat{x}, \hat{\alpha})$ then

$$\hat{g}^{-1} = \hat{g}(-\hat{x}_{-\hat{\alpha}}, 2\pi - \hat{\alpha}). \tag{15.13}$$

Another useful parametrization can be represented by isoEuler angles. On the plane, we parametrize the vector $\hat{x} = (a, b)$ by isopolar angles $a = r\Delta g_{11}^{-1/2}\cos[\varphi\Delta^{1/2}]$ and $b = g_{22}^{-1/2}\sin[\varphi\Delta^{1/2}]$. The set of parameters for $\hat{g}$ is then $(\hat{r}, \hat{\varphi}, \hat{\alpha})$, with the rabges

$$0 \leq \hat{r} < \infty, quad 0 \leq \hat{\varphi} < 2\pi, quad 0 \leq \hat{\alpha} < 2\pi. \tag{15.14}$$

Decomposition for element of $\hat{M}(2)$ reads

$$\hat{g}(\hat{r},\hat{\varphi},\hat{\alpha}) = \hat{g}(0,\hat{\varphi},0)\Delta\hat{g}(\hat{r},0,0)\Delta\hat{g}(0,0,\hat{\alpha}-\hat{\varphi}). \tag{15.15}$$

Transformations corresponding to $\hat{g}(0,0,\hat{\varphi})$ and $\hat{g}(0,0,\hat{\alpha}-\hat{\varphi})$ are rotations while $\hat{g}(\hat{r},0,0)$ defines a parallel transport along the axis $O\hat{x}$. For $\hat{g}_1 = \hat{g}(\hat{r},0,\hat{\alpha}_1)$ and $\hat{g}_2 = \hat{g}(\hat{r}_2,0,0)$, we have from eqs.(15.9)-(15.11)

$$\hat{g}_1\Delta\hat{g}_2 = \hat{g}(\hat{r},\hat{\varphi},\hat{\alpha})$$

, where

$$\hat{r} = \sqrt{\hat{r}_1^{\hat{2}} + \hat{r}_2^{\hat{2}} + 2\Delta\hat{r}_1\Delta\hat{r}_2\Delta g_{11}^{-1/2}\cos[\alpha\Delta^{1/2}]} \tag{15.16}$$

and

$$\hat{r}^{\hat{2}} = xb_1^2x + yb_2^2y + zb_3^2z; \hat{r}_1^{\hat{2}} = x_1b_1^2x_1 + y_1b_2^2y_1 + z_1b_3^2z_1, \tag{15.17}$$

$$\hat{r}_2^{\hat{2}} = x_2b_1^2x_2 + y_2b_2^2y_2 + z_2b_3^2z_2,$$

$$\exp i\Delta^{3/2}\varphi = \frac{\hat{r}_1 + \hat{r}_2\Delta\exp\{i\Delta^{3/2}\alpha_1}{\}}\hat{r}, \tag{15.18}$$

$$\hat{\alpha} = \hat{\alpha}_1. \tag{15.19}$$

To find the parameters of the composition $\hat{g}_1\Delta\hat{g}_2$ for $\hat{g}_1 = \hat{g}(\hat{r}_1,\hat{\varphi}_1,\hat{\alpha}_1)$ and $\hat{g}_2 = \hat{g}(\hat{r}_2,\hat{\varphi}_2,\hat{\alpha}_2)$, one should replace $\hat{\alpha}_1$ by $\hat{\alpha}_1 + \hat{\varphi}_2 - \hat{\varphi}_1$, $\hat{\alpha}$ by $\hat{\varphi} - -\hat{\varphi}_1$, and $\hat{\alpha}$ by $\hat{\alpha} - \hat{\alpha}_2$ in (15.16)-(15.19).

From decomposition (15.15) and equation

$$\hat{g}(0,0,\hat{\alpha}_1 + \hat{\varphi}_2 - \hat{\varphi}_1) = \hat{g}(0,0,\hat{\alpha}_1 - \varphi_1)\Delta\hat{g}(0,0,\hat{\varphi}_2) \tag{15.20}$$

we get

$$\hat{g}_1\hat{g}_2 = \hat{g}(0,0,\hat{\varphi}_1)\hat{g}(\hat{r}_1,0,0,)\hat{g}(0,0,\hat{\alpha}_1 + \hat{\varphi}_2 - \hat{\varphi}_1)\hat{g}(\hat{r}_2,0,0,)\hat{g}(0,0,\hat{\alpha}_2 + \hat{\varphi}_2)\Delta^3. \tag{15.21}$$

16 Irreps of $\hat{M}(2)$

16.1 Description of the irreps

Denote the space of smooth functions $f(\hat{x})$ on circle $x_1b_1^2x_1 + x_1b_1^2x_1 = \Delta^{-1}$ by D. To every element $\hat{g}(a,\hat{\alpha}) \in \hat{M}(2)$ we associate the operator $T_c(\hat{g})$ acting on $f(\hat{x})$,

$$\hat{T}_c(\hat{g})\hat{f}(\hat{x}) = e^{c\Delta(a,\hat{x})}\hat{f}(\hat{x}_{-\hat{\alpha}}). \tag{16.1}$$

Here, c is fixed complex number, $\hat{x}_{-\hat{\alpha}}$ is vector to which the vector $\hat{x}$ is transformed by rotation on angle $-\hat{\alpha}$, and $(a, \hat{x}) = a_{11}x_1 g_{11}^{1/2} + a_2 x_2 g_{11}^{1/2}$. Let us show that $T_c(\hat{g})$ is the representation of $\hat{M}(2)$. For $\hat{g}_1 = \hat{g}(a, \hat{\alpha})$ and $\hat{g}_2 = \hat{g}(b, \hat{\beta})$ we have

$$\hat{T}_c(\hat{g}_1)\hat{T}_c(\hat{g}_2)\hat{f}(\hat{x}) = \hat{T}_c(\hat{g}_1)e^{c\Delta(b,\hat{x})}\hat{f}(\hat{x}_{-\hat{\beta}})e^{c(a,\hat{x})}e^{c(b,\hat{x}_{-\hat{\alpha}}}\hat{f}(\hat{x}_{-\hat{\alpha}-\hat{\beta}}). \tag{16.2}$$

Since $(b, \hat{x}_{-\alpha}) = (\hat{b}_{\hat{\alpha}}, \hat{x})$ the following equation is valid:

$$\hat{T}_c(\hat{g}_1)\Delta\hat{T}_c(\hat{g}_2)\hat{f}(\hat{x}) = e^{c\Delta(a+b_{\hat{\alpha}},\hat{x})}\hat{f}(\hat{x}_{-\hat{\alpha}-\hat{\beta}}). \tag{16.3}$$

On the other hand, owing to (15.12)

$$\hat{g}_1\hat{g}_2 = \hat{g}(a, \hat{\alpha})\hat{g}(b, \hat{\beta}) = \hat{g}_(a + b_{\hat{\alpha}}, \hat{\alpha} + \hat{\beta}), \tag{16.4}$$

so that

$$\hat{T}_c(\hat{g}_1\Delta\hat{g}_2)\hat{f}(\hat{x}) = e^{c\Delta(a+b_{\hat{\alpha}},\hat{x})}\hat{f}(\hat{x}_{-\hat{\alpha}-\hat{\beta}}). \tag{16.5}$$

Thus, $\hat{T}_c(\hat{g}_1\Delta\hat{g}_2) = \hat{T}_c(\hat{g}_1\Delta\hat{T}_c\hat{g}_2)$, i.e. $\hat{T}_c(\hat{g})$ is representation of $\hat{M}(2)$.

Parametrical equations of the circle, $x_1 b_1^2 x_2 + x_2 b_1^2 x_2 = \Delta^{-1}$, have the form

$$x_1 = g_{11}^{-1}\cos[\psi\Delta^{1/2}], \quad x_2 = \Delta^{-1/2}\sin[\psi\Delta^{1/2}], \quad 0 \le \psi < 2\pi, \tag{16.6}$$

so that one can think of functions $f(\hat{x}) \in D$ as functions depending on $\hat{\psi}$,

$$\hat{f}(\hat{x}) = \hat{f}(\hat{\psi}). \tag{16.7}$$

The operator can be rewritten as

$$\hat{T}_c(\hat{g})\hat{f}(\hat{\psi}) = \exp\{c\Delta^2\hat{r}g_{11}^{-1/2}\cos[(\psi - \varphi)\Delta^{1/2}]\}\hat{f}(\hat{\psi} - \hat{\alpha}), \tag{16.8}$$

where

$$a = (\hat{r}\Delta g_{11}^{-1/2}\cos[\varphi\Delta^{1/2}], \hat{r}\Delta g_{22}^{-1/2}\sin[\varphi\Delta^{1/2}]), \quad \hat{g} = \hat{g}(a, \hat{\alpha}).$$

By introducing scalar product,

$$(\hat{f}_1 \hat{;} \hat{f}_2) = \frac{\Delta^2}{2\pi}\int_0^{2\pi} \hat{f}_1(\hat{\psi})\bar{\hat{f}}_2(\hat{\psi})d\hat{\psi}, \tag{16.9}$$

we make the space D to be isoHilbert space $\mathcal{E}$. Then, $\hat{T}_c(\hat{g})$ is isounitary in respect to the scalar product (16.9) if and only if $c = \hat{i}\rho$ is an imaginary number.

16.2 Infinitesimal operators

The operator $\hat{T}_c(\hat{w}_1(\hat{t}))$, where

$$\hat{w}_1(\hat{t}) = \begin{pmatrix} \Delta^{-1} & 0 & t\Delta^{-1} \\ 0 & \Delta^{-1} & 0 \\ 0 & 0 & \Delta^{-1} \end{pmatrix}, \tag{16.10}$$

$w_1 \in \Omega_2$, transforms function $\hat{f}(\hat{\psi})$ to

$$\hat{T}_c(\hat{w}_1(\hat{t}))\hat{f}(\hat{\psi}) = \exp\{c\Delta^2 t\hat{r} g_{11}^{-1/2}\cos[\psi\Delta^{1/2}]\}\hat{f}(\hat{\psi}), \tag{16.11}$$

so that

$$\hat{A}_1 = \frac{d\hat{T}_c(\hat{w}_1(\hat{t}))}{d\hat{t}}|_{\hat{t}=0} = c\Delta g_{11}^{-1/2}\cos[\psi\Delta^{1/2}], \tag{16.12}$$

i.e. $\hat{A}_1$ acts as a multiplication operator.

Similarly, one can prove that the infinitesimal operator $\hat{A}_2$ corresponding to the subgroup Ω_2 represented by the matrices

$$\hat{w}_2(\hat{t}) = \begin{pmatrix} \Delta^{-1} & 0 & 0 \\ 0 & t\Delta^{-1} & 0 \\ 0 & 0 & \Delta^{-1} \end{pmatrix} \tag{16.13}$$

is given by

$$\hat{A}_2 = C(g_{11}g_{22}^{1/2})\sin[\psi\Delta^{1/2}]. \tag{16.14}$$

Also, for the subgroup Ω_3 consisting of the matrices

$$\hat{w}_3(\hat{t}) = \begin{pmatrix} g_{11}^{-1/2}\cos[t\Delta^{1/2}] & -g_{22}^{-1/2}\sin[t\Delta^{1/2}] & 0 \\ g_{22}^{-1/2}\sin[t\Delta^{1/2}] & g_{11}^{-1/2}\cos[t\Delta^{1/2}] & 0 \\ 0 & 0 & \Delta^{-1} \end{pmatrix} \tag{16.15}$$

we have

$$\hat{A}_3 = -\frac{d}{d\hat{\psi}}. \tag{16.16}$$

16.3 The irreps

The prove of irreducibility of the representation $\hat{T}(\hat{g})$ of the group $\hat{M}(2)$ can be given in the same way as it of $\tilde{T}(\hat{g})$, and we do not present it here.

Below, we consider two choices of c.

(a) $c \neq 0$. We have

$$\hat{T}_c(\hat{w}_3(\hat{\alpha}))\hat{f}(\hat{\psi}) = \hat{f}(\hat{\psi} - \hat{\alpha}). \tag{16.17}$$

(b) $c = 0$. We have

$$\hat{T}_c(\hat{g})\Delta\hat{f}(\hat{\psi}) = \hat{f}(\hat{\psi} - \hat{\alpha}), \tag{16.18}$$

where $g = (\hat{x}\hat{;}\hat{\alpha})$. This representation is reducible since it can be decomposed into direct sum of the one-dimensional representations

$$\hat{T}_{0n}(\hat{g}) = e^{i\Delta^{3/2}n\alpha}. \tag{16.19}$$

Note that $\hat{T}_c(\hat{g})$ with $c \neq 0$ and $\hat{T}_{0n}(\hat{g})$, where n is integer number, constitute all possible irreps of $\hat{M}(2)$.

17 Matrix elements of the irreps of $\hat{M}(2)$ and isoBessel functions

17.1 Matrix elements

In the space $\mathcal{E}$, we choose the orthonormal basis $\{\exp(i\Delta^{5/2}n\psi)\}$ consisting of eigenfunctions of the operator $\hat{T}_c(\hat{w})$, $\hat{w} \in \Omega_3$. The matrix elements are written in this basis as

$$\hat{t}^c_{mn}(\hat{g}) = (\hat{T}_c(\hat{g})e^{in\psi\Delta^{3/2}}\hat{,}e^{im\psi\Delta^{3/2}}). \tag{17.1}$$

Taking into account definition (16.9) and eq.(16.8) we get

$$\hat{t}^c_{mn}(\hat{g}) = \frac{\exp\{-in\alpha\Delta^{3/2}\}}{2\pi}\Delta^3\int_0^{2\pi} d\hat{\psi}\, e^{c\Delta^2\hat{r}g_{11}^{-1/2}\cos[(\psi-\varphi)\Delta^{1/2}]}e^{i(n-m)\psi\Delta^{3/2}}. \tag{17.2}$$

Let $\hat{r} = \hat{\varphi} = 0$, *i.e.* $\hat{g}$ defines rotation on isoangle $\hat{\alpha}$. Due to orthogonality of the functions $\exp\{-in\psi\Delta^{5/2}\}$, we have

$$\hat{t}^c_{mn}(\hat{g}) \equiv \hat{t}^c_{mn}(\hat{\alpha}) = \exp\{-in\alpha\Delta^{3/2}\})\delta_{mn}. \tag{17.3}$$

Thus, the rotation is represented by a diagonal matrix $\hat{T}_c(\hat{\alpha})$, with non-zero elements being $\exp\{-in\alpha\Delta^{5/2}\}$, $\infty < n < \infty$.

Let $\hat{\varphi} = \hat{\alpha} = 0$. In this case, $\hat{g}$ defines transplacement on $\hat{r}$ along the $O\hat{x}$ axis so that (17.2) takes the form

$$\hat{t}^c_{mn}(\hat{g}) \equiv \hat{t}^c_{mn}(\hat{r}) = \frac{\Delta^2}{2\pi}\int_0^{2\pi} d\hat{\psi}\, \exp\{C\Delta^2 \hat{r} g_{11}^{-1/2}\cos[\psi\Delta^{1/2}] + i(n-m)\psi\Delta^{3/2}\}. \tag{17.4}$$

Replacing $\hat{\psi}$ by $\pi/2 - \hat{\theta}$, we then have

$$\hat{t}^c_{mn}(\hat{r}) = \frac{\Delta^2}{2\pi} i^{n-m} \int_0^{2\pi} d\theta\, \exp\{c\Delta^2 \hat{r} g_{22}^{-1/2}\sin[\theta\Delta^{1/2}] - i(n+m)\theta\Delta^{3/2}\}. \tag{17.5}$$

Let us denote

$$\hat{J}_n(\hat{x}) = \frac{\Delta^2}{2\pi} i^{n-m} \int_0^{2\pi} d\theta\, \exp\{\Delta^2 g_{22}^{-1/2}\sin[\theta\Delta^{1/2}] - in\theta\Delta^{3/2}\}, \tag{17.6}$$

and refer to $\hat{J}_n(\hat{x})$ as *isoBessel function.*

Using this definition we have from (17.5), in a compact writting,

$$\hat{t}^c_{mn}(\hat{r}) = i^{n-m}\Delta \hat{J}_{n-m}(-ic\Delta^2 \hat{r}). \tag{17.7}$$

Now, to obtain $\hat{t}^c_{mn}(\hat{g})$ in a eneral case it is suffice to make the replacement $\hat{\psi} - \hat{\varphi} = \frac{\pi}{2} - \hat{\theta}$ in the integral (17.2). Namely, using (17.6) we obtain

$$\hat{t}^c_{mn}(\hat{g}) = i^{n-m}\Delta \exp\{-in\alpha\Delta^{3/2} + i(n+m)\varphi\Delta^{3/2}\}\hat{J}_{n-m}(-ic\Delta^2\hat{r}). \tag{17.8}$$

Indeed, from (17.8) it follows that

$$\hat{T}_c(\hat{g}) = \hat{T}_c(\hat{\varphi})\hat{T}_c(\hat{r})\hat{T}_c(\hat{\alpha} - \hat{\varphi}). \tag{17.9}$$

Since the matrices $\hat{T}_c(\hat{g})$ and $\hat{T}_c(\hat{\alpha} - \hat{\varphi})$ are both diagonal, with the nonzero elements $\exp\{i\Delta^{3/2}n\varphi\}$ and $\exp\{-i\Delta^{3/2}n(\varphi - \varphi)\}$ respectively, while $\hat{t}^c_{mn}(\hat{r}) = i^{n-m}\Delta\hat{J}_{n-m}(-ic\Delta^2\hat{r})$ we come to (17.8).

If $\hat{g}$ is an identity transformation, $\hat{g} = \hat{g}(0,0,0)$, then $\hat{T}_c(\hat{g})$ is the isounit matrix. Consequently, we have the following relations:

$$\hat{J}_{n-m}(0) = \hat{\delta}, \quad \hat{J}(0) = \Delta^{-1}, \quad \hat{J}_n(0) = 0, \qquad (n \neq 0).$$

17.2 IsoBessel functions with opposite sign indeces

In this section, we find the relation between the isoBessel functions with opposite sign indeces.

In the space of functions $\hat{f}(\hat{\psi})$, introduce the operator $\hat{Q}$ acting according to

$$\hat{Q}\Delta\hat{f}(\hat{\psi}) = \hat{f}(-\hat{\psi}). \tag{17.10}$$

This operator commutes with operator $\hat{T}_c(\hat{g}) \equiv \hat{T}_c(\hat{r})$; where $\hat{g} = \hat{g}(\hat{r}, 0, 0)$. Indeed,

$$\hat{T}_c(\hat{r})\Delta\hat{Q}\hat{f}(\hat{\psi}) = \hat{T}_c(\hat{r})\hat{f}(-\hat{\psi}) = \exp\{c\Delta^2\hat{r}g_{11}^{-1/2}\cos[\psi\Delta^{1/2}]\}\hat{f}(-\hat{\psi}).$$

Consequently,

$$\hat{Q}\hat{T}_c(\hat{r}) = \hat{T}_c(\hat{r})\hat{Q}. \tag{17.11}$$

Operator $\hat{Q}$ acts by changing the basis element, $\exp\{in\alpha\Delta^{5/2}\}$ to $\exp\{-in\alpha\Delta^{5/2}\}$, so the matrix has the form $(\hat{q}_{mn})$, where $\hat{q}_{m,-m} = \Delta^{-1}$ and $\hat{q}_{mn} = 0$ for $m + n \neq 0$. Thus, from (17.11) we obtain

$$\hat{t}^c_{-m,n}(\hat{r}) = \hat{t}^c_{m,-n}(\hat{r}). \tag{17.12}$$

Then, taking into account (17.7) we get

$$i^{m+n}\hat{J}_{n+m}(-i\Delta^2 c\hat{r}) = i^{-m-n}\hat{J}_{-n-m}(-i\Delta^2 c\hat{r}). \tag{17.13}$$

Putting in (17.13) $m = 0$ and $\hat{z} = -i\Delta^2 c\hat{r}$, we finally have

$$\hat{J}_n(\hat{z}) = -\Delta^{1-n}\hat{J}_{-n}(\hat{z}). \tag{17.14}$$

17.3 Expansion series for IsoBessel functions

Our aim is to derive the expansion series for isoBessel function in $\hat{x}$. To this end, we use integral representation (17.6). Expanding the exponent $\exp\{i\Delta^2\hat{x}g_{-1/2}\sin[\psi\Delta^2]\}$ and integrating over all the terms we obtain

$$\hat{J}_n(\hat{x}) = \sum_{k=0}^{\infty} a_k x^k (g_{11}^{k+1/2} g_{22}^k), \tag{17.15}$$

where

$$a^k = \frac{\Delta^{3k-s}}{2\pi k!}\int_0^{2\pi} d\psi\,\exp\{-i\Delta^{5/2}n\psi\}(i(g_{11}g_{22}^{1/2}\sin[\psi\Delta^2])^{\hat{k}}. \tag{17.16}$$

Here, $s = 1, 2, 3, \ldots$ On the other and, owing to the Euler formula,

$$(ig_{22}^{-1/2}\sin[\psi\Delta^2])^k = \frac{\exp\{i\Delta^{3/2}\psi}{-}e^{-i\Delta^{3/2}\psi\}}2^k\Delta^{2k} \tag{17.17}$$

$$= \sum_{m=0}^{k} \frac{(-1)^m \Delta^{2(1-k)-m} \hat{C}_k^m \exp\{i(k-2m)\Delta^{3/2}\psi\}}{2^k}.$$

Inserting this formula into (17.16), one can observe that a_k is non-zero iff $(k-n)$ is an even number, *i.e.* $k-n=2m$, $m \geq 0$. If $k=n+2m$ then

$$a_k = \frac{(-1)\Delta^m}{2^k m!(k-m)!\Delta^{k+2s}} = \frac{(-1)\Delta^{-n-3m-2s}}{2^{n+2m} m!(n+m)!}. \tag{17.18}$$

So, we finally have

$$\hat{J}_n(\hat{x}) = (g_{11}^{5/2-2s} g_{22}^{2-2s})(x/2)^n \sum_{m=0}^{k} \frac{-\Delta^{n-m} x^{2m}}{2^{2m} m!(n+m)!}. \tag{17.19}$$

18 Functional relations for isoBessel function

18.1 Theorem of composition

Theorem of composition for isoBessel function can be derived in the same manner as it for isoLegendre function $\hat{P}^l_{mn}$. One should use the equality $\hat{T}_c(\hat{g}_1\Delta\hat{g}_2) = \hat{T}_c(\hat{g}_1)\Delta\hat{T}_c(\hat{g}_2$, that is

$$\hat{t}^c_{mn}(\hat{g}_1\Delta\hat{g}_2) = \sum_{k=-\infty}^{\infty} \hat{t}^c_{mk}(\hat{g}_1)\Delta\hat{t}^c_{kn}(\hat{g}_2). \tag{18.1}$$

Let us put $\hat{g}_1 = \hat{g}(\hat{r}_1, 0, 0)$ and $\hat{g}_2 = \hat{g}(\hat{r}_2, \hat{\varphi}_2, 0)$. Then the parameters $\hat{r}$, $\hat{\varphi}$, and $\hat{\alpha}$ corresponding to the composition $\hat{g} = \hat{g}_1\Delta\hat{g}_2$ can be expressed via parameters $\hat{r}_1$, $\hat{r}_2$, and $\hat{\varphi}_2$ as

$$\hat{r} = \sqrt{\hat{r}_1^{\hat{2}} + \hat{r}_2^{\hat{2}} + 2\hat{r}_1\Delta\hat{r}_2\Delta g_{11}^{-1/2} cos[\varphi_2\Delta^{1/2}]}, \tag{18.2}$$

$$e^{i\Delta^{3/2}\varphi} = \hat{r}_1 + \hat{r}_2\Delta e^{i\Delta^{3/2}\varphi_2}, \tag{18.3}$$

$$\alpha = 0, \tag{18.4}$$

where $\hat{r}_1^{\hat{2}}$, $\hat{r}_2^{\hat{2}}$, and $\hat{r}^{\hat{2}}$ are defined due to (15.17).

Inserting the matrix elements (17.8) into (18.1) and putting $m=0$ and $R = i\Delta^{-1}$ we have after some algebra

$$e^{i\Delta^{5/2}n\varphi}\hat{J}_n(\hat{r}) = \sum_{k=-\infty}^{\infty} e^{i\Delta^{5/2}k\varphi}\hat{J}_{n-k}(\hat{r}_1)\Delta\hat{J}_k(\hat{r}_2), \tag{18.5}$$

where $\hat{r}$, $\hat{r}_1$, $\hat{r}_2$, $\hat{\varphi}$, and $\hat{\varphi}_2$ are defined according to (18.2)-(18.3).

The formula (18.4) represents the *theorem of composition of isoBessel functions.*

Particularly, at $n = 0$ we have from (18.4)

$$\hat{J}_0(\hat{r}) = \sum_{k=-\infty}^{\infty} (\frac{-1}{\Delta^k}) e^{i\Delta^{5/2}k\varphi_2} \hat{J}_k(\hat{r}_1)\Delta^3 \hat{J}_k(\hat{r}_2). \tag{18.6}$$

Below, we consider some useful particular cases of the theorem.

(a) At $\hat{\varphi}_2 = 0$, we have $\hat{r} = \hat{r}_1 + \hat{r}_2$ and $\hat{\varphi} = 0$, so that

$$\hat{J}_n(\hat{r}_1 + \hat{r}_2) = \sum_{k=-\infty}^{\infty} \hat{J}_{n-k}(\hat{r}_1)\Delta \hat{J}_k(\hat{r}_2). \tag{18.7}$$

(b) At $\hat{\varphi}_2 = \pi$ and $\hat{r}_1 \geq \hat{r}_2$, we have $\hat{\varphi} = 0$ and $\hat{r} = \hat{r}_1 - \hat{r}_2$, so that

$$\hat{J}_n(\hat{r}_1 - \hat{r}_2) = \sum_{k=-\infty}^{\infty} (-1)\hat{J}_{n-k}(\hat{r}_1)\Delta^{1-k} \hat{J}_k(\hat{r}_2). \tag{18.8}$$

(c) For $\hat{\varphi} = \pi/2$, we have

$$(\frac{\hat{r}_1 + i\hat{r}_2}{\hat{r}_1 - i\hat{r}_2})^{\frac{n}{2}} \Delta^{\frac{n}{2}+1} \hat{J}_n(\sqrt{\hat{r}_1^{\hat{2}} + \hat{r}_2^{\hat{2}}}) = \sum_{k=-\infty}^{\infty} i^k \hat{J}_{n-k}(\hat{r}_1)\Delta^{1+k} \hat{J}_k(\hat{r}_2). \tag{18.9}$$

(d) For $\hat{r} = \hat{r}_1 = \hat{r}_2$ we have

$$\sum_{k=-\infty}^{\infty} \hat{J}_{n+k}(\hat{r})\Delta \hat{J}_k(\hat{r}) = \hat{J}_n(0) = \begin{matrix} \Delta^{-1}, & n = 0 \\ 0, & n \neq 0 \end{matrix} \tag{18.10}$$

18.2 Theorem of multiplication

Multiplying both sides of equation (18.5) by $\exp\{-i\Delta^{3/2}m\varphi_2\}/2\pi$ and integrating over $\hat{\varphi}_2$ in the range $(0, 2\pi)$, we have

$$\frac{\Delta^2}{2\pi} \int_0^{2\pi} e^{i\Delta(n\hat{\varphi} - m\hat{\varphi}_2} \hat{J}_n(\hat{r}) d\hat{\varphi}_2 = \hat{J}_{n-m}(\hat{r}_1)\hat{J}_m(\hat{r}_2), \tag{18.11}$$

where $\hat{r}$, $\hat{r}_1$, $\hat{r}_2$, $\hat{\varphi}$, and $\hat{\varphi}_2$ are defined according to (18.2)-(18.4). Here, we have used the fact that $\exp\{i\Delta^{3/2}n\varphi_2\}$ are orthogonal so that all the terms are zero except for those with $k = m$.

The equation (18.11) represents the *theorem of product for isoBessel functions.*

Let us consider specific case of the theorem characterized by $\hat{r}_1 = \hat{r}_2 = R$. From this condition it follows that $\hat{r} = 2R\Delta^2 g g_{11}^{-1/2} cos[\frac{\varphi_2}{2}]$ and $\hat{\varphi} = \hat{\varphi}_2$, so that

$$\hat{J}_{n-m}(\hat{r}_1)\hat{J}_m(\hat{r}_2) = \frac{\Delta^2}{2\pi}\int_0^{\pi} e^{i\Delta(n-2m)\hat{\varphi}}\hat{J}_n(2R\Delta^2 g_{11}^{-1/2}\cos[\frac{\varphi_2}{2}])d\hat{\varphi}. \quad (18.12)$$

Replacing the variable in the above integral by $\hat{r}$, we note that when $\hat{\varphi}_2$ varies from 0 to π the variable $\hat{r}$ varies from $\hat{r}_1 + \hat{r}_2$ to $|\hat{r}_1 - \hat{r}_2|$, while when $\hat{\varphi}_2$ varies from π to 2π the variable $\hat{r}$ varies from $|\hat{r}_1 - \hat{r}_2|$ to $\hat{r}_1 + \hat{r}_2$. In addition,

$$\frac{d\hat{r}}{d\hat{\varphi}_2} = \pm\frac{\sqrt{4\hat{r}_1^2\Delta\hat{r}_2^2 - (\hat{r}^2 - \hat{r}_1^2 - \hat{r}_2^2)}}{2\Delta\hat{r}}, \quad (18.13)$$

where minus and plus signs correspond to $0 \leq \hat{\varphi}_2 \geq \pi$ and $\pi \leq \hat{\varphi}_2 \geq 2\pi$ respectively. Thus, we have

$$\hat{J}_{n-m}(\hat{r}_1)\hat{J}_m(\hat{r}_2) = \frac{2\Delta^2}{\pi}\frac{\int_{|\hat{r}_1-\hat{r}_2|}^{\hat{r}_1+\hat{r}_2} e^{i\Delta(n\hat{\varphi}-m\hat{\varphi}_2}\hat{J}_n(\hat{r})\hat{r}d\hat{r}}{\sqrt{4\hat{r}_1^2\Delta\hat{r}_2^2 - (\hat{r}^2 - \hat{r}_1^2 - \hat{r}_2^2)^2}}, \quad (18.14)$$

where $\hat{\varphi}$ and $\hat{\varphi}_2$ are related to $\hat{r}$ according to (18.2)-(18.4).

At $m = n = 0$ the formula (18.14) takes the most simple form,

$$\hat{J}_0(\hat{r}_1)\hat{J}_0(\hat{r}_2) = \frac{2\Delta^2}{\pi}\frac{\int_{/\hat{r}_1-\hat{r}_2/}^{\hat{r}_1+\hat{r}_2}\hat{J}_0(\hat{r})\hat{r}d\hat{r}}{\sqrt{4\hat{r}_1^2\Delta\hat{r}_2^2 - (\hat{r}^2 - \hat{r}_1^2 - \hat{r}_2^2)}}. \quad (18.15)$$

19 Recurrency relations for $\hat{J}_n(\hat{z})$

As it for isoLegendre functions $\hat{P}^l_{mn}(\hat{z})$, recurrency relations for isoBessel functions follow from the composition theorem. Namely, we should first put $\hat{r}_2$ in this theorem to be infinitesimal.

Let us find derivatives of the isoBessel function on $\hat{x}$ at the point $\hat{x} = 0$. Differentiating (17.6) we have

$$\hat{J}'_n(0) = \frac{i\Delta^2}{2\pi}\int_0^{2\pi}\exp -i\Delta^2 n\hat{\psi}g_{22}^{-1/2}\sin[\hat{\psi}]d\hat{\psi} \quad (19.1)$$

$$= \frac{\Delta^2}{4\pi}\int_0^{2\pi}[\exp -i\Delta^2(n-1)\hat{\psi} - \exp -i\Delta^2(n+1)\hat{\psi}]d\hat{\psi}.$$

This integral is non-zero only when $n = \pm\Delta^{-1}$. Also,

$$\hat{J}'_1(0) = \hat{J}'_{-1}(0) = \frac{\Delta^{-1}}{2}. \quad (19.2)$$

Differentiating both sides of (18.9) on $\hat{r}_2$ and putting $\hat{r}_2 = 0$ we find

$$2\hat{J}_n'(\hat{x}) = \hat{J}_{n-1}(\hat{x}) - \hat{J}_{n+1}(\hat{x}). \tag{19.3}$$

Here, we used (19.2) and replace $\hat{r}$ by $\hat{x}$.

Similarly, from (18.11) we find

$$\frac{2n}{\hat{x}}\hat{J}_n(\hat{x}) = \hat{J}_{n-1}(\hat{x}) + \hat{J}_{n+1}(\hat{x}). \tag{19.4}$$

Combining (19.3) and (19.4) we finally obtain

$$\hat{J}_{n-1}(\hat{x}) = \frac{n}{\hat{x}}\Delta\hat{J}_n(\hat{x}) + \hat{J}_n'(\hat{x}), \tag{19.5}$$

$$\hat{J}_{n+1}(\hat{x}) = \frac{n}{\hat{x}}\Delta\hat{J}_n(\hat{x}) - \hat{J}_n'(\hat{x}) \tag{19.6}$$

. These formulas can be presented also in the following form:

$$\hat{J}_{n-1}(\hat{x}) = (\frac{n}{\hat{x}} + \frac{d}{d\hat{x}})\Delta\hat{J}_n(\hat{x}), \tag{19.7}$$

$$\hat{J}_{n+1}(\hat{x}) = (\frac{n}{\hat{x}} - \frac{d}{d\hat{x}})\Delta\hat{J}_n(\hat{x}). \tag{19.8}$$

20 Relations between IsoBessel functions and $\hat{P}^l_{mn}(\hat{z})$

20.1 IsoEuclidean plane and sphere

Two-dimensional sphere can be mapped to isoEuclidean plane in a standard way. Namely, this can be done in taking the limit $\hat{R} \to \infty$ for the radius of the sphere. Accordingly, $\hat{M}(2)$ can be considered as some limit of $\hat{SO}(3)$. More precisely, replacing $\hat{\varphi}$, $\hat{\psi}$, $\hat{\theta}$, $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\varphi}_2$ by $\hat{\varphi}$, $\hat{r}/\hat{R}$, $\hat{\alpha}$, $\hat{r}_1/\hat{R}$, $\hat{r}_2/\hat{R}$, and $\hat{\alpha}_1$ in (8.9) defining multiplications in $\hat{SO}(3)$ we should retain leading terms in the limit $\hat{R} \to \infty$. Simple calculations show that the result is exactly the formulas (15.16)-(15.19) defining multiplications in $\hat{M}(2)$.

20.2 IsoBessel and isoJacobi functions

The relation between the groups $\hat{M}(2)$ and $\hat{SO}(3)$ makes it possible to relate matrix elements of its irreducible isounitary representations. Thus, isoBessel functions, as matrix elements of representations $\hat{T}_{I\rho}(\hat{g})$ of $\hat{M}(2)$, can be

derived from $\hat{P}^l_{mn}$, which are matrix elements of representations $\hat{T}_l(\hat{g})$ of $\hat{SO}(3)$. The limiting procedure is $\hat{R} \to \infty$ and $l \to \infty$.

To obtain concrete formulas we note first that $\hat{P}^l_{mn}$ has the integral representation,

$$\hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\hat{\theta}]) = (\frac{\Delta^7}{2\pi})\sqrt{\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!}}\int_0^{2\pi} d\hat{\varphi}\,(g_{11}^{-1/2}\cos[\frac{\hat{\theta}}{2}]e^{i\hat{\varphi}/2}$$
$$+ig_{22}^{-1/2}\sin[\frac{\hat{\theta}}{2}]e^{-i\hat{\varphi}/2})(g_{22}^{-1/2}\sin[\frac{\hat{\theta}}{2}]e^{i\hat{\varphi}/2} + ig_{11}^{-1/2}\cos[\frac{\hat{\theta}}{2}]e^{-i\hat{\varphi}/2})e^{im\varphi}. \tag{20.1}$$

Putting $\hat{\theta} = \hat{r}/l$ and taking the limit $l \to \infty$ we find

$$\lim_{l\to\infty} \hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\frac{\hat{r}}{l}]) = \frac{\Delta^{2(l+1)}}{2\pi}\int_0^{2\pi}(1+\frac{ir\Delta^{5/2}}{2l}\exp -i\varphi)^{l-n} \tag{20.2}$$
$$\times(1+\frac{ir\Delta^{5/2}}{2l}\exp i\varphi)\exp i\Delta^{3/2}(m-n)\varphi d(\varphi\Delta^{1/2}).$$

Note that at $m = n = 0$ the above relation takes the following simple form:

$$\lim_{l\to\infty} \hat{P}_l(g_{11}^{-1/2}\cos[\frac{\hat{r}}{l}\Delta^{1/2}]) = \hat{J}_0(\hat{r}). \tag{20.3}$$

so that $\hat{J}_0(\hat{r})$ appears as the limit from the isoLegendre polynomial.

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7: LIE-ADMISSIBLE THEORY

7.1: STATEMENT OF THE PROBLEM

A central assumption of hadronic mechanics is that one operator alone, the Lagrangian or the Hamiltonian, is *insufficient* to represent physical reality, which needs instead (3N + 1)–quantities, the Lagrangian or Hamiltonian, plus the 3N diagonal elements of the isounit $\hat{1}$.

By no means is the above assumption new, because its origins go back to Lagrange [1], Hamilton [2], Jacobi [3] and other founders of analytic dynamics. The novelty is merely in the *realization* of the 3N additional quantities via the elements of the isounit.

The equations originally conceived by Lagrange and Hamilton *are not* those available in the contemporary mathematical and physical literature, but equations *with external terms*. In fact, the *true Lagrange's equations* for a system of N particles in three–dimensional Euclidean space are given by [1]

$$\frac{d}{dt}\frac{\partial L(t, r, \dot{r})}{\partial \dot{r}^{ka}} - \frac{\partial L(t, r, \dot{r})}{\partial r^{ka}} = F_{ka}(t, r, \dot{r}), \qquad (7.1.1a)$$

$$L = K(\dot{r}) - V(t, r, \dot{r}), \quad k = 1, 2, 3\ (= x, y, z), \quad a = 1, 2, ..., N, \qquad (7.1.1b)$$

the *true Hamilton's equations* are [2]

$$\dot{r}^{ka} = \frac{\partial H(t, r, p)}{\partial p_{ka}}, \quad \dot{p}_{ka} = -\frac{\partial H(t, r, p)}{\partial r^{ka}} + F_{ka}(t, r, p), \qquad (7.1.2a)$$

$$H = K(p) + V(t, r, p), \qquad (7.1.2b)$$

and the *true Jacobi theorem* [3] is also that with external terms.

As one can see, Eq.s (7.1.1) and (7.1.2) require precisely (3N + 1)–quantities for the representation of physical reality, a Lagrangian or a Hamiltonian, plus the 3N external forces F_{ka}.

By comparison, the analytic equations of the contemporary literature are the so-called *"truncated Lagrange's and Hamilton's equations"*; i.e., those *without* external terms. As a result of a scientific process still ignored by contemporary historians, the external terms were progressively removed from the analytic equations sometime by the end of the past century, to acquire the form almost universally used nowadays.

The origin of this "truncation" appears to be the birth of Lie's theory [4] in the second part of the past century. In fact, the brackets of the true Hamilton's equations, not only violate the Lie algebras axioms, but actually violate all conditions to constitute an algebra, whether Lie or not (see below). The achievement of a classical realization of Lie algebras therefore required the elimination of the external terms. The historical successes of the truncated Hamilton's equations in the description, first, of planetary systems (see, e.g. ref. [5]) and then of the atomic structure (see, e.g., ref. [6]) provided a major drive toward the current elimination of the external terms.

However, by no means, has this scientific process suppressed the vision of Lagrange and Hamilton. In fact, the "truncated analytic equations" can directly represent[50] only conservative systems and a restricted class of other systems. By comparison, the "true analytic equations" are directly universal for all possible systems of the physical reality, whether conservative or not. In fact, the Lagrangian and Hamiltonian represent all conservative forces, while all remaining forces are directly represented with the external terms.

The primary difference is that, while the truncated equations represent *closed-isolated systems* with conserved total energy, the true equations represent instead *open-nonconservative systems* with the following *time rate of variation of the energy*[51]

$$d H / dt = \sum_{ka} v_{ka} F_{ka} \neq 0, \tag{7.1.3}$$

due to interactions with systems interpreted as external.

This author conducted his graduate studied at the Universitá di Torino, Italy, where Lagrange did most of his work, thus having the opportunity of studying Lagrange's original papers and and comparing them with contemporary literature. The latter is essentially based on the trend to reduce all physical systems, whether classical or quantum mechanical, to a form representable by the truncated analytic equations. By contract, Lagrange and Hamilton were fully aware that the quantities today called "Lagrangian" and "Hamiltonian" cannot

[50] We are here referring to a representation with a direct physical meaning of all algorithms at hand, whereby r represents the actual frame of the observer, H represents the actual total energy K + V, p represents the actual linear momentum $m\dot{r}$, etc.

[51] We here adopt the definition of *nonconservation* of ref.s [8,9] in which the energy can monotonically *increase or decrease* in time, while *dissipation* is referred to the case when the energy solely *decreases* in time.

represent the totality of the physical reality, but only a very small part of it and, for this reason, they introduced the external terms to their equations.

Subsequent rigorous studies on the integrability conditions for the existence of a Lagrangian or a Hamiltonian, *Helmholtz's conditions of variational selfadjointness* [7,8], proved Lagrange's and Hamilton's vision in its entirety. In fact, the broadest possible class of Newtonian systems (those of the *interior dynamical problem*) result in being nonrepresentable with a Lagrangian or a Hamiltonian in the coordinates of the observer (These are the so-called *essentially nonselfadjoint systems* [8]).

We mentioned earlier the contemporary trend of using the Lie-Koening theorem (see ref. [9] and quoted literature) to turn nonlagrangian-nonhamiltonian systems into equivalent forms which are representable with the truncated analytic equations. Yet, Lagrange and Hamilton's vision remains broader then the contemporary one in this respect too for the reasons indicated earlier (lack of general applicability of the Lie-Koening theorem, e.g., for integral or discontinuous systems, lack of realization of the transformed frame in the laboratory, loss of conventional relativities because of the highly noninertial character of the transformed frame, etc.).

But even ignoring all this, and assuming that some artificial construction permits the construction of a Lagrangian or a Hamiltonian for the truncated equations, the physical significance of these quantities is unclear, controversial and manifestly misleading, particularly in the operator treatment of nonconservative forces.[52]

Because of the above occurrences, this author spent his research life studying the true, historical, Lagrange and Hamilton equations *with* external terms, according to the following two main lines:

Isotopies [7] These are the methods outlined in the preceding chapters possessing a *Lie-isotopic structure*, which can now be reinspected from a different viewpoint. In fact, these methods were conceived to preserve the basic

[52] The literature in particle physics is full of models in which the physical structure of the Hamiltonian

$$H = K + V = \text{Kin. Energy} + \text{Pot. Energy}. \tag{1}$$

is generalized into canonical expressions of the type

$$H = p^2/2m + V(r), \quad p = \alpha\, e^{\beta r \dot{r}}, \quad \alpha, \beta \in R. \tag{2}$$

Yet H is continued to be interpreted as "the total energy" while in reality H is a pure mathematical quantity (a first integral). The real total energy $E = K + V$ is nonconserved because of interactions not properly identified as being external, i.e. $[E, H] \neq 0$. The "physical conclusions" of these models are unsettled at best. This is another illustration of *the paramount importance of solely using "direct representations" (as identified in the preceding footnote) whenever studying nonpotential forces.*

axioms of the truncated analytic equations, yet requiring (3N+1)–quantities for the representation of physical reality and permitting the same direct representation of the true analytic equations with external terms.

Genotopies [7]. These are the more general methods outlined in this chapter with the covering *Lie–admissible structure.*

A central property represented by Eq.s (7.1.1) and (7.1.2) is that conventional closed–conservative systems are a particular case of the more general open–nonconservative ones. In fact, the conventional conservation law of the energy is a particular case of the more general laws (7.1.3) on the *time–rate–of–variation of the energy.* As a result, we expect the existence of covering methods for the treatment of open nonconservative systems which admit the conventional Lie and Lie–isotopic methods as particular cases.

A *central problem for a quantitative study of open nonconservative systems is therefore the identification of a covering of both, Lie and Lie–isotopic theories which permits a direct representation of the time–rate–of–variation of physical quantities; that is, a representation in which all quantities* H, r, p, r∧p, *etc., have a direct physical meaning, change their value in time and admit conservation laws as particular cases.*

Recall that the conservation of the energy for Lie and Lie–isotopic formulations,

$$dH/dt = [H, H] = H\,H - H\,H \equiv 0, \tag{7.1. 3a}$$

$$dH/dt = [H \hat{,} H] = H\,T\,H - H\,T\,H \equiv 0, \tag{7.1.3b}$$

are a consequence of the anticommutativity of the products [A, B] and [B,$\hat{}$ A]. Thus, the above requirements can be expressed by the following conditions, originally submitted in ref.s [12–14] and then studied in detail in ref.s [7,10,11]:

Condition 7.1.1: The brackets, say A⊙B, of the analytic equations characterizing interactions under *external* forces must not be anticommutative, $A \odot B \neq - B \odot A$, as a necessary condition to represent the time–rate–of–variation of the energy and of other physical quantities

$$i\, dH / dt = H \odot H = f(t) \neq 0; \tag{7.1.4}$$

Condition 7.1.2: The new brackets A⊙B must recover the isotopic [A,$\hat{}$H] or conventional Lie product [A, H] when all external forces are null

$$\lim_{\text{Ext. Forces} = 0} A \odot B \equiv [A \hat{,} B] \text{ or } [A, B] ; \tag{7.1.5}$$

Condition 7.1.3: The new brackets A⊙B must, first, define a consistent

algebra, and, second, that algebra must be a covering of the Lie and Lie–isotopic algebras, therefore admitting the latter in their classification.

As originally identified in ref.s [12–14], and confirmed in the subsequent studies [7,10,11], the algebras which verify all the above conditions are the *Lie–admissible algebras* preliminarily presented in App. I.4.A.

When joint with studies on the isotopic formulations, this occurrence permits the identification of the following chain of covering formulations:

LIE FORMULATIONS: Closed–isolated local–diff. Hamiltonian systems	**LIE–ISOTOPIC FORMULATIONS: Closed–isolated nonlocal–integral nonhamiltonian systems**	**LIE–ADMISSIBLE FORMULATIONS: Open–nonconserv. nonlocal–integral nonhamiltonian systems**

In fact, the Lie–isotopic formulations were introduced in ref. [7] precisely as a particular form of the more general Lie–admissible structures because the antisymmetric algebras generally attached to the Lie–admissible ones are *not* Lie, but Lie–isotopic.

In this chapter we shall outline the rudiments of the Lie–admissible formulations with the understanding that they are considerably less developed than the corresponding Lie–isotopic methods, and so much remains to be done.

The mathematical relevance of the Lie–admissible theory is self–evident from their covering character over the conventional Lie and Lie–isotopic theories. Their physical relevance can be understood only after a knowledge of the problematic aspects of current formulations of nonconservative systems outlined in Sect. 7.2 below. The Lie–admissible formulations result to be as rather unique for a number of applications, including nonconservative systems, q–deformations, nonlinear and nonlocal theories, and others.

However, the primary mathematical and physical relevance of the Lie–admissible theory for which it was conceived [7–11] rests in the capability of providing an *axiomatic formulation on the origin of irreversibility.*

The scientific scene prior to the advent of hadronic mechanics is well known. On one side, experimental evidence establishes that macroscopic structures generally are irreversible at the Newtonian, statistical, thermodynamical and other levels. On the other side, only *one* theory for the macroscopic world, the reversible quantum mechanics, was then available. All past efforts in irreversibility have therefore been centered in attempting a *reconciliation of the macroscopic evidence of irreversibility with the only*

available microscopic theory (see, e.g., the recent account [15] and quoted literature).

The advent of hadronic mechanics has altered this scientific scenario because of the structural irreversibility of its Lie-admissible branch. In fact, quantum mechanics emerges as being exact for the *exterior particle problem in vacuum* (such as the atomic structure) which is reversible also at the classical level (such as the planetary structure). Hadronic mechanics then emerges as the applicable theory for the *interior particle problem* which is irreversible at the particle level (such as the structure of a neutron star) and remains irreversible at its classical counterpart (such as the structure of Jupiter).

ORIGIN OF IRREVERSIBILITY

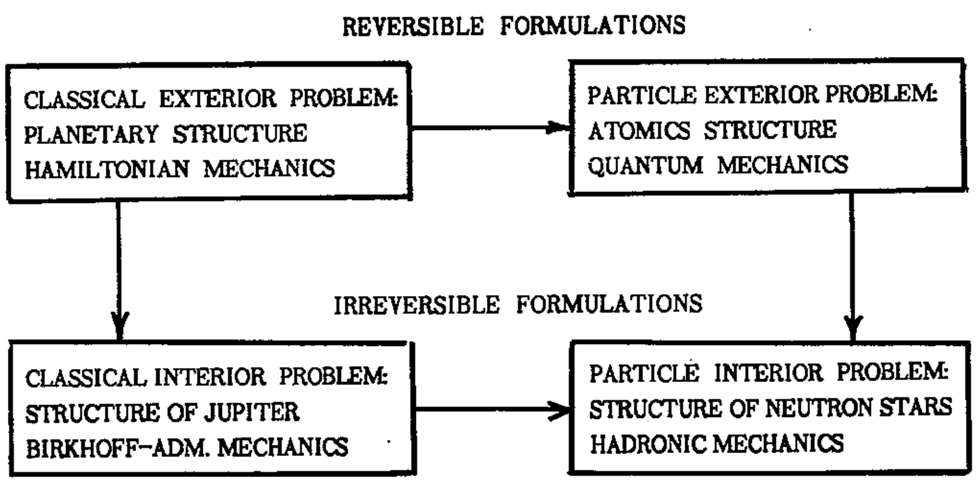

FIGURE 7.1.1. A schematic view of the scenario on irreversibility after the advent of hadronic mechanics. As well known, *exterior dynamical problems* of point particles in vacuum are reversible at both the classical and particle levels, and so are the corresponding mechanics, the Hamiltonian and quantum mechanics. Irreversibility emerges in nature for *interior dynamical problems.* Once this fundamental distinction is understood, the scenario in irreversibility is completely altered. One begins with the need of a covering mechanics at the *purely classical level* because Hamiltonian mechanics cannot represent all interior Newtonian systems in the frame of the experimenter [8,9]. These studies have resulted in the construction of a generalization of Hamiltonian mechanics submitted in ref.s [10,11] under the

name of *Birkhoff-admissible mechanics* which is structurally irreversible and therefore directly compatible with irreversibility at the subsequent statistical and thermodynamical levels. In these volumes we shall study the Lie-admissible branch of hadronic mechanics as the particle counterpart of the Birkhoff-admissible mechanics. Irreversibility then emerges as originating at the ultimate elementary level of *interior* particle problems, and then merely persists at the macroscopic level.

At any rate, the current studies on irreversibility cannot resolve the paradox caused by the *No-Reduction Theorems* of Ch. I.1, according to which an irreversible interior system, such as a satellite during re-entry in a monotonically decaying orbit, simply cannot be decomposed in any consistent way into a collection of elementary particles all in stable-reversible orbits, while such paradox is indeed resolved by hadronic mechanics (see Fig. 7.1.1).

The epistemological origins are the limitations of quantum mechanics (Sect. I.1.2) caused by its local-differential structure which does not permit an exact description of the nonlocal-integral conditions of interior problems. The advent of a structurally irreversible mechanics specifically built for interior problems evidently alter the scenario.

7.2: PROBLEMATIC ASPECTS OF CONTEMPORARY FORMULATIONS OF OPEN SYSTEMS

The best to way initiate the study of the Lie-admissible formulations is to see (and admit) rather serious problematic aspects in the contemporary formulation of open-nonconservative systems *beginning at the purely classical level,* which then persist at different levels.

They can be identified by inspecting the brackets of the time evolution at the various levels off description, such as in:

Classical mechanics, where nonconservative systems of N particles (labeled with a = 1, 2, ..., N) in Euclidean space with local coordinates x^k (k = 1, 2, 3)represented via external forces F_{ka}, result in the following dynamical evolution of a quantity A(r,p)

$$dA / dt \overset{\text{def}}{=} A \times H = [A , H] + \frac{\partial A}{\partial p_{ka}} F_{ka} , \qquad (7.2.1)$$

with

$$[A, H] = \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k} \frac{\partial A}{\partial p_k} \qquad (7.2.2)$$

being the conventional Poisson brackets;

Quantum mechanics, where nonconservative systems are generally represented by nonhermitean Hamiltonians of the type

$$H = H_0 + i V \neq H^\dagger , \tag{7.2.3}$$

as rather popular in nuclear physics, resulting in the dynamical equations

$$i\, dA / dt = A \times H = A H - H^\dagger A \tag{7.2.4}$$

Statistical mechanics, where collisions and other effects are expressed also with external terms, classically and quantum mechanically, resulting in the following dynamical evolution of the density matrix ρ

$$i\, d\rho / dt = \rho \times H = [\rho , H] + C \tag{7.2.5}$$

with

$$[\rho, H] = \rho H - H \rho \tag{7.2.6}$$

being the classical or quantum, canonical brackets.

Note that all the above formulations correctly describe the time-rate-of-variation of the energy,

$$i\, dH / dt = H \times H = f(t) \neq 0 \tag{7.2.7}$$

Therefore, the brackets $A\times H$ do indeed describe an open nonconservative system, by verifying Condition 7.1.1. The admission of the conventional Lie brackets as a particular case is trivial, and brackets $A\times H$ also verify Condition 7.1.2. The central point is that the above formulations violate the crucial Condition 7.1.3.

Proposition 7.2.1 [7,10,11]: *The brackets of conventional formulations of open nonconservative systems, Eq.s (7.2.1), (7.2.4) and (7.2.5), do not constitute an "algebra" as commonly understood in mathematics (see Sect. 2.4 and App. 4.A) because they verify the right scalar and distributive laws,*

$$\alpha \times (A \times B) = (\alpha \times A) \times B = A \times (\alpha \times B) , \tag{7.2.8a}$$

$$(A + B) \times C = A \times B + B \times C, \tag{7.2.8b}$$

but violate the left distributive and scalar laws, i.e., for any scalar $\alpha \neq 0$, $\alpha \in F$, *and elements* A, B, C, *we have*

$$(A \times B) \times \alpha \neq A \times (B \times \alpha) \neq (A \times \alpha) \times B, \tag{7.2.9a}$$

$$A \times (B + C) \neq A \times B + A \times C, \tag{7.2.9b}$$

In different terms, *in the transition from the conventional Lie formulations characterized by brackets* [A, H] *to the above classical, quantum or statistical brackets* A×H, *we have not only the loss of all Lie algebras, but in actuality we have the loss of all possible consistent algebraic structures.*

Additional mathematical properties are the following.

Proposition 7.2.2 [loc. cit.]: *Eq.s (7.2.1), (7.2.4) and (7.2.5) do not admit a consistent enveloping algebra.*

This can be seen in a number of ways, the most effective one being the fact that *Eq.s (7.2.5) cannot be exponentiated as for conventional Lie equations,* because they do not admit a consistent infinite basis (no Poincaré–Birkhoff–Witt theorem–see Sect. I.4.3).

Proposition 7.2.3 [loc. cit.]: *Eq.s (7.2.1), (7.2.4) and (7.2.5) do not admit a consistent unit.*

This can also be seen in a number of ways, e.g., from the lack of a consistent envelope needed to define the unit of the theory.

Rather than being mere mathematical curiosities, the physical implications of the above occurrences are rather serious, and can be summarized as follows (for a detailed study see ref. [11,16]):

Problematic aspect 7.2.1: *Eq.s (7.2.1), (7.2.4) and (7.2.5) do not admit a consistent measurement theory.* The fundamental notion of all measurements theories, whether classical, or quantum mechanical or statistical, is the unit with respect to which the measurements are referred to[53]. No formulation without a unit can therefore have a measurement theory usable for contemporary experiments. Note that one may indeed conduct measures. However, the insidious aspect is that they have no rigorous relationship to the theory at hand. The lack of existence of the unit for the equations considered can be established on numerous independent counts, e.g., from the lack of the envelope itself in which

[53] The issue is technically deeper. In fact, a Hilbert space can certainly be defined over a field which, as such, possesses the unit 1, even for nonconservative systems. The point is that the enveloping operator algebra of the theory here considered has no unit, which implies that no operator can be "measured" in a consistent way. To understand the occurrence one should think at a quantum mechanical measurement in which Planck's constant cannot be defined.

the unit is defined. The physical implications for plasma physics and other fields are self-evident.

Problematic aspect 7.2.2: *The angular momentum, spin, and other physical quantities characterized by Lie's theory cannot be consistently defined under the generalized brackets $A \times H$ of the equations considered.* As well known, the angular momentum and spin are centrally dependent on the exact O(3) and SU(2) theory, respectively. Then, the same quantities are manifestly meaningless, mathematically and physically, for Eq.s (7.2.1), (7.2.4) and (7.2.5), trivially, because they have lost not only the entire Lie's theory, but the very notion of algebra. This is another occurrence which should not be taken lightly. As an example, the use of the terms "protons and neutrons with spin $\frac{1}{2}$" has no mathematical or physical meaning when referred to Eq.s (7.2.4) in nuclear physics or Eq.s (7.2.5) in plasma physics.

Problematic aspect 7.2.3: *Loss of the conventional notion of particle.* Eq.s (7.2.4) have been generally used in nuclear physics over the past decades to describe nonconservative processes of nucleons. However, the quantum mechanical notion of protons and neutrons can be rigorously proved to be inapplicable to these equations and, if applied, to imply a host of inconsistencies. They are technically due to the loss of all means to characterize the conventional notion of particle.

Problematic aspect 7.2.4: *Loss of the rotational, Lorentz and other fundamental space–time symmetries.* This is evidently due to the lack of a consistent exponentiation and other technical reasons. Stated explicitly, *the open–nonconservative systems generally represented in the contemporary literature imply the inapplicability of Galilei's, Einstein's spécial and Einstein's general relativities.*

Problematic aspect 7.2.5: *Eq.s (7.2.1), (7.2.4) and (7.2.5) cannot consistently represent irreversibility.* As well known, Lie's theories for Hermitean Hamiltonians verify the *Theorem of Detailed Balancing* (see, e.g., ref. [17]) and, as such, they do consistently represent reversibility from first principles (Fig. 7.1.1). Such a theorem becomes manifestly inapplicable under nonunitary transformations as those underlying Eq.s (7.2.4), but no consistent generalization of the theorem of detailed balancing exists for brackets $A \times H$, to our best knowledge. Thus, the equations considered cannot consistently represent irreversibility (see Vol. II for a Lie–admissible, irreversible generalization of the Theorem of Detailed Balancing [17]).

For additional problematic aspects the interested reader may consult ref.s [10,11,16].

It is hoped the reader can see *the need for a fundamental structural*

revision in the treatment of open nonconservative systems in their classical, particle and statistical formulation, because any attempt at reconciling these systems with old knowledge will inevitably lead to inconsistencies.

7.3: HISTORICAL NOTES

The notion of *Lie-admissible algebra* was introduced by A. A. Albert in paper [18] of 1948. A generally nonassociative algebra U with elements a, b, c, ... and (abstract) product ab over a field $F(\alpha,+,\times)$[54] is called Lie-admissible when the attached algebra U^-, which is the same vector space as U (that is, the elements of U and U^- coincide) but equipped with the product $[a, b]_U = ab - ba$, is Lie.

Since the attached product $[a, b]_U$ is antisymmetric, the sole condition for a product ab to be Lie-admissible is that the attached product $[a,b]_U$ verifies the *Jacobi identity,* i.e., the following axiom, called *axiom of general Lie-admissibility*, is identically verified

$$(a, b, c) + (b, c, a) + (c, a, b) - (c, b, a) - (b, a, c) - (a, c, b) = 0, \qquad (7.3.1)$$

where

$$(a, b, c) = (a\,b)\,c - a\,(b\,c) \qquad (7.3.2)$$

is called the *associator* (see also App. 4.A), and represents the departure of the algebra from an associative one.

Albert [18] identified only one nontrivial subcase of Lie-admissible algebras called *flexible Lie-admissible algebras* and characterized by the axioms

$$(a, b, a) = 0, \qquad (7.3.3a)$$

$$(a, b, c) + (b, c, a) + (c, a, b) = 0. \qquad (7.3.3b)$$

where condition (7.3.3a), called the *flexibility law,* is a simple generalization of the anticommutative law. No additional study, e.g., of the structure theory, was conducted by Albert in his original paper [18].

In the subsequent two decades, only two additional brief notes appeared by mathematicians in Lie-admissible algebras, one in 1957 and one in 1962 (see the general bibliography [19]), but without any detailed mathematical study.

The Lie-admissible algebras made their first appearance in classical mechanics in paper [12–14] 1967–68 via their identification in the fundamental brackets of the time evolution of Hamilton's equations with external terms, when properly written (see bellow). The algebras were then studied in more details in

54 Assumed throughout this presentation as of characteristic zero.

ref.s [7,10,11]

By introducing the unified notation $a = (a^\mu) = (r_{ka}, p_{ka})$, $\mu = 1, 2, ..., 6N$, the main result of ref.s [7,13–15] can express via the re-formulation of brackets A⊙B of Eq.s (7.1.5)

$$dA / dt = [A,H] + \frac{\partial A}{\partial p_{ka}} F_{ka} \equiv (A, H) := \frac{\partial A}{\partial a^\mu} S^{\mu\nu} \frac{\partial B}{\partial a^\nu}, \quad (7.3.4a)$$

$$S^{\mu\nu} = \omega^{\mu\nu} + s^{\mu\nu} = \omega^{\mu\alpha} \hat{1}_\alpha{}^\nu, \quad s^{\mu\nu} = \text{diag.}\{ 0, F/(\partial H / \partial p) \}, \quad (7.3.4b)$$

$$\hat{1}_\alpha{}^\nu = I_\alpha{}^\nu + s_\alpha{}^\nu, \quad (7.3.4c)$$

where $\omega^{\mu\nu}$ is Lie's tensor characterizing the Poisson brackets, and $\hat{1}$ is a quantity to be identified shortly. It is then easy to verify the existence of: the consistent exponentiation of Eq.s (7.3.4a) into the finite form

$$A(t) = e^{t\, S^{\mu\nu} (\partial_\nu H)(\partial_\mu)} A(0); \quad (7.3.5)$$

the direct representation of the time-rate-of-variation of the energy

$$\dot{H} = H - e^{t\, S^{\mu\nu} (\partial_\nu H)(\partial_\mu)} H = v_{ka} F^{NSA}{}_{ka}; \quad (7.3.6)$$

and underlying equations of motion in explicit form

$$\dot{r}_{ka} = \frac{\partial H}{\partial p_{ka}} = p_{ka} / m_a, \quad (7.3.7a)$$

$$\dot{p}_{ka} = - \frac{\partial H}{\partial r_{ka}} + s_{ka} \frac{\partial H}{\partial p_{ka}} = F^{SA}{}_{ka} + F^{NSA}{}_{Ext{\cdot}ka}, \quad (7.3.7b)$$

where SA stands for the *conditions of variational selfadjointness* and NSA stands for their violation [7,8].

The verification of the right and left scalar and distributive laws by brackets (A, B) is evident.[55] Equally evident is their Lie-admissibility because their attached antisymmetric brackets are Lie,

$$(A, B) - (B, A) \equiv 2 [A, B]. \quad (7.3.8)$$

[55] Note that the addition of a second term in the equation for $\dot{r}_{ka}$ would imply the loss, in general, of the physical meaning of the linear momentum $p_{ka} = m_1 \dot{r}_{ka}$.

Thus, Lie–admissible equations (7.3.7) resolve all problematic aspects of Eq.s (7.1.2).

Very few additional papers appeared in the decade following ref.s [12–14] (see the "genealogical tree" on Lie–admissible algebras, ref. [7], p. 304 and quoted literature in pp. 414–415). However, following paper [7] of 1978, the study of Lie–admissible algebras increased considerably, also as a result of a series of *Workshops on Lie–admissible Formulations* organized by this author (see the general bibliography [19]).

The Lie–admissible algebras made their first appearance in operator mechanics ref. [20], p. 746, of 1978 as the central structural algebras of hadronic mechanics via the basic dynamical equations

$$i\, dA / dt = (A \hat{,} B) = A\, R\, B - B\, S\, A, \tag{7.3.9a}$$

$$R, S, R + S \neq 0, \quad R \neq S^{\dagger}, \tag{7.3.9b}$$

with exponentiated form (ref. [20], Sect. 4.18, p. 779 ff.)

$$A(t) = e^{+ i H S t}\, A(0)\, e^{- i t R H}, \tag{7.3.10}$$

and time–rate–of–variation of the energy operator

$$i\, dH / dt = (H \hat{,} H) = H\, (R - S)\, H\, . \tag{7.3.11}$$

It is evident that the product (A, B) characterizes a general Lie–admissible algebra because the attached algebra is Lie–isotopic (rather than Lie)

$$(A \hat{,} B) - (B \hat{,} A) = [A \hat{,} B] = A\, T\, B - B\, T\, A\, , \quad T = R + S\, . \tag{7.3.12}$$

The algebra characterized by the following brackets

$$(A, B) = p\, AB - q\, BA, \tag{7.3.13}$$

with p and q non–null scalars (or functions, was introduced by the author [12] back in 1967[56] as a realization of flexible Lie–admissible and Jordan–admissible algebras (App. I.4.A).

Subsequently, the Lie–admissible algebras made their first appearance in statistical mechanics in paper [21] of 1979 via the master equation for the density matrix

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56 The sole realization of the Lie–admissible product introduced by Albert [18] is $(a, b) = \lambda ab - (1 - \lambda) ba$ which does not include the so–called *q–deformations* (App. I.7.A) as a particular case, while the latter are indeed a particular case of the Lie–admissible algebras with product (7.3.13).

$$i\, d\rho / dt \equiv [\rho, H] + C \equiv (\rho, H) = \rho R H - H S \rho, \qquad (7.3.14)$$

which admits conventional equations of type (7.2.5) as a particular case with the identifications

$$\rho H - H \rho + C \equiv \rho R H - H S \rho, \; R = 1, \; S = 1 + H^{-1} \rho^{-1} C, \qquad (7.2.15)$$

although Eq. (7.3.15) are transparently more general than (7.2.5).

Since that time (1979), Lie-admissible algebras have been submitted to considerable, mathematical and physical studies by numerous authors. A comprehensive bibliography until 1984 can be found in ref. [19]. More recent accounts can be found in Vol.s II and III.

Monograph [11] presents the Lie-admissible theory in classical realization. In this chapter we shall outline the foundations of the *Lie-admissible theory* in its operator realization. Applications will be studied in the subsequent volumes.

This is a line of study conducted by this author [7,10,11] which is considerably different than the studies generally listed in bibliography [19]. In fact, the latter were conducted within the context of abstract nonassociative algebras, while the former refer, specifically, to a step-by-step generalization of enveloping algebras, Lie algebras, Lie groups, representation theory, etc. The understanding is that all studies in Lie-admissibility, whether explicitly or implicitly oriented for the generalization of Lie's theory, are relevant for these volumes because they deal with the mathematical structure of hadronic mechanics.

The inspection of classical studies [11] is recommended for the reader interested in acquiring a technical knowledge of the field, because all the basic concepts of the the Lie-admissible formulations already exist at the classical level, where they find theirs clearest realization.[57]

In summary, we have the following *three* notion of Lie-admissibility:

1) Albert Lie-admissibility [18] which characterizes a nonassociative algebra U such that U^- is Lie *without any condition that conventional Lie algebras are contained as particular cases of* U. In fact, Albert was primarily concerned with realizations of *Jordan-admissible* algebras with *quasiassociative realization* (see App. I.4.A for details)

$$(A, B) = \lambda\, Ab - (1 - \lambda)\, BA\,. \qquad (7.2.16)$$

where λ is a non-null parameter, which is indeed Lie-admissible (and Jordan-

[57] Particularly important is the classical realization of the *Lie-admissible symmetries* which provide a structural generalization of Noether's theorem whereby the Lie-admissible symmetries characterize the time-rate-of-variation of physical quantities, thus admitting as a particular case Lie symmetries and conservation laws. Rather oddly, these covering notions have remained virtually ignored in the physical literature.

admissible), yet it does not contain any Lie algebra for finite values of λ.

2) Lie–Santilli Lie–Admissibility [18,12] which is Albert's notion of Lie-admissibility, *plus the condition that Lie algebras are a particular case of* U [12]. This latter condition was evidently necessary for physical applications. In fact, Santilli introduced, apparently for the first time in paper [12] the realization

$$(A, B) = \lambda A B - \mu B A, \tag{7.2.17}$$

which is indeed Lie-admissible (and Jordan-admissible) and admits conventional Lie algebras for the values of the parameters $\lambda = \mu = 1$.

3) Albert–Santilli Lie–Isotopic–Admissibility [18,12,20] which is the preceding notion *under the condition that the attached algebras is not Lie, but Lie–Santilli* [20], i.e., the nonassociative algebra U must be such that U^- is Lie-isotopic and U must contain as particular case conventional Lie algebras, as it is the case for the product first introduced in memoir [20]

$$(A, B) = A R B - B S A, \tag{7.2.18}$$

where R and S are non-null operators. The latter condition resulted to be necessary for physical applications because of insufficiencies of the preceding definition and it is that used in these volumes.

7.4: GENONUMBERS

The technical understanding of the Lie-admissible formulations requires the knowledge that they are based on a generalized theory of numbers beyond that of isonumbers.

Let $F(\alpha,+,\times)$ be a conventional field (Sect. 2.3) with multiplication $\alpha\beta := \alpha\times\beta$. In Ch. I.2 we have reviewed a generalization of this basic operation into the isotopic form $\alpha*\beta = \alpha T\beta$. Both products $\alpha\beta$ and $\alpha*\beta$ are based on the assumption that they apply irrespective of whether α multiplies β from the left, or β multiplies α from the right. We can therefore introduce the following:

Definition 7.4.1 - Ordering of the multiplication [22]: *The multiplication of two numbers α and β is ordered to the right, and denoted $\alpha>\beta$, when α multiplies β to the right, while it is ordered to the left, and denoted $\alpha<\beta$ when β multiplies α from the left.*

Note that the above ordering is compatible with other properties and axioms of number theory. As an example, if the original field F is commutative, it remains commutative after the above ordering, that is, if $\alpha\beta = \beta\alpha$, then $\alpha>\beta = \beta>\alpha$ and $\alpha<\beta = \beta<\alpha$. The same occurrence holds for other properties, such as

associativity while the verification of the left and right distributive laws is evident. Thus, the entire Definition 2.3.1 can therefore be reformulated under ordering by characterizing fully acceptable fields.

The point at the foundations of the Lie-admissible theory is that *the multiplications of the same numbers in different orderings are generally different,* $\alpha > \beta \neq \beta < \alpha$. In turn, this implies the possibility of introducing *two ordered isounits,* called *genounits,* one per each ordering

$$\hat{1}^{>}: \ \hat{1}^{>} > \alpha = \alpha > \hat{1}^{>} \equiv \alpha, \tag{7.4.1a}$$

$$^{<}\hat{1}: \ ^{<}\hat{1} < \alpha = \alpha < {^{<}\hat{1}} \equiv \alpha, \tag{7.4.1b}$$

The above features permit a dual generalization of Definition 2.3.1, one for ordering to the right, yielding the *right genofield*

$$\hat{F}^{>}(\hat{\alpha}^{>},+,\hat{*}^{>}), \qquad \hat{\alpha}^{>} = \alpha \hat{1}^{>}, \tag{7.4.2}$$

whose elements $\hat{\alpha}^{>}$ are called *right genonumbers,* and one to the left, yielding the *left genofield*

$$^{<}\hat{F}(^{<}\hat{\alpha},+,^{<}\hat{*}), \qquad ^{<}\hat{\alpha} = {^{<}\hat{1}}\, \alpha, \tag{7.4.3}$$

whose elements $^{<}\hat{a}$ are called *left genonumbers.* The above two different genofields are often denoted with the unified symbol $^{<}\hat{F}^{>}(^{<}\hat{\alpha}^{>},+,^{<}\hat{*}^{>})$, with the understanding that the orderings can solely be used individually and not jointly.

The realization of the genoproducts used in these volumes is given by the following two different isotopic multiplications, one to the right and one to the left,

$$\alpha > \beta := \alpha R \beta, \tag{7.4.4a}$$

$$\alpha < \beta := \alpha S \beta, \tag{7.4.4b}$$

where $R \neq S$, with realization of the genounits

$$\hat{1}^{>} = R^{-1}, \qquad \hat{1}^{>} > \alpha \equiv \alpha, \tag{7.4.5a}$$

$$^{<}\hat{1} = S^{-1}. \qquad \alpha < {^{<}\hat{1}} \equiv \alpha. \tag{7.4.5b}$$

The entire theory of isonumbers of Ch. 2, including isoreal, isocomplex, isoquaternions and isooctonions numbers, then admits a generalization into the *theory of genonumbers* first introduced in ref. [22].

Note the need for a prior isotopy $\alpha\beta \to \alpha T \beta$ in order to construct genotopies (7.4.4). In fact, no ordering is evidently meaningful for the conventional

multiplication $\alpha\beta = \alpha 1 \beta$.

So far we have presented in this section the right and left genomultiplications and related isounits as disjoint, in which case the isounits can indeed be Hermitean and real-valued, thus admitting of Kadeisvili classification into Classes I, II, III, IV, V.

Nevertheless, the realizations used in physics are those when the right and left genounits are inter-related by a conjugation, such as the Hermitean conjugation

$$\hat{1}^{>} = ({}^{<}\hat{1})^{\dagger}, \tag{7.4.6}$$

In this case Kadeisvili's classification still holds, but must be referred to the Hermitean parts of the genounits. More specifically, we shall decompose the genounits into the products

$$\hat{1}^{>} = \hat{1}\, P, \qquad {}^{<}\hat{1} = Q\hat{1}, \qquad \hat{1} = \hat{1}^{\dagger}, \qquad P^{\dagger} = Q\,, \tag{7.4.7}$$

where $\hat{1}$ is the maximal Hermitean part. We can then classify the theory of genonumbers into Kadeisvili's Classes I, II, III, IV and V now referred to the maximal Hermitean part of the genounits.

As it will be soon evident, *under the above interconnection, the product ordered to the right can be interpreted as characterizing motion forward in time, while that ordered to the left can represent motion backward in time.* In different term, the ordering of Definition 7.4.1 can represent *Eddington's "arrows of time"*, and we have the following:

Lemma 7.4.1 [22]: *An axiomatization of irreversibility in number theory is given by: A) the ordering of the multiplications to the right and to the left, representing motion forward and backward in time, respectively; B) the differentiation of these two multiplications; and C) the assumption of an interconnecting map representing time-reversal.*

As we shall see in the rest of this chapter and in Vol.s II and III, the theory of genonumbers that with interconnecting map is the true foundation of the Lie-admissible branch of hadronic mechanics,

Note that the simpler theory of isonumbers is a subcase of that of genonumbers under the simple condition

$$R = S = R^{\dagger}, \tag{7.4.8}$$

This illustrates that the origin of the reversibility of the Lie and Lie-isotopic theories can be seen in their respective theories of numbers and isonumbers and, more specifically, from the fact that their multiplications to the right and to the

left are identical, $\alpha > \beta \equiv \alpha < \beta$.

We close this section with a few mathematical comments. Realization (7.4.4) is evidently not unique. In fact, other realizations of ordered multiplications are given by

$$\alpha > \beta = W \alpha W R W \beta W, \quad W^2 = W, \tag{7.4.9a}$$

$$\alpha < \beta = Z \alpha Z S Z \beta Z, \quad Z^2 = Z, \tag{7.4.9b}$$

where $R \neq S$ and $W \neq Z$. The latter realizations are not used in physics to our best knowledge at this writing, because they do not verify the Fundamental Condition 4.4.1 of admitting unique, left and right units.

Conjugation (7.4.8) is used in physics, but in mathematics one can introduce any other conjugation, such as that characterized by isoduality

$$\hat{1}^{>} = ({}^{<}\hat{1})^{d} = -{}^{<}\hat{1}, \tag{7.4.10}$$

or have no conjugation at all.

Finally, note that the notion of isoduality applies also to genofields, yielding the *isodual genofields* ${}^{<}\hat{F}^{> d}({}^{<}\hat{a}^{> d}, +, {}^{<}\hat{*}^{> d})$.

In the preceding chapters we have indicated the truly remarkable, novel mathematical developments permitted by the theory of isonumbers. The yet broader theory of genonumbers permit additional mathematical developments that are simply inconceivable with conventional theories.

As an illustration, the Lie product AB −BA originates from two envelopes, one for the multiplication to the right with product BA, and one to the left with product AB, as we shall see, even though these two multiplications are evidently identical. Then, the theory of genonumbers permits *the reinterpretation of Lie algebras as commutative Jordan algebras defined on two genofields interconnected by isoduality*, i.e.

$$A B - B A \equiv A I B + B I^{d} A. \tag{7.4.11}$$

The remark is important to indicate that *Jordan legacy (i.e., a possible quantum mechanical content of Jordan algebras) is still open.*

We finally note that the ordering of the multiplication can also be extended to the addition, although it must necessarily be lifted to be nontrivial. This further generalization is not used in physics because it violates the distributive law as studied in Sect. I.2.3. Nevertheless, the extension is significant to point out that the most general notion of "numbers" introduced by this author [22], the *theory of genonumbers and their isoduals.* It can be expressed by the unified symbol ${}^{<}\hat{F}^{>}({}^{<}\hat{\alpha}^{>}, {}^{<}\hat{+}^{>}, {}^{<}\hat{\times}^{>})$, representing: three separate generalizations of the numbers $\alpha \to \hat{\alpha} \to \hat{\alpha}^{>} \to {}^{<}\hat{\alpha}$; characterized by three separate generalizations of

the operations $+ \rightarrow \hat{+} \rightarrow \hat{+}^> \rightarrow {}^<\hat{+}$ and $\times \rightarrow \hat{\times} \rightarrow \hat{\times}^> \rightarrow {}^<\hat{\times}$; with three separate generalizations of the additive units $0 \rightarrow \hat{0} \rightarrow \hat{0}^> \rightarrow {}^<\hat{0}$, and multiplicative units $1 \rightarrow \hat{1} \rightarrow \hat{1}^> \rightarrow {}^<\hat{1}$, plus the image of all these structures under isoduality.

Such genonumbers can be not only of dimension 1 *(genoreals)*, 2 *(genocomplex)*, 4 *(genoquaternions)* and 8 *(genooctonions)*, but also have dimension 3, 5, 6, 7 (called *"hidden numbers"* because they hidden in the operations as for the case of isonumbers (see App. I.2.A and ref. [22] for brevity).

7.5: GENOSPACES

The entire theory of isospaces of Ch. I.3 admits a consistent and significant genotopic covering. Let $S(x,g,R)$ be a conventional metric or pseudo-metric space and $\hat{S}(x,\hat{g},\hat{R})$ its family of isotopes. Then, the following *left and right genospaces* hold

$$\hat{S}^>(x,\hat{g}^>,\hat{R}^>): \ \hat{g}^> = g\,R\,, \ x^{\hat{2}>} = x^t\,\hat{g}^>\,x, \quad \hat{1}^> = R^{-1}\,, \tag{7.5.1a}$$

$$^<\hat{S}(x,{}^<\hat{g},{}^<\hat{R}) : \ {}^<\hat{G} = S\,g\,, \quad x^{<\hat{2}} = x\,{}^<\hat{g}\,x^t\,, \quad {}^<\hat{1} = S^{-1}\,. \tag{7.5.1b}$$

$$\hat{1}^> = (\,{}^<\hat{1}\,)^\dagger\,, \tag{7.5.1c}$$

A most visible difference between genospaces and isospaces is therefore that the invariant in the former is unique, while in the latter we have two different invariants, one for the multiplication to the right and one to the left.

When the two multiplications are interconnected by conjugation (7.5.1c), we have *two different genospaces one for motion forward in time, and one for motion backward in time.*

The most significant genospaces, denoted with a unified notation $^<\hat{S}^>(x,{}^<\hat{g}^>,{}^<\hat{R}^>)$, are given by:

I) genoeuclidean spaces $^<\hat{E}^>(x,{}^<\hat{\delta}^>,{}^<\hat{R}^>)$ and their isoduals;

II) genominkowskian spaces $^<\hat{M}^>(x,{}^<\hat{\eta}^>,{}^<\hat{R}^>)$ and their isoduals;

III) genoriemannian spaces $^<\hat{\mathfrak{R}}^>(x,{}^<\hat{g}^>,{}^<\hat{R}^>)$ and their isoduals,

where $\hat{\delta}$, $\hat{\eta}$, $\hat{g}$ are the isometric of the corresponding isospaces of Ch. I.3.

It should be noted that conventional spaces, such as the Euclidean space $E(r,\delta,R)$, admit a nontrivial isodual images $E^d(r,\delta^d,R^d)$. However, their genoimages $^<E^>(r,{}^<\delta^>,{}^<R^>)$ without a joint isotopy are trivial, evidently because $^<\delta^> \equiv \delta$. This occurrence is similar to that of the preceding section whereby ordinary fields $F(a,+,\times)$ admit nontrivial isoduals $F^d(a^d,\times d)$ without isotopies, but trivial genotopes, $^<F^> \equiv F$, because $a>b \equiv b<a$ for ordinary fields.

The lack of a significant "arrow of time" in the conventional numbers and

spaces is the axiomatic origin of their *reversibility*. By comparison, the presence of a structural "arrow of time" in the theory of genonumbers and genospaces renders them particularly suited to represent *irreversibility*.

The use of conventional transformation theory for genospaces also violates linearity, transitivity and other basic laws. For this reason it must be lifted into the *right and left genotransformations*

$$x' = \ U^{>} > x \ = U^{>} R\, x\,, \tag{7.5.2a}$$

$$x' = x < {}^{<}U \ = x\, S\, {}^{<}U\,. \tag{7.4.2b}$$

The above transformations are one-sided isolinear, isolocal and isocanonical as it occurs for the isotransformations. This illustrates again that the ordering of the multiplication does indeed preserve all basic axioms. The remaining aspects of isospaces (Ch. I.3) and their transformation theory therefore admit a consistent and intriguing generalization into left and right theories.

7.6: LIE-ADMISSIBLE THEORY

Recall that the conventional unit I is at the foundation of Lie's theory, and the same occurrence holds for the Lie-isotopic theory.

The distinction of the multiplication to the right from that to the left with corresponding different genounits implies an evident generalization of the entire Lie and Lie-isotopic theories whose study has only been initiated at this writing [11]. We here indicate the existence of two genoassociative enveloping algebras $\xi^{>}$ and ${}^{<}\xi$ with the same elements A, B, C, ... denoted with the joint symbol ${}^{<}\xi^{>}$, buy different genoproducts and genounits

$$\xi^{>} \,:\, A > B := A\,R\,B\,, \quad 1^{>} = R^{-1}\,, \tag{7.6.1a}$$

$${}^{<}\xi \,:\, A < B := A\,S\,B\,, \quad {}^{<}1 = S^{-1}\,. \tag{7.6.1b}$$

defined over corresponding genofields ${}^{<}F^{>}({}^{<}\hat{\alpha}^{>},+,{}^{<}*^{>})$.

It is easy to see that the isotopic Poincaré-Birkhoff-Witt theorem (Sect. 4.3) can be consistently generalized for each direction of the multiplication, yielding an infinite-dimensional base for each genoassociative envelope.[58]

This allows the introduction of the unique, fundamental notions of

[58] This is possible because, again, the genoalgebras admit well defined right and left units. By comparison, q-deformation have no such unit and, therefore, do not admit a unique basis for their exponentiation (App. I.7.A).

genoexponentiation

$$\hat{U}^{>} = e_{\xi >}^{iXw} = \{ e^{iXRw} \} \hat{1}^{>}, \tag{7.6.2a}$$

$$^{<}\hat{U} = e_{<\xi}^{iwX} = {}^{<}\hat{1} \{ e^{iwSX} \}, \tag{7.6.2b}$$

which, in turn, permit the formulation of the *Lie-admissible group* first introduced in refg. [7] (see also ref.s (10,11]), which is given by the left and right genotransformations of a generic quantity $Q \in {}^{<}\xi^{>}$

$$Q(w) = \hat{U}^{>} > Q(0) < {}^{<}\hat{U} = \{ e_{\xi >}^{iXw} \} > Q(0) < \{ e_{<\xi}^{iwX} \} =$$

$$= \{ e^{iXRw} \} Q(0) \{ e^{iwSX} \}, \tag{7.6.2b}$$

Its most fundamental feature is of admitting a non-Lie/non-Lie-isotopic but Lie-admissible algebra in the neighborhood of the genoidenties

$$i \frac{dQ}{dw} = (Q \hat{,} X) = Q < X - X > Q, \tag{7.6.3}$$

thus confirming the existence of a Lie-admissible generalization of Lie's theory at all various levels (enveloping algebras, Lie algebras, Lie groups, etc.). Structure (7.6.3) also confirms that the l.h.s. of the product $(Q \hat{,} X)$ is characterized by the backward genoenvelope, while the r.h.s. is characterized by the forward genoenvelope, as anticipated earlier.

An important point for the correct interpretation and use of the theory is that *the envelopes underlying the Lie-admissible formulations remain associative*, thus verifying Fundamental Condition I.4.4.1. In different terms, structure (7.6.3) is a generalization of the corresponding Lie and Lie-isotopic structures

$$i \frac{dQ}{dw} = [Q, H] = QH - HQ, \tag{7.6.4a}$$

$$i \frac{dQ}{dw} = [Q \hat{,} H] = QTH - HTQ, \tag{7.6.4b}$$

where, as now familiar, the brackets [,] and [$\hat{,}$] are nonassociative, but their envelopes with respective product QX and QTX are indeed associative.

Exactly the same occurrence holds for the more general Lie-admissible formulations. In fact, the brackets ($\hat{,}$) are evidently nonassociative, but the underlying envelopes with products Q>H and H<Q are isoassociative.

In Vol. II we shall study the basic laws of the Lie-admissible representation

of interior systems. In particular, we shall identify the *Lie-admissible symmetries* and show that they characterize *time-rate-of-variation of physical quantities*, by providing in this way an operator counterpart of the corresponding classical notions [11], and by reaching an intriguing covering of the corresponding notions for Lie and Lie-isotopic theories.

The most important application of the Lie-admissible theory is the characterization of the most general known notion of particle, called *genoparticle*, as studied in more details in Vol.s II and III. At this moment we simply list the notions of particles used in hadronic mechanics:

Conventional particles, which is characterized by Lie symmetries in a stable-reversible orbit, such as an electron of an atomic structure;

Isoparticles which is characterized by the Lie-isotopic symmetries also on stable orbits, such as the constituents of few-body nuclei and hadrons; and

Genoparticles, which is characterized by Lie-admissible symmetries on the most general known nonconservative, unstable and irreversible orbit, such as an electron in the core of a star undergoing gravitational collapse

plus all their isoduals.

The best way to understand the conceptual, mathematical and physical advances permitted by the Lie-admissible theory is by inspecting the underlying representations called *genorepresentations.*

In Sect. I.4.7. we have studied the isorepresentation theory which is based on the notion of module implying only one action, e.g., that to the right. By comparison genorepresentations of Lie-admissible algebras require a *two-sided isobimodule* called *genomodule.*

Consider an algebra U over a field $F(\alpha,+,\times)$. Let V be a vector space over F and introduce the direct sum

$$S = U \oplus V \tag{7.6.5}$$

in such a way that S is an algebra verifying the same axioms of U while V is a two sided ideal of S.

This can be done as follows [23]:

1) retain the product of U;

2) introduce a left and a right composition av and va, for all elements a
$\in$ U and $v \in V$ which verify all axioms of U (including the right and
left scalar and distributive laws); and

3) to complete the requirement that V is an ideal of S, assume vt = tv = 0 for all elements of V.

When all the above properties are verified, V is called a *two-sided, left and right module,* or a *bimodule* of U, and the algebra S is called a *split null extension* of U [loc. cit.].

Bimodules clearly provide a generalized, left and right representation theory of all algebras, whether associative or nonassociative. It is important to understand why bimodules are not needed for the representation theory of conventional Lie algebras (i.e., for the conventional notion of particle) as well as of Lie–isotopic algebras (i.e., for the generalized notion of isoparticle), but they become essential for the covering Lie–admissible algebras (i.e., for the most general possible notion of genoparticle).

A *bimodule* V *of a Lie algebra* L or *Lie–bimodule* [24] is characterized by left and right compositions av and va, $a \in L$, $v \in V$, verifying the properties

$$a\,v = -\,v\,a, \tag{7.6.5a}$$

$$v(a\,b) = (v\,a)\,b - (v\,b)\,a, \tag{7.6.5b}$$

which can be identically expressed via the left and right multiplications

$$L_a = -R_a, \tag{7.6.7a}$$

$$R_{ab} = R_a R_b - R_b R_a, \tag{7.6.7b}$$

The mappings $a \Rightarrow R_a$ and $a \Rightarrow L_a$ then provide *a left and right representation,* or a *birepresentation,* of the Lie algebra L over the bimodule V as a $\mathrm{Hom}^L{}_F(V_R, V_L)$.

However, owing to property (7.6.6a), the left representation is trivially equivalent to the right representation, $R_a = -L_a$.[59] This is the reason why only one–sided representations of Lie algebras are significant in quantum mechanics.

The notions of isomodules and isobimodules were introduced for the first time in ref. [24] of 1979, although they do not appear to have been studied thereafter in the mathematical or physical literature. In essence, a *Lie–isobimodule* is an isovector space $\hat{V}$ over an isofield $\hat{F}(\alpha,+,*)$ with left and right isocompositions $a * v$ and $v * a$ verifying the distributive and scalar laws, and the rules

$$a * v = -\,v * a, \tag{7.6.8a}$$

$$v * (a * b) = (v * a) * b - (v * b) * a, \tag{7.6.8b}$$

or, equivalently in terms of isomultiplications

$$\hat{R}_a = -\,\hat{L}_a, \tag{7.6.9a}$$

$$\hat{R}_{a*b} = \hat{R}_a * \hat{R}_b - \hat{R}_b * \hat{R}_a, \tag{7.6.9b}$$

---

[59] Note again the intriguing possibility of reinterpreting the left representation as an isodual of the right here left to the interested reader.

An *isobirepresentation* of a Lie-isotopic algebra $\hat{L}$ is then given by $Hom^{\hat{L}}{}_{\hat{F}}(\hat{V}_R, \hat{V}_L)$.

However, the left and right isorepresentations are again equivalent because of the property $\hat{R}_a = - \hat{L}_a$. Thus, *only one-sided isomodules and one-sided isorepresentations are needed for the Lie-isotopic branch of hadronic mechanics*, and this explains the reason for our silence on them in Ch. I.4.

Note also that the above equivalence between the right and left isomodular actions is an axiomatic representation of reversibility. This implies that isoparticles as characterized by one-sided isorepresentations are on stable-reversible orbits.

The *two-sided isorepresentations*, or *genorepresentations*, become *necessary* when studying Lie-admissible algebras evidently because of the loss of the antisymmetric (or symmetric) character of the product. As a result, the representation theory of the Lie-admissible algebras is much richer than those of the Lie and Lie-isotopic algebras.

A *Lie-admissible bimodule* $\hat{V}$, or *genomodule* for short, is a vector isospace over a genofield $^<\hat{F}^>$ equipped with two, inequivalent, right and left compositions $a>v$ and $v<a$ such that the attached composition $a \odot v = a<v - v>a$ verifies the axioms

$$a \odot v = - v \odot a, \tag{7.6.10a}$$

$$v \odot (a \odot b) = (v \odot a) \odot b - (v > b) \odot a. \tag{7.6.10b}$$

Thus, genomodules are characterized by their attached composition $a>b - v<a$, rather then each individual actions $a>v$ and $v<a$. They can be equivalently expressed via the right and left isomultiplications

$$\hat{R}_{a>v - v<a} + \hat{L}_{a>b - b<a} = [(\hat{r}_a - \hat{L}_a), (\hat{R}_b - \hat{L}_b)]. \tag{7.6.11}$$

A *genorepresentation of a Lie-admissible algebras* U over the genofields $^<\hat{F}^>$ is therefore given by the $Hom^U{}_{<\hat{F}>}(^<\hat{V}_{\hat{L}}, \hat{V}^>{}_{\hat{R}})$.

Further advances in the Lie-Isotopic-Admissible theory (Sect. 7.3) were presented by this author at the International Congress of Mathematicians (ICM) held in Zurich in August 1994 [53], and can be expressed via the following:

Theorem 7.6.1: *The general Lie-admissible product* $(A, B) = A<B - B>A$ *is neither antisymmetric nor symmetric when projected in the conventional vector space of the original Lie algebra over conventional fields* V(F),

$$(A, B) = \{ A<B - B>A \}_{V(F)} \neq \pm (B , A) \tag{7.6.12}$$

but the same product becomes totally antisymmetric and verify the Lie axioms

when each ordered product is computed in its own genovector space over the corresponding genofield, $^<\hat{V}^>(^<\hat{F}^>)$, i.e.,

$$(A, B) = \{ A<B \}_{^<\hat{V}(^<\hat{F})} - \{ B>A \}_{\hat{V}^>(\hat{F}^>)} = -(B, A). \qquad (7.6.12)$$

Proof. When computed as a genovector space over a genofield, the genoenvelope $\hat{\xi}^>$ with product A>B as well as the genoenvelope $^<\hat{\xi}$ with product A<B are both isomorphic to the conventional envelope ξ with product AB

$$\xi(F) \approx \hat{\xi}^>(\hat{F}^>) \approx {}^<\hat{\xi}(^<\hat{F}). \qquad (7.6.13)$$

. Under these conditions, the product (A, B) = A<B − B>A verifies the Lie axioms in exactly the same measure as the conventional product [A, B] = AB − BA. **q.e.d.**

To see this result from a different perspective, recall from Ch. I.4 that the isotopy ξ: AB → $\hat{\xi}$: A*B = ATB satisfies the local isomorphism $\xi \approx \hat{\xi}$ when T is positive-definite and $\hat{\xi}$ is computed with respect the isofield whose isounit is the *inverse* of the deformation, i.e., $\hat{1} = T^{-1}$.

Exactly the same situation occurs for the product (A, B) = A<B − B>A. In fact, the genotopy ξ: AB → $\hat{\xi}^>$: A>B = ASB also verifies the isomorphism $\xi \approx \hat{\xi}^>$ when the latter is computed with respect to a genofield whose genounits is the *inverse* of the deformation, $\hat{1}^> = S^{-1}$ and exactly the same situation occurs for the conjugate genoenvelope. The verification of the Lie axioms by the Lie-Isotopic-Admissible product (A,. B) then follows.

It is evident that Theorem 7.6.1 confirms the possible construction of the Lie-Isotopic-Admissible theory as a step-by-step genotopy of Lie's theory suggested in ref.s [10,11], as we hope to study at some future time.

The physical meaning of the Lie-admissible theory is identified by the following:

Lemma 7.6.1 [11]: *An axiomatization of irreversibility at the algebraic-group theoretical level is provided by the differentiation of enveloping associative algebras of Lie's theory into two genotopic forms of the Lie-admissible theory* $\hat{\xi}^>$ *and* $^<\hat{\xi}$ *and related genorepresentations characterizing motion forward and backward in time, respectively, with a corresponding interconnecting conjugation, and related forward and backward genounits* $\hat{1}^>$, $^<\hat{1}$, *for corresponding right and left actions.*

The axiomatic nature of the above characterization is expressed by the fact that irreversibility is intrinsic in the theory, i.e., it holds also for time-reversible Hamiltonians, as we shall see better in Vol. II. By comparison, both Lie and Lie-isotopic theories are structurally reversible.

The implications of the above axiomatization of irreversibility are far reaching. In fact, as we shall study in detail in Vol.s II and III, the lifting of the

Poincaré symmetry into its Lie-admissible covering (first proposed at the classical level in ref. [11]) characterizes the most complex known notion of particle with locally varying intrinsic characteristics, as expected to represent the most complex known physical conditions in Nature, such as for a neutron in the core of a neutron star.

7.7: GENOGEOMETRIES

As stressed throughout our studies, physical theories in general, and relativities in particular, are a symbiotic expression of deeply interconnected and mutually compatible analytic, algebraic and geometric formulations.

In the preceding sections we have presented the analytic and algebraic structures of the Lie-admissible theory. It is therefore important to show that, exactly as it occurs for the Lie and Lie-isotopic theories, the Lie-admissible theory also admits a fully defined geometric counterpart.

This problem was studied in ref.s [7,11] and resulted in the submission of new geometries, more general than the isogeometries of Ch. I.5, called *genogeometries*, according to the following main lines.

7.7.A: Genoeuclidean and genominkowskian geometries. They are the geometries of the genospaces ${}^{<}\hat{E}^{>}(r,{}^{<}\hat{\delta}^{>},{}^{<}\hat{R}^{>})$ and ${}^{<}\hat{M}^{>}(r,{}^{<}\hat{\delta}^{>},{}^{<}\hat{R}^{>})$, respectively, and are essentially given by the isoeuclidean and isominkowskian geometries of Sect. I.5.2 and I.5.3, in different realizations for each "arrow of time".

The most important difference between the iso- and genogeometries is therefore that the metric of the former is unique for both directions of time, while the metric of the latter is differentiated depending on the assumed direction of time, $\hat{\delta}^{>} \neq {}^{<}\hat{\delta}$ and $\hat{\eta}^{>} \neq {}^{<}\hat{\eta}$.

However, the base unit is lifted in correspondence to each of these generalized metrics according to the rules

$$g \Rightarrow \hat{g}^{>} = g\,T^{>}, \quad I \Rightarrow \hat{1}^{>} = (T^{>})^{-1}, \tag{7.7.1a}$$

$$g \Rightarrow {}^{<}\hat{g} = {}^{<}T\,g, \quad I \Rightarrow {}^{<}\hat{1} = ({}^{<}T)^{-1}, \tag{7.7.1b}$$

as a result of which all the peculiar properties of the isogeometries are preserved for forward and, separately, backward genogeometries.

This implies the existence of two different deformations of the sphere, the light cone, etc., for interior dynamical problems, one per each direction of time, each of which is mapped into the perfect sphere and the perfect cone in genospace.

The extension of the remaining properties of isogeometries into the

genotopic form is an instructive exercise for the interested reader, and it will be assumed hereon.

7.7.B: Genosymplectic geometry. Recall that the symplectic geometry is the geometry underlying Lie's theory, while the isosymplectic geometry (Sect. I.5.5) is that underlying the Lie-isotopic theory. In ref.s [7,11] this author showed that the yet more general Lie-admissible theory also admits a fully defined underlying geometry, evidently of a generalized nature submitted under the name of *symplectic-admissible geometry*, or *genogeometry* for short.

Recall from App. I.5.A Birkhoff's brackets in $T^*E(r,\delta,\Re)$ and related exact symplectic two-form in the now familiar unified notation $a = (a^\mu) = (r, p)$, $\mu = 1, 2, ..., 2n$,

$$[A \hat{,} B] = \frac{\partial A}{\partial a^\mu} \Omega^{\mu}(a) \frac{\partial B}{\partial a^\nu}, \tag{7.7.2a}$$

$$\Omega = \tfrac{1}{2} \Omega_{\mu\nu}(a)\, da^\mu \wedge da^\nu, \tag{7.7.2b}$$

where the algebraic-contravariant and geometric-covariant tensors are interconnected by the familiar rule

$$\Omega^{\mu\nu} = (\|\Omega_{\alpha\beta}\|^{-1})^{\mu\nu}. \tag{7.7.3}$$

In the transition to the Birkhoff-isotopic brackets on isospaces $T^*\hat{E}_2 r,\hat{\delta},\hat{R})$ with isounit $\hat{1}_2$ (Sct. I.5.4),

$$[A \hat{,} B] = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\alpha}(a) \hat{1}_{2\alpha}{}^{\nu}(t, a, \dot{a}, ...)) \frac{\partial B}{\partial a^\nu}, \tag{7.7.4}$$

we have the transition to the isosymplectic geometry characterized by the isoexact two-isoform

$$\hat{\Omega} = \tfrac{1}{2} T_\mu{}^{\alpha}(t, a, \dot{a}, ...)\, \Omega_{\alpha\nu}(a)\, \hat{d}a^\mu \hat{\wedge} \hat{d}a^\nu \quad , \tag{7.7.5}$$

where, again, the algebraic and geometric tensors are interconnected by the rule

$$\Omega^{\mu\alpha} \hat{1}_{2\alpha}{}^{\nu} = [\ (T_{2\alpha}{}^{\rho}\, \Omega_{\rho\beta})^{-1}\ ]^{\mu\nu}. \tag{7.7.6}$$

The problem of the geometry underlying the *Birkhoff-admissible brackets* [7,11]

$$(A \hat{,} B) \;=\; \frac{\partial A}{\partial a^{\mu}} \; {}^{<}\hat{S}^{>\mu\nu}(t,a) \; \frac{\partial B}{\partial a^{\nu}} \;, \qquad (7.7.7a)$$

$$\hat{S}^{\mu\nu} \;=\; \Omega^{\mu\alpha} \; {}^{<}\hat{1}^{>}{}_{\alpha}{}^{\nu}, \qquad (7.7.7b)$$

$$\Omega^{\mu\nu} \;=\; -\; \Omega^{\nu\mu}, \qquad (7.7.7c)$$

$${}^{<}\hat{1}^{>}{}_{\alpha}{}^{\nu} \;=\; {}^{<}\hat{1}^{>}{}_{\nu}{}^{\alpha}, \qquad (7.A.7d)$$

was resolved via the introduction of a geometry more general than the symplectic and the isosymplectic ones.

The first point to realize is that *the symplectic geometry and related exterior calculus, whether in their conventional or isotopic formulations, are intrinsically unable to characterize the Lie-admissible algebras.*

This is due to the fact that the calculus of exterior forms is essentially *antisymmetric* in the indices, and so remains under isotopies by assumption, while the Lie-admissible tensors ${}^{<}\hat{S}^{>\mu\nu}$ are not antisymmetric, and the same occurs for the covariant counterpart

$${}^{<}\hat{S}^{>}{}_{\mu\nu}(t,\, a) \;=\; (\| {}^{<}\hat{S}^{>\alpha\beta} \|^{-1})_{\mu\nu} \;\neq\; \pm\, {}^{<}\hat{S}^{>}{}_{\nu\mu} \qquad (7.7.8)$$

. In fact, the construction of a conventional exterior two-form with the above tensor implies the reduction

$${}^{<}\hat{S}^{>}{}_{\mu\nu}\, da^{\mu} \wedge da^{\nu} \;\equiv\; \tfrac{1}{2}\, \hat{\Omega}_{\mu\nu}\, da^{\mu} \wedge da^{\nu}, \qquad (7.7.9)$$

namely, the symplectic geometry automatically eliminates the symmetric component of the S-tensor, thus characterizing only its Lie content.

The main idea of the symplectic-admissible geometry is that of generalizing the conventional exterior calculus, say, of two differentials

$$da^{\mu} \wedge da^{\nu} \;=\; -\; da^{\nu} \wedge da^{\mu}, \qquad (7.7.10)$$

into a more general calculus, called *exterior-admissible calculus*, or *genoexterior calculus*, which is defined over the genofield of real numbers ${}^{<}\hat{R}^{>}({}^{<}\hat{n}^{>},+,{}^{<}*^{>})$ based on a product, say $\odot$, which is neither totally symmetric nor totally antisymmetric, but such that its antisymmetric component is the conventional exterior one [7,11],

$$da^{\mu} \odot da^{\nu} \;=\; da^{\mu} \wedge da^{\nu} \;+\; da^{\nu} \times da^{\nu}, \qquad (7.7.11a)$$

$$da^{\mu} \wedge d\bar{a}^{\nu} \;=\; -\; da^{\nu} \wedge da^{\mu}, \qquad (7.7.11b)$$

$$da^{\mu} \times da^{\nu} \;=\; da^{\nu} \times da^{\mu}, \qquad (7.7.11c)$$

The isocotangent bundle is then further generalized into the *genocotangent bundle* $T^{*<\hat{E}>}(r, {}^<\hat{\delta}^>, {}^<\hat{R}^>)$ upon selection of one given ordering in the multiplication.

This allows the introduction of the *exterior-admissible forms* or *genoforms,* via the sequence

$$^<\hat{S}^>_0 = {}^<\phi^>(a), \tag{7.7.12a}$$

$$^<\hat{S}^>_1 = {}^<\hat{S}^>_\mu \, da^\mu, \tag{7.7.12b}$$

$$^<\hat{S}^>_2 = {}^<\hat{S}^>_{\mu\nu} \, da^\mu \odot da^\nu, \tag{7.7.12c}$$

....

The *exact exterior-admissible forms* or *exact genoforms,* are then given by

$$^<\hat{S}^>_1 = d\,{}^<\hat{S}^>_0 = \frac{\partial {}^<\phi^>}{\partial a^\mu} \, da^\mu, \tag{7.7.13a}$$

$$^<\hat{S}^{>2} = d\,{}^<\hat{S}^>_1 = \frac{\partial {}^<A^>_\nu}{\partial a^\mu} \, da^\mu \odot da^\nu, \tag{7.7.13b}$$

....

The calculus of exterior-admissible forms can indeed characterize the Lie-admissible algebras, because it characterizes not only the antisymmetric component of the Lie-admissible brackets, but also their symmetric part, via the two-forms

$$\begin{aligned} {}^<\hat{S}^>_2 &= {}^<\hat{S}^>_{\mu\nu}(t,a) \, da^\mu \odot da^\nu = \\ &= \hat{\Omega}_{\mu\nu}(a) \, da^\mu \wedge da^\nu + {}^<\hat{\tau}^>_{\mu\nu}(t, a) \, da^\mu \times da^\nu, \end{aligned} \tag{7.7.14}$$

Structures (7.7.14) are *symplectic-admissible two-forms* because their antisymmetric component is symplectic, in a way fully parallel to the property whereby the antisymmetric part of the Lie-admissible algebras is Lie. Structure (7.7.14) are also called *genosymplectic two-forms*, when emphasis is needed on the loss of the original antisymmetric axiom. The spaces $T^{*<\hat{E}>}(r, {}^<\hat{\delta}^>, {}^<\hat{R}^>)$, again selected either for the multiplication to the right or to the left, when equipped with two-form (7.7.14) are called *symplectic-admissible manifolds* or *genosymplectic manifolds,* and the related geometry is called *symplectic-admissible geometry* or *genosymplectic geometry.*

As incidental comments, note that the dependence on time appears only in the symmetric part, as needed for consistency in the symplectic component. Also,

under inversion (7.7.8), we generally have

$$(\tilde{\Omega}_{\mu\nu}) \neq (\Omega^{\alpha\beta})^{-1}, \quad (\langle T \rangle_{\mu\nu}) \neq (\langle T \rangle^{\alpha\beta})^{-1}, \tag{7.7.15}$$

which is a rather intriguing feature of the generalized geometry here considered, whereby the symplectic content of a contravariant tensor is more general than the symplectic counterpart of the covariant tensor. (see ref. [11] for details)

The most salient departure from the exterior calculus in its conventional or isotopic formulation is that *the Poincare' Lemma no longer holds for the genosymplectic geometry,* i.e., for exact symplectic-admissible two-forms we have

$$\langle \hat{S} \rangle_2 = d\langle \hat{S} \rangle_1, \tag{7.7.16a}$$

$$d\langle \hat{S} \rangle_2 = d(d\langle \hat{S} \rangle_1) \neq 0. \tag{7.7.16b}$$

In actuality, within the contest of the exterior-admissible calculus, the Poincare' Lemma is generalized into a rather intriguing geometric structure which evidently admits the conventional Lemma as a particular case when all symmetric components are null.

The geometric understanding of the Lie-isotopic algebras requires the understanding that *the validity of the Poincaré Lemma within the context of the isosymplectic geometry is a necessary condition for the representation of the conservation of the total energy under nonhamiltonian internal forces,* as studied in the main sections of this volume.

By the same token, the geometric understanding of the more general Lie-admissible formulations requires the understanding that *the lack of validity of the Poincaré Lemma within the context of the symplectic-admissible geometry is a necessary condition for the representation of the nonconservation of the energy of an interior dynamical system.*

7.7.C: Genoriemannian geometry. Despite impressive and historical advances in gravitation during this century, gravitation is still at its first infancy, particularly when compared to the problems yet to be addressed, let alone solved.

In Ch. I.5 we identified the need of an integral generalization of the Riemannian geometry for a more adequate representation of interior gravitational problems, such as gravitational collapse, "black holes", "big bang", etc., and submitted a generalization of the Riemannian geometry of the desired integral type called isoriemannian geometry.

The point to be stressed here is that physics is a discipline that will never admit final theories. No matter how advanced the isoriemannian geometry is over the Riemannian one, it is not expected to be "the" final geometry. Instead, the isoriemannian geometry is "one" geometry specifically conceived for one purpose,

the treatment of closed–isolated interior systems with total conservation laws under a generalized interior structure.

Another fundamental physical problem in gravitation which has not even been addressed so far, let alone solved, is the dichotomy expressed by experimental evidence in the observation, say, of Jupiter, according to which the center–of–mass of the celestial body is time–reversal invariant, while its interior dynamics is manifestly irreversible. It is evident that the conventional Riemannian geometry is insufficient to represent the interior irreversibility in the needed axiomatic form.

It is at this point that the dual Lie–isotopic and Lie–admissible formulations become useful. In fact, as indicated earlier, the Lie–isotopic formulations are structurally reversible while the Lie–admissible formulations are intrinsically irreversible. The dual representation of reversible center–of–mass–trajectories *versus* irreversible interior dynamics, is then permitted by the complementarity of the Lie–isotopic and Lie–admissible formulations because of their inter–relation discussed in this chapter (see also Fig. 7.1.1).

Note the necessity of the Lie–isotopic formulations for this complementarity. In fact, reversible, conventionally Lie formulations for the global–exterior description are not compatible with irreversible, Lie–admissible, interior descriptions because their attached Lie algebra is not Lie but Lie–isotopic.

It may therefore be of some value to indicate a conceivable generalization of the Riemannian geometry, under the name of *Riemannian–admissible geometry* or *genoriemannian geometry*, originally submitted in ref. [11] which provides an irreversible description of interior gravitation in a way compatible with and complementary to the reversible description of the isoriemannian geometry of Sect. I.5.6. The understanding is that, unlike the isoriemannian geometry, the genoriemannian extension is vastly unexplored at this writing.

The notion of genospace of Sect. I.7.3 can be specialized to that of *genoaffiine manifolds* as the manifolds ${}^{<}\hat{M}^{>}(x, {}^{<}\hat{R}^{>})$ which possess the same dimension, local coordinates and continuity properties of a conventional affine manifold $M(x,R)$, but are defined over an isofield ${}^{<}\hat{R}^{>}$ with two different isounits $\hat{1}^{>}$ and ${}^{<}\hat{1}$ for the modular–isotopic action to the right and to the left, respectively,

$$x'^{>} = A > x = AT^{>}x, \quad \hat{1}^{>} = (T^{>})^{-1}, \tag{7.7.17a}$$

$${}^{<}x' = x < A = x\,{}^{<}TA, \quad {}^{<}\hat{1} = ({}^{<}T)^{-1}, \tag{7.A.17.1b}$$

$$\hat{1}^{>} = ({}^{<}\hat{1})^{\dagger}. \tag{7.7.17.1c}$$

A *"Riemannian–admissible manifold" or genoriemannian manifold* can then be thought as an isoriemannian manifold (Definition I.5.6.1) with inequivalent isomodular actions to the right (forward in time) and to the left (backward in

time), here denoted $\langle\hat{R}\rangle(x,\langle\hat{g}\rangle, \langle\hat{R}\rangle)$; namely, a manifold characterized by the "genometrics for motions forward and backward in time"

THE DUAL ISORIEMANNIAN AND GENORIEMANNIAN REPRESENTATION OF INTERIOR GRAVITATION

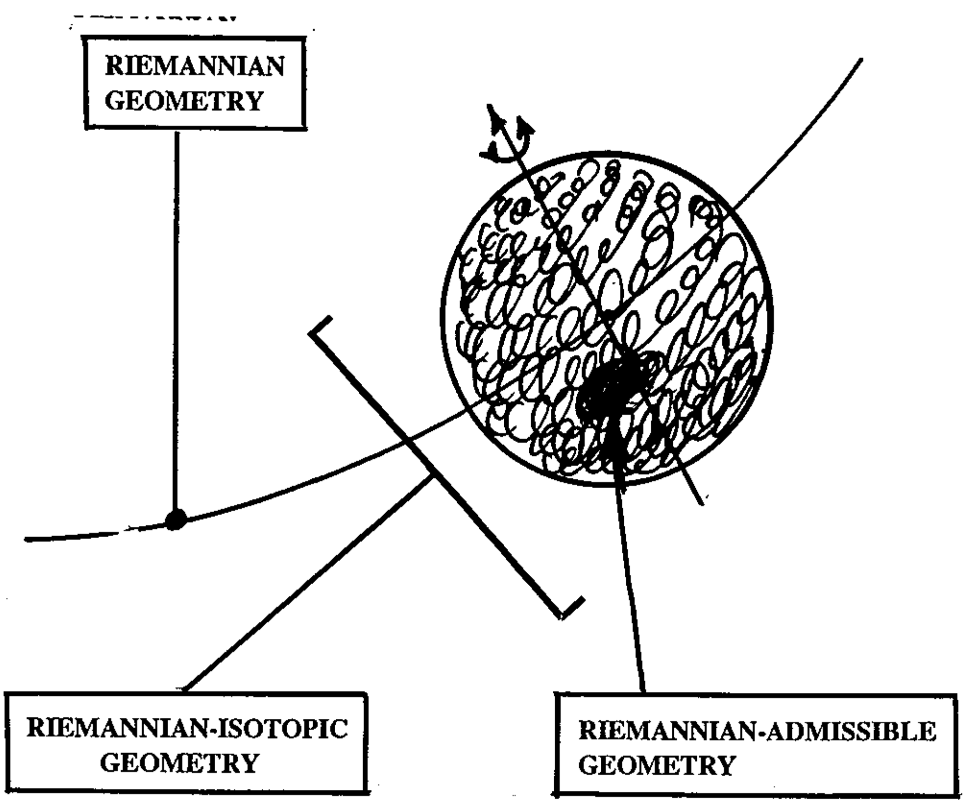

FIGURE 7.7.1: A schematic view of the geometric treatment of gravitation studied in these volumes. The Riemannian geometry is local-differential, thus being exact for the exterior problem of point-like test bodies in vacuum, but only approximate for the interior one. Moreover, a fundamental condition of the geometry is the *symmetric* character of the metric, $g = g^t$, thus implying its *reversible* character, with consequential inability to represent the experimental evidence of the interior irreversibility, say, of Jupiter. The Riemannian-isotopic geometry does solve the first problem, by permitting a direct representation of internal nonlocal-nonlagrangian effects in a way conform with total conservation laws. However, the isometric $\hat{g} = Tg$ of this latter geometry is also *symmetric*, $T = T^t$, thus being also structurally *reversible*. A *necessary* condition for the construction of an interior geometry for the direct representation of irreversibility is the use of a *nonsymmetric* metric. The construction of a generalization of the Riemannian geometry with a nonsymmetric metric via the use of conventional methods (those over a conventional field) presents simply unsurmontable difficulties for current mathematical

knowledge, which explains its absence at this writing. However, the same objective can be achieved via isotopic techniques in such a simple way to appear elementary. The main idea is based on two different nonsymmetric liftings of the isometric, one for motion forward in time $\hat{g} \Rightarrow \hat{g}^{>} = \hat{g}T^{>}$ and one backward in time $\hat{g} \Rightarrow {}^{<}T\hat{g}$, with $T^{>}$ and ${}^{<}T$ being different nonsymmetric real-valued tensors (evidently of the same dimension of $\hat{g}$) interconnected by the conjugation $T^{>} = ({}^{<}T)^{t}$. The definition of the forward isometric $\hat{g}^{>}$ over the genofield $\hat{R}^{>}(\hat{n}^{>},+,*^{>})$, and of the backward isometric ${}^{<}\hat{g}$ over ${}^{<}\hat{R}({}^{<}\hat{n},+,{}^{<}*)$ for the conjugation ${}^{<}T = (T^{>})^{t}$ or over its isodual for ${}^{<}T = (T^{>})^{d}$ then removes all technical difficulties indicated earlier because it implies the deformation $\hat{g} \Rightarrow T^{>}g$ while jointly deforming the unit of the amount inverse of the deformation, $I \Rightarrow \hat{1}^{>} = (T^{>})^{-1}$, and the same occurs for motion backward in time. Recall that deformed spheres, cones etc. are perfect spheres, cones, etc. at the level of the isogeometry (Ch. I.5). Along similar lines, the understanding of the genogeometry requires the knowledge that its nonsymmetric character appears only when the genogeometry is projected in the conventional Riemannian space because at the abstract level conventional, isotopic and genotopic geometries coincide.

$$g^{>} = T^{>}(s, x, \dot{x}, \ddot{x},...)\, g(x), \tag{7.7.18a}$$

$$^{<}g = {}^{<}T(s, x, \dot{x}, \ddot{x},...)\, g(x), \tag{7.7.18b}$$

where the two motions (multiplications) are interconnected by a suitable conjugation, e.g.,

$$T^{>} = ({}^{<}T)^{t} \quad \text{or} \quad ({}^{<}T)^{d}, \tag{7.7.19}$$

and equipped with two nonequivalent isoaffine connections, one for the modular-isotopic action to the right and the other to the left, the *Christoffel-admissible symbols of the first kind*

$$\hat{\Gamma}^{>1}{}_{hlk} = \tfrac{1}{2}\left(\frac{\partial g^{>}{}_{kl}}{\partial x^{h}} + \frac{\partial g^{>}{}_{lh}}{\partial x^{k}} - \frac{\partial g^{>}{}_{hk}}{\partial x^{l}}\right) \neq \hat{\Gamma}^{>1}{}_{klh}, \tag{7.7.20a}$$

$$^{<}\hat{\Gamma}^{1}{}_{hlk} = \tfrac{1}{2}\left(\frac{\partial {}^{<}g_{kl}}{\partial x^{h}} + \frac{\partial {}^{<}g_{lh}}{\partial x^{k}} - \frac{\partial {}^{<}g_{hk}}{\partial x^{l}}\right) \neq {}^{<}\Gamma^{1}{}_{klh}, \tag{7.7.20b}$$

with corresponding *Christoffel-admissible symbols of the second kind*

$$\hat{\Gamma}^{>2}{}_{h}{}^{i}{}_{k} = g^{>ij}\, \Gamma^{>1}{}_{hjk} = \hat{\Gamma}^{>2}{}_{k}{}^{i}{}_{h}, \tag{7.7.21a}$$

$$^{<}\hat{\Gamma}^{2}{}_{h}{}^{i}{}_{k} = {}^{<}g^{ij}\; {}^{<}\hat{\Gamma}^{1}{}_{hjk} = {}^{<}\hat{\Gamma}^{2}{}_{k}{}^{i}{}_{h}. \tag{7.7.21b}$$

The capability of a genometric to raise and lower the indices is understood (as in any affine space), and

$$g^{>ij} = |(g^{>}_{rs})^{-1}|^{ij}. \tag{7.7.22a}$$

$$^{<}g^{ij} = |(^{<}g_{rs})^{-1}|^{ij}. \tag{7.7.22b}$$

The *Riemannian-admissible geometry* or *genoriemannian geometry* for short, is the geometry of genospaces $^{<}\hat{R}^{>}(x, {}^{<}\hat{g}^{>}, {}^{<}\hat{R}^{>})$.

Its explicit construction can be done via the appropriate generalization of the isoriemannian geometry, with particular reference to the isoconnections which, besides being different for the right and left modular-isotopic action, can now be *nonsymmetric* depending on the assumed characteristics of the genotopic elements $T^{>}$ and $^{<}T$.

What is important is the mechanism of the lifting, which consists of a deformation of the original metric while jointly lifting the unit by the inverse of the deformation, Eq.s (7.7.1). The consistency of the new geometry is then consequential (see Fig. 7.7.1).

The above results permits the following

Lemma 7.7.1 [11]: *An axiomatization of irreversibility in interior gravitation is provided by inequivalent deformations of modular actions, metrics and connections to the right (forward in time) and to the left (backward in time) under a joint lifting of the unit per each direction of time characterized by the inverses of the deformations.*

As we shall see in Vol.s II and III, the above geometrization does indeed permit the representation of open-nonconservative-irreversible interior trajectories in Jupiter, such as a representation of interior vortices with monotonically varying angular momenta, although in a way compatible with the reversibility of the closed-isolated systems.

The interconnection of Lemmas 7.5.1, 7.6.1 and 7.7.1 should be kept in mind.

It is hoped that geometers in the field will be intrigued by the Riemannian-admissible geometry and develop it in the necessary technical details needed for quantitative studies of irreversible interior gravitation.

7.8: FUNCTIONAL GENOANALYSIS

In Sect. I.7.2 we have pointed out a number of problematic aspects of the current representation of nonconservative systems, such as their representation via the addition of a fictitious "imaginary potential" to the Hamiltonian, $H = K + iV$. This evidently implies a trajectory different from the physical one because the forces originating the nonconservation are generally of *nonpotential* type.

This approach to open nonconservative systems has yet another fundamental problematic aspects, and it is given, on one side, by the evident lack of Hermiticity of the Hamiltonian with consequential loss of observability, while, on the other side, the loss of energy is indeed observed and physically measured.

Hadronic mechanics has been conceived and constructed to resolve these evident problematic aspects. In fact, the nonpotential forces responsible for the nonconservation are not represented with a potential but with other means. Moreover and most importantly, the nonconserved Hamiltonian remains fully Hermitean and, thus observable, during the nonconservative process.

As we shall see in Vol. II, these are not mere mathematical curiosities, because they have direct experimental consequence. As an example, the achievement of the Hermiticity of the Hamiltonian during its time–rate–of–variation requires a suitable, corresponding revision of the data elaboration (such as a structural alteration of the expectation values) resulting in different numerical predictions and interpretations for the same event, as we shall see.

The preservation of the Hermiticity/observability of a Hamiltonian when nonconserved is achieved by a further generalization of the functional isoanalysis of the preceding chapter, this time, of genotopic character.

Recall that the Lie–isotopic theory admits a formulation via operators on a *conventional* Hilbert space $\mathcal{H}$, as originally proposed in ref. [20]. However, in so doing the observability is lost even for *conservative* processes. The observability can however be preserved if one lifts the Hilbert space via the same isotopic element T of the enveloping algebra.

A fully similar situation occurs for the more general Lie–admissible theory. In fact, it can be well defined on both a conventional Hilbert space $\mathcal{H}$ and its isotope $\hat{\mathcal{H}}_T$. However, a Hamiltonian is generally nonhermitean in both.

We reach in this way the *left and right genohilbert spaces*

$$\hat{\mathcal{H}}^{>} : \quad (\psi \hat{,} \Phi)^{>} = \hat{1}^{>} \int d^3r \, \psi^{\dagger}(r) > \phi(r) \in \hat{R}^{>} , \tag{7.8.1a}$$

$$^{<}\hat{\mathcal{H}} : \quad ^{<}(\psi \hat{,} \phi) = {}^{<}\hat{1} \int d^3r \, \phi(r) < \psi^{\dagger}(r) \in {}^{<}\hat{R} . \tag{7.8.1b}$$

It is easy to prove that a Hamiltonian which is conventionally Hermitean, remains Hermitean under the above genotopies, thus being observable, even though it is nonconserved,

$$i\,dH/dt = H(R-S)H \neq 0. \tag{7.8.2}$$

A simple example can be instructive here. Consider the free quantum mechanical particle with Hamiltonian $H_0 = \frac{1}{2}p_0^2$, $m = 1$, which is evidently Hermitean over $\mathcal{H}$. Suppose now that this particle at a given instant of time t_0 enters within a resistive medium, thus losing energy to the medium itself. Assume the simplest possible decay, the linearly damped one

$$H = e^{-\gamma t} H_0 = e^{-\gamma t}\,\tfrac{1}{2}\,p_0^2. \tag{7.8.3}$$

As we shall see in Vol. II, the Lie-admissible branch of hadronic mechanics permits an *axiomatic representation* of the above system; that is, a formulation derivable from first principles which is invariant under its own time evolution.

At this point we are merely interested in illustrating the basic dynamical equations, the underlying genohilbert spaces, and the Hermiticity-observability of the Hamiltonian.

It is easy to see that the desired Lie-admissible representation of system (7.8.3) is characterized by the realizations of the R–S quantities

$$R = -\tfrac{1}{2}\,i\,\gamma\,H_0^{-1}, \quad S = +i\,\tfrac{1}{2}\,\gamma\,H_0^{-1}, \quad R = S^{\dagger}, \tag{7.8.4}$$

The Lie-admissible group of the time evolution of a quantity Q is then given by

$$\begin{aligned} Q(t) &= \{ e_{\xi>}^{\,i\,H\,(t_0 - t)} \} > Q(t_0) < \{ e_{<\xi}^{\,i\,(t - t_0\,H} \} = \\ &= e^{\,i\,H_0\,S\,(t_0 - t)}\, Q(t_0)\, e^{\,-\,i\,(t_0 - t)\,R\,H}\,. \end{aligned} \tag{7.8.5}$$

with infinitesimal Lie-admissible equation

$$i\,dQ/dt = Q < H_0 - H_0 > Q \tag{7.8.6}$$

which becomes for the energy

$$i\,dH_0/dt = -\,i\,\gamma\,H_0, \tag{7.8.7}$$

thus verifying law (7.8.4).

The underlying genohilbert spaces are then given by

$$\mathcal{H}^{>}: \quad (\psi \hat{,} \phi)^{>} = 1^{>} \int d^3r\, \psi^{\dagger}(r) > \phi(r) \in \hat{R}^{>}, \; 1^{>} = i\,2\,\gamma^{-1} H_0 \tag{7.8.8a}$$

$$^{<}\mathcal{H}: \quad ^{<}(\psi \hat{,} \phi) = {}^{<}1 \int d^3e \;\; \phi(r) < \psi^{\dagger}(r) \in {}^{<}\hat{R}, \;\; {}^{<}1 = -\,i\,2\,\gamma^{-1} H_0\,. \tag{7.8.8b}$$

The Hermiticity/observability of the Hamiltonian during the decaying process can

be easily verified.

Note the formal identities for the case considered

$$(\psi \hat{,} \phi)^{>} \equiv {}^{<}(\psi \hat{,} \phi) \equiv (\psi , \phi) = \int d^3r \, \psi^{\dagger}(r) \, \phi(r) , \qquad (7.8.9)$$

namely, the compositions of the genohilbert spaces coincide with the conventional one. The Hermiticity/observability under decay emerges from the definition of the same composition in an invariant form on a genofield, that is, the decay is represented by the *operations* on H_0 and not by the Hamiltonian. In turn, this simple example illustrates the truly fundamental character of the theory of isonumbers and genonumbers for hadronic mechanics. More complex nonconservative systems will be studied Vol. II and III along structurally the same lines.

All isotopic generalizations of trigonometry, Dirac's δ–function, Fourier series, Fourier transforms, etc. then admit a significant and intriguing genotopic extension which is hereon assumed.

7.9: FUNDAMENTAL EQUATIONS OF HADRONIC MECHANICS AND THEIR DIRECT UNIVERSALITY

We are finally in a position to identify the fundamental equations of the two branches of hadronic mechanics indicated in Sect. 1.5

7.9.A: Lie-isotopic branch of hadronic mechanics. This branch describes *closed-isolated, composite systems with conserved total energy and other physical quantities and nonlinear-nonlocal-nonpotential internal forces.*

The nonrelativistic characterization of systems via this branch requires two operators, the Hamiltonian H and one isotopic operator T . The mathematical structure of the branch is characterized by one single isotopic product for both the right and the left with one single space isounit

$$\hat{1} = \hat{1}^{\dagger} = T^{-1} , \quad \hbar = 1 , \qquad (7.9.1)$$

and it is based on the following main structures:

A–1) Isofields of isoreal or isocomplex numbers $\hat{F}(\hat{a},+,*)$,

A–2) Enveloping isoassociative operator algebras $\hat{\xi}_T$,

A–3) Isohilbert spaces $\hat{\mathcal{H}}_G$, $G = G^{\dagger} > 0$, $G \neq T$,

which characterize the fundamental dynamical equations in the infinitesimal form

$$i\hat{1}_t \frac{dQ}{dt} = [Q\hat{,}H] = Q * H - H * Q = Q\,T\,H - H\,T\,Q, \qquad (7.9.2)$$

where $\hat{1}_t = T_t^{-1} \neq 1$ is the time isounit and $[Q\hat{,}H]$ are the Lie-isotopic brackets, with finite form

$$Q(t) = \hat{U} * Q(0) * \hat{U}^{\dagger} = \{e_{\hat{\xi}}^{\,iHt}\} * Q(0) * \{e_{\hat{\xi}}^{\,-itH}\} = \\ = \{e^{iHTt}\}\, Q(0)\, \{e^{-itTH}\}, \qquad (7.9.3)$$

yielding a Lie-isotopic group of isounitary transformations on $\hat{\mathcal{H}}_G$.

The corresponding, isoequivalent Schrödinger-type representation in isospaces $\hat{E}(t,\hat{R}_t)\times\hat{E}(r,\hat{\delta},\hat{R})$ [60] are given by

$$i\hat{1}_t \frac{\partial}{\partial t}\hat{\psi}(t,r) = H * \hat{\psi}(t,r) = H\,T\,\hat{\psi}(t,r) \qquad (7.9.4a)$$

$$-\,i\,\hat{\psi}^{\dagger}(t,r)\frac{\partial}{\partial t}\hat{1}_t = \hat{\psi}^{\dagger}(t,r) * H = \hat{\psi}^{\dagger}(t,r)\,TH\,. \qquad (7.9.4b)$$

and fundamental isocommutation rules

$$[a^{\mu}\hat{,}a^{\nu}] = \begin{pmatrix} [r^i\hat{,}r^j] & [r^i\hat{,}p_j] \\ [p_i\hat{,}r^j] & [p_i\,,p_j] \end{pmatrix} = \begin{pmatrix} 0 & i\hat{1} \\ -i\hat{1} & 0 \end{pmatrix} \qquad (7.9.5)$$

The above Lie-isotopic branch is structurally reversible, can be therefore used for either direction of time, and is divided into Kadeisvili's Classes I, II, III, IV and V depending on the characteristics of the basic isounit $\hat{1}$ (see Vol. II for details and relativistic extensions).

7.9.B: Lie-admissible branch of hadronic mechanics, which characterizes *open-nonconservative systems with nonconserved energy and other physical quantities under the most general possible nonlinear-nonlocal-nonpotential external interactions.*

Physical systems are represented nonrelativistically in this branch by three operators, the Hamiltonian H and the genotopic elements R and S which are however interconnected by the conjugation $R^{\dagger} = S$. This second branch is characterized by two different space genounits, one for motion forward, and one for motion backward in time interconnected by Hermitean conjugation

$$\hat{1}^{>} = R^{-1}, \ A > B := A\,R\,B, \ \ {}^{<}\hat{1} = S^{-1}, \ A < B := A\,S\,B, \ \ \hat{1}^{>} = ({}^{<}\hat{1})^{\dagger}, \qquad (7.9.6)$$

---

[60] We introduce here and in the following isounits not dependent explicitly in the local coordinates to avoid gravitational considerations at this time.

and it is based on the following main structures:

B–1) Genofields of genoreal or genocomplex numbers $^{<}\hat{F}^{>}(^{<}\hat{\alpha}^{>},+,^{<}*^{>})$,

B–2) Enveloping genoassociative operator algebras $^{<}\xi^{>}$,

A–3) Genohilbert spaces $^{<}\mathcal{H}^{>}$,

which characterize the fundamental dynamical equations in the infinitesimal form

$$i\,^{<}1^{>}{}_t \frac{dQ}{d^{<}t^{>}} = (Q\,\hat{,}\,H) = Q < H - H > Q = Q\,R\,H - H\,S\,Q \qquad (7.9.7)$$

where $t^{>}$ ($^{<}t$) represents forward (backward) time with corresponding genounits $1^{>}{}_t$ ($^{<}1_t$) and $(Q\,\hat{,}\,H)$ are the Lie-admissible brackets, with finite form

$$Q(t) = U^{>} > Q(0) < {}^{<}U^{\dagger} = \{ e_{\xi^{>}}{}^{iHt} \} > Q(0) < \{ e_{{}^{<}\xi}{}^{-itH} \} =$$

$$= \{ e^{iHRt} \} Q(0) \{ e^{-itSH} \}, \qquad (7.9.8)$$

yielding a Lie-admissible group of genounitary transformations on $^{<}\mathcal{H}^{>}$,

The corresponding genoequivalent Schrödinger-type representation in genospaces $^{<}\hat{E}^{>}(t,^{<}\hat{R}^{>}{}_t) \times {}^{<}\hat{E}^{>}(r,^{<}\hat{\delta}^{>},^{<}\hat{R}^{>})$ is characterized by the equations

$$i\,1^{>}{}_t \; \frac{\partial}{\partial t^{>}} \psi^{>}(t, r) = H > \psi^{>}(t, r) = H\,R\,\psi^{>}(t, r) \qquad (7.9.9a)$$

$$- i\,^{<}\psi(t, r) \frac{\partial}{\partial^{<}t} \,^{<}1_t = {}^{<}\psi(t, r) < H = {}^{<}\psi(t, r)\,S\,H . \qquad (7.9.9b)$$

with fundamental genocommutation rules

$$(a^{\mu} \,\hat{,}\, a^{\nu}) = \begin{pmatrix} (r^i \,\hat{,}\, r^j) & (r^i \,\hat{,}\, p_j) \\ (p_i \,\hat{,}\, r^j) & (p_i \,,\, p_j) \end{pmatrix} = i\,^{<}S^{>\mu\nu} \qquad (7.9.10)$$

where $^{<}S^{>\mu\nu}$ is the operator image of the corresponding classical Lie-admissible tensor also originating from the fundamental genocommutation rules.

The Lie-admissible branch is intrinsically irreversible, must be used for each given direction of time, and is also divided into Kadeisvili's Classes I, II, III, IV and V referred to the maximal Hermitean part of the genounits $^{<}1^{>}$ and $^{<}1^{>}$.

The crucial Lie-isotopic and Lie-admissible equations for the linear momentum will be presented in Vol. II.

It is evident that the Lie-isotopic branch is a particular case of the Lie-admissible branch, and this illustrates the reason why hadronic mechanics was

originally submitted [20] in terms of the fundamental Lie-admissible equations (7.9.7) and (7.9.8). A detailed study of the derivation, properties and basic axioms of the above equations is presented in Vol. II.

7.7.C: Direct Universality of Hadronic Mechanics. We shall now outline the "direct universality" of hadronic mechanics, that is, its capability to represent all possible linear and nonlinear, local and nonlocal, Hamiltonian and nonhamiltonian, continuous or discrete, and other systems (universality), directly in the frame of the observer (direct universality).

It should be stressed from the outset that this *does not* means that hadronic mechanics is the only applicable mechanics, because numerous other approaches are indeed possible for the elaboration of the same system (see the appendix).

The direct universality however implies the remarkable occurrence that, while other theories generally treat only *one* class of systems, hadronic mechanics can treat them all. The selection of one theory versus another does not evidently depend on personal taste, but rather on the intrinsic consistency of the theory at hand, as well as the experimental verification.

The identification whether a given theory is a particular case of hadronic mechanics implies:

1) The identification of possible departures from conventional quantum mechanical laws which are inherent in the theory considered;

2) The identification of corresponding generalized physical laws, as well as the physical conditions for their applicability, as a basis for experimental resolution; and

3) The availability of rigorous axiomatic methods for the quantitative treatment of the theory considered in a way demonstrably consistent with the basic assumption. As we shall see, this basic condition is lacking for a number of theories which, while possessing a generalized structure, elaborate data with conventional quantum mechanical assumptions, thus leading to insidious problematic aspects in their physical interpretation and applications.

In this final section, it may be recommendable to provide the primary guidelines for detailed study later on, as expressed by the following two theorems.

Theorem 7.9.1 - Direct universality for systems with conserved energy: *All possible linear or nonlinear, local-differential or nonlocal-integral, continuous or discrete operator, nonrelativistic or relativistic, equations representing a system with conserved total energy admit a direct representation via the Lie-isotopic branch of hadronic mechanics in the frame of the experimenter in one of the Classes I, II, III,. IV and V.*

The above theorem is transparently proved, e.g., by Eq.s (7.9.4) when written

explicitly

$$i\,\hat{1}_t(t, r, p, \psi, \psi\dagger, \partial\psi, \partial\psi, \ldots)\frac{\partial}{\partial t}\hat{\psi} = H\,T(t, r, p, \psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \ldots\ldots)\,\hat{\psi}, \quad (7.9.11a)$$

$$i\,dH/dt \;\equiv\; 0. \quad (7.9.11b)$$

which provide a direct representation of any given operator equations with conserved total energy. A similar situation occurs at the relativistic level (Vol. II).

Note that the Lie-isotopic equations permit an *infinite number of different representations of the same system* evidently due to the availability of two operators H and T for the same equation. However, such an infinity is reduced to only one, up to isoequivalence, when H is restricted to represent the total energy of the system considered.

Theorem 7.9.2 - Direct universality for systems with nonconserved energy: *All possible linear or nonlinear, local-differential or nonlocal-integral, continuous or discrete, nonrelativistic or relativistic, operator equations representing a system with nonconserved energy admit a direct representation via the Lie-admissible branch of hadronic mechanics in the frame of the experimenter in one of the Classes I, II, III,. IV and V.*

Again, the property is transparently exhibited by Eq.s (7.9.9) in their explicit form

$$i\,1^{>}_t(t, r, p, \psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \ldots)\frac{\partial}{\partial t}\hat{\psi} = H\,R(t, r, p, \psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \ldots)\,\hat{\psi}, \quad (7.9.12a)$$

$$i\,dH/dt \;\neq\; 0. \quad (7.9.12b)$$

As an illustration, one, among the infinitely possible reformulation of Eq.s (7.2.4) in terms of Lie-admissible equations submitted since the original proposal of the hadronic mechanics [20] is given by

$$(A \hat{,} H) \equiv A\,RH - HSA \;\equiv\; AH - H\dagger A \;=\; A \times H, \quad (7.9.13a)$$

$$R = 1, \qquad S = H^{-1} H^{\dagger}. \quad (7.9.13b)$$

We now close this section with the necessary conditions for the existence of a bona-fide generalized mechanics. When inspecting any generalized theory, the fundamental issue is whether conventional quantum mechanical laws and

axioms are preserved or generalized. In turn, this issue sets the stage for the elaboration via conventional or generalized methods, thus resulting in different numbers predicted by the theories for the same system.

The above issue can be answered via the following:

Basic criterion 7.9.1 - Identification of conventional vs generalized theories: *Any theory whose fundamental commutation rule coincide with or are unitarily equivalent to the canonical commutation rules*

$$[a^\mu, a^\nu] = \begin{pmatrix} [r^i, r^j] & [r^i, p_j] \\ [p_i, r^j] & [p_i, p_j] \end{pmatrix} = \begin{pmatrix} 0 & i\,I \\ -i\,I & 0 \end{pmatrix}, \tag{7.9.14}$$

is structurally equivalent to quantum mechanics, with corresponding cases occurring for relativistic and field theoretical theories. A necessary condition for the existence of a generalization of quantum mechanics is therefore the presence of generalized fundamental commutation rules which are not unitarily equivalent to those of quantum mechanics.

As one can see, the situation is clear–cut, without possibilities of using generalized theories while preserving old physical laws: generalized fundamental canonical commutation rules demand the use of generalized physical laws and methods. A good example is given by generalized commutation rules of the type

$$[r, p] = r\,p - p\,r = i\,f(r,p) \tag{7.9.15}$$

where $f(r,p)$ is a function or even a number different than $\hbar = 1$ (see also App. I.7.A.). Then the theory is noncanonical and must be reformulated via the re–definition of the unit and of the commutators themselves into the isotopic form

$$[r \hat{,} p] = r\,T\,p - p\,T\,r = i\,\hat{1} = i\,f(r, p)\ , \quad T = [\,f(r, p)\,]^{-1}, \tag{7.9.16}$$

which is now axiomatic, that is, derivable from first principles and invariant under its own time evolution. On the contrary, it is easy to prove that the "noncanonical" brackets (7.9.15) expressed in terms of the "conventional" Lie product $rp - pr$, do not preserve their form, and are in actuality mapped precisely into the isotopic form (7.9.16), as shown in Eq.s (4.1.3).

This is due to the fact that the only possible transformations capable of reducing the noncanonical value $f(r, p)$ to 1 are nonunitary, even when the function f reduces to a constant.

Equivalently, we can say that noncanonical brackets (7.9.15) are based on a generalization of the unit precisely of the fundamental form (1.1.1). The reconstruction of the entire structure of quantum mechanics into that of the

covering hadronic mechanics is then necessary for consistency of the formalism, as well as for its axiomatic, form-invariant character.

The physically relevant issue here is that *the quantum mechanical data elaboration of the theory based on isocommutation rules (7.9.15) are different from those based on rule (7.9.16).* The formulation of rule (7.9.15) via the conventional Lie product therefore gives only a misleading *impression* of having preserved quantum mechanics.

We should insist in this important point and indicate some of the problematic aspects of formulation (7.9.15), such as: the belief that conventional quantum mechanical energy, linear momentum, etc., remain Hermitean and thus observable under generalized commutation rules (7.9.15) at all times. It is easily proved that, under the nonunitary time evolution, the enveloping algebra becomes isotopic, while the Hilbert space remains unchanged, and this implies the general loss of Hermiticity, as familiar from Ch. I.6.

In short, the "fundamental canonical commutation rules" are truly "fundamental". Any structural deviation from them implies a necessary, consequential and compatible generalization of the structure of quantum mechanics. This is the case of the large variety of models of type (7.9.15) and other models (App. I.7.A).

After having understood (and, most importantly, admitted) the generalized character of a given theory, the next basic issue is the determination whether the total energy is conserved or not, so as to determine which methods to use as per Theorem 7.9.1 and 7.9.2.

Basic criterion 7.9.2 - Conservation of the energy in generalized theories: *A necessary condition for the total energy* H *of a generalized theory (as per Criterion 7.9.1) to be conserved is that the generalized fundamental commutation rules are isounitarily equivalent to the Lie-isotopic rules (7.9.5).*

Note that the above condition is necessary but not sufficient. In fact, the establishing that the total energy is conserved requires the additional conditions that: 1) H is the generator of the time evolution; 2) the canonical algorithm "p" represents the physical linear momentum, $p = m\dot{r}$; 3) H consists of the sum of two terms, $H = K + V$, the physical kinetic energy K and the physical potential energy V, etc. (for a detailed study of this aspect one may consult ref. [8,9].

Note also the necessary use of isounitary transformation. In fact, the use of unitary transformations would be futile, inasmuch as fully within conventional quantum mechanical settings.

Basic criterion 7.9.3 - Nonconservation of the energy in generalized theories: *A necessary condition for the operator* H *of a generalized theory (as per Criterion 7.9.1) to represent the nonconserved energy of the system is that the generalized fundamental commutation rules are isounitarily equivalent to the*

Lie-admissible rules (7.9.10).

Stated in different terms, the identification of an essential Lie-admissible structure guarantees the nonconservation of the energy. It is evident that, by no means does this implies any violation of any basic law of physics. The Lie-admissible formulations merely identify the *external* character of the interactions represented via the R and S operators. The understanding is that when the nonconservative system is completed with these external interactions, one regains the conservation of the total energy in full.

The above outline should be sufficient for the identification, first, whether a given theory is a generalization of quantum mechanics or not and, second, whether a generalized theory has a Lie-isotopic or Lie-admissible structure according to Theorem 7.9.1 or 7.9.2. Once these basic identifications have been made, then the methods of Vol. II are applicable for an axiomatic, form-invariant characterization of the theory, the identification of their physical laws, and the correct elaboration of data for experimental verifications.

APPENDIX 7.A: CONNECTION BETWEEN HADRONIC MECHANICS AND OTHER GENERALIZED THEORIES

As indicated since the Preface of this volume, hadronic mechanics has a direct connection with *all* generalizations of quantum mechanics attempted until now, with no exception known to this author. This is due to the universality theorems 7.9.1 and 7.9.2 which imply the inclusion of generalizations of nonlinear, nonlocal, discrete, algebraic, geometric, or any other type.

All existing generalized theories have been conceived and developed in a way *independent* from hadronic mechanics. Such independence is here confirmed as well as supported because of the polyhedric nature of mathematical and physical inquiries indicated earlier.

At the same time, another aspect of scientific inquiries is the need to study inter-relationship among different theories, because of the evident scientific gains reached in the comparison.

Along the latter lines, the primary contribution expected by the reformulation of a given generalized theory in terms of hadronic mechanics is of primary *physical character,* and deals with the identification of the axiomatic form invariant under time evolution, the applicable physical laws, and the applicable formalism for the data elaboration, so as to reach predictions with the necessary consistency needed for experimental consideration.[61]

[61] The reader should always keep in mind the numerous papers existing in the literature with *noncanonical* commutation rules, yet the elaboration of data via *conventional quantum mechanics,* whose predictions have no credibility warranting a consideration for experiments.

The axiomatic formulation, the applicable basic laws and the methods for the data elaboration are studied in Vol. II, jointly with primary applications such as to the origin of irreversibility, gauge theories, and the like. In this appendix we merely illustrate the connection between hadronic mechanics and a few representative generalized theories.

7.A.1: Hadronic mechanics and q-deformations. Albert's paper [18] of 1948 studied the generalized product

$$a \times b = p\, a\, b - (1 - p)\, b\, a, \qquad (7.A.1)$$

where ab can be assumed for simplicity to be associative and p is an element of the field, as a realization of the *noncommutative Jordan algebras*[62] which were of particular interest in the mathematics of the time.

Besides being a realization of noncommutative Jordan algebras, the above product is Lie-admissible, Jordan-admissible and admits the commutative Jordan algebras as a particular case for $p = \frac{1}{2}$, but it does not admit Lie algebras for any (finite) value of p. For this reason, this author introduced [12] back in 1967 as part of his graduate studies in physics at the University of Torino, Italy, and apparently for the first time in both mathematical and physical literature, the generalized product

$$(a, b) = p\, a\, b - q\, b\, a, \qquad (7.A.2)$$

where p and q are elements of the base field or functions, under the name of *(p,q)-mutations of associative algebras.* As one can see, product (7.A.2) is Lie-admissible, Jordan-admissible, admits both Lie algebras and commutative Jordan algebras as particular cases for finite values of p and q, and constitutes a realization of the noncommutative Jordan algebras (see ref. [12] for details).

The above initial studies were then expanded by the author [20] in 1978 into the Lie-admissible time evolution (.7.9.7), i.e.,

$$i\, dA / dt = A\, P\, H - H\, Q\, A, \qquad (7.A.3)$$

where P and Q are now unrestricted integro-differential operators, and in the fundamental Lie-admissible commutation rules (7.9.10), i.e. ,

$$(a^{\mu} \hat{,} a^{\nu}) = \begin{pmatrix} (r^i \hat{,} r^j) & (r^i \hat{,} p_j) \\ (p_i \hat{,} r^j) & (p_i , p_j) \end{pmatrix} = i \, \langle \hat{S} \rangle^{\mu\nu} \qquad (7.A.4))$$

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[62] Those are algebras with product a×b ≠ b×a verifying Jordan's axiom (a×b)×(a×a) = (a×(b×(a×a)) [18].

Subsequent studies along Albert's notion of Lie-admissibility have been reported in this chapter.

Independently from the above, various authors studied in the early 80's a generalization of canonical commutation rules of the type

$$(r, p) = r\,p - q\,p\,r\,, \tag{7.A.5}$$

under the name *q-deformation*[63], and more recently referred in a highly improper way as *quantum groups*[64] (see the recent ref.s [25] and literature contained therein). As one can see, product (7.A.5) is the particular case (0, q) of the (p, q)-mutations (7.A.2), but *it is not* a particular case of product (7.A.1). As such, product (7.A.5) is also Lie-admissible, Jordan-admissible, admits Lie and commutative Jordan algebras as particular case, and it is a realization of the noncommutative Jordan algebras.

The studies in the field have recently multiplied and extended to various parts of quantum mechanics, including the q-deformation of the Poincaré algebra (see, e.g., ref.s [26]). [65]

The "q-deformations" are an ideal example to illustrate the relationship between generalized theories and hadronic mechanics. In fact, their *mathematical* consistency is impeccable, their *independence* from hadronic mechanics is established, e.g., by comparing q-special functions and isospecial functions (Ch. I.6), and their *beauty* is undeniable as shown by the number of researchers attracted to the field.

However, the q-deformations are afflicted by a number of problematic aspects of a *physical nature* which cannot be ignored. To identify them, let us recall that the terms "q-deformations" are now refereed to a variety of generalized theories all generally defined at a fixed value of time, such as:

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[63] In his original proposal of 1967 [12], this author had intentionally used a term other than "deformation" (and suggested the term "mutation" because most of the so-called q-deformations *are not* "deformations" as conventionally understood in mathematics. Nevertheless, the terms "q-deformations" are now widely used, and they will be kept in this volume to avoid confusion.

[64] The use of the terms "quantum groups" is discouraged, and will not be adopted in these volumes because excessively misleading. In fact, the terms were historically referred, first, to a structure forming a conventional *Lie group* and, second, to the realization of such group in quantum mechanics. The use of the same terms for the q-deformation is therefore misleading on at least two counts, first, because the q-deformations do not yield a group as conventionally understood, and, second, because their structure is incompatible with the very notion of quantum of energy.

[65] It should be noted that the first Lie-admissible, P-Q-operator deformation of the Poincaré symmetry was introduced by the author in ref. [11] via the notion of Lie-admissible isobimodules or genomodules.

I) **Deformation of the enveloping associative algebra.** Let $\xi(L)$ be the universal enveloping associative algebra of a Lie algebra L (Sect. I.4.3) with elements A, B, ... and conventional associative product AB over a field $F(a,+,\times)$. This first type is characterized by the following generalization of the associative product AB[66]

$$A\,B \quad \Rightarrow \quad A * B = q\,A\,B, \tag{7.A.6}$$

where q is an element of the base field (or a function), without the joint lifting of the basic field as adopted by isotopic theories (Ch.s I.1, and I.2);

II) **Deformation of the Lie product.** Let L be a Lie algebra in quantum mechanical realization on a Hilbert space $\mathcal{H}$ over a field $F(a,+,\times)$ with generators A, B, ...and fundamental commutation rules $rp - pr = i$ ($\hbar = 1$). This second type of q-deformation is based on the generalization of the canonical commutators

$$r\,p - p\,r \quad \Rightarrow \quad r\,p - q\,p\,r = i\,f(q, ...) \tag{7.A.7}$$

which is evidently of type (7.A.5).

III) **Deformation of the structure constants.** let L be an n-dimensional Lie algebra with ordered basis X_i, envelope $\xi(L)$ and commutation rules $[X_i, X_j] = C_{ij}{}^k X_k$ over a field $F(a,+,\times)$. This third type of deformations is based on the preservation of the original product X_iX_j of $\xi(L)$ and of the original Lie product $X_iX_j - X_jX_i$ of L, while deforming this time the structure constants

$$X_i\,X_j - X_j\,X_i = C = C_{ij}{}^k\,X_k \;\Rightarrow\; X_i\,X_j - X_j\,X_i = F_{ij}{}^k(q, ..)\,X_k, \tag{7.A.8}$$

where the quantities $F_{ij}{}^k$ are similar to the "structure functions" of the Lie-isotopic theory (this type includes deformations characterized by the *Hopf algebras* and numerous others);

plus additional deformations, such as those characterized by the combination of *deformed* commutators (7.A.7) and *conventional* Heisenberg equations for the time evolution,[67] the deformation of creation-annihilation operators of the

[66] The reader should be aware that the form "qAB" of the product is correct only for q-numbers or functions and *not* for q-operators, in which case the product must be written "AqB", as done throughout this volume. In fact, if AB is an associative algebra, the product $A \times B = qAB$ with q a fixed operator violates the left scalar and distributive laws and, as such, it does not constitute any algebra of any kind.

[67] This latter class evidently requires *two different envelopes*, a generalized one for the characterization of the generalized commutation rules, and a conventional one for the characterization of the conventional time evolution. Even though *mathematically*

above Types I, II, III, etc. (see ref.s [25,26] or brevity).

Again, all the above q-deformations have an impeccable mathematical consistency and un undeniable beauty. However, when considered for *physical applications* they require the necessary use of the dynamical time evolution, in which case a number of problematic aspects emerge as recently studied by Lopez [27], such as:

1) General loss of the Hermiticity/observability of the Hamiltonian. As now familiar from the studies presented in this volume, deformations of the Types I, II, III above generally imply a nonunitary time evolution, as necessary from the lack of canonicity of the commutation rules, and demonstrable, e.g., via quantization of the corresponding, classical, noncanonical theories. In turn, nonunitary time evolutions imply the lifting of the envelope into the isotopic form for all Types I, II, III,

$$\xi: \; A\,B \;\Rightarrow\; \hat{\xi}: \; A' * B' = A'\,T\,B', \quad A' = U\,A\,U^{\dagger}, B' = U\,B\,U^{\dagger}, \quad (7.A.9a)$$

$$U\,U^{\dagger} = \hat{1} \neq I, \quad T = (U\,U^{\dagger})^{-1}, \quad \hat{1} = T^{-1}. \qquad (7.A.9b)$$

Still in turn, this implies the loss of the Hermiticity/observability of the Hamiltonian and of other physical quantities because q-deformations are defined on a *conventional Hilbert space* $\mathcal{H}$, while the preservation of Hermiticity under lifting (7.A.4) demands the joint lifting of the base field $F \Rightarrow \hat{F}_T$ and of the Hilbert space $\mathcal{H} \Rightarrow \hat{\mathcal{H}}_T$ (Sect. I.6.3).[68]

2) General loss of the measurement theory. Most q-deformations are deformations of the basic associative product AB and/or of Planck's constant $\hbar =$, and/or of the structure constants without a corresponding redefinition of the unit as done in the isotopic theories. Therefore, *q-deformations are theories without a left and right unit which remains invariant under the time evolution.*

correct, this class multiplies, rather than reduces the *physical problematic aspects* discussed below.

[68] It should be indicated for clarity that, when nonunitary time evolutions are admitted also for the Hilbert space, Hermiticity can be preserved. In fact, in this case the conventional inner product is lifted into the form

$$<\phi\,|\,\psi> = \int d^2r\, \phi^{\dagger}(r)\,\psi(r) = \int d^3r\, \phi'^{\,\dagger}\, T\, \psi', \;\; \phi' = U\phi, \; \psi' = U\psi, \;\; T = (U\,U^{\dagger})^{-1},$$

which is precisely of the isotopic type. However, the correct preservation of Hermiticity requires the joint lifting of the base field into the isofield with isounit $\hat{1} = T^{-1}$, in which case the correct form of the isoinner product is given by

$$<\phi' \hat{|}\, \psi'> = \hat{1} \int d^3r\, \phi'^{\dagger}\, T\, \psi',$$

(and coincides with the original product for T independent of the integration variables), thus implying the entire structure of hadronic mechanics.

This occurrence is transparent in lifting (7.A.6) which deforms the product $AB \Rightarrow A*B = qAB = ATB$ without jointly deforming the unit as done in the foundations of hadronic mechanics

$$I \Rightarrow \hat{1} = T^{-1} = q^{-1}. \tag{7.A.10}$$

The lack of basic unit can also be established for deformations of Types II and III, e.g., under time evolution with ensuing nonunitary structure, and unification of all envelopes into isotopic form (7.A.4). The loss of the unit then implies the evident loss of the measurement theory, owing to the necessary condition of the existence of a well defined, left and right unit for the very concept of measurement.[69]

3) General lack of uniqueness of Gaussian distributions and related physical laws. One of the strengths of quantum mechanics is the *uniqueness* of its various formulations (such as the Gaussian) which evidently implies the known uniqueness of its physical predictions (such as the uniqueness of Heisenberg's uncertainties, see Sect. I.6.1). This uniqueness can be mathematically traced to the uniqueness of the basic unit of the theory, Planck's constant, as well as to the existence of a right and left unit of the universal enveloping operator algebra $\xi(L)$. The general lack of the basic unit then implies that *q-deformations do not possess a consistent formulation of the Poincaré-Birkhoff-Witt theorem which is applicable at all times.* In fact, a necessary condition for the very formulation of the theorem is the existence and uniqueness of a left and right unit.

This means *the lack of existence of a unique, infinite-dimensional basis for the envelopes of q-deformations* and, therefore, *the lack of existence of a unique form of exponentiation.* In fact, q-deformations are known for their variety of "exponentiations".

The above occurrences add to the mathematical beauty of the theory, but have rather serious physical consequences, such as *the lack of uniqueness of a Gaussian distribution with consequential lack of uniqueness of the generalized uncertainties.* A similar situation occurs for other physical laws.

4) General loss of special functions under time evolution. As recalled earlier, q-deformations are formulated at a fixed value of time, and so are their special functions (Ch. I.6). But under time evolution the q-*number* is replaced by the isotopic *operator* T. The inapplicability of the q-special functions under time evolution is then consequential.

From a *mathematical* viewpoint, this occurrence may be irrelevant. The

69 We are here intentionally silent, as a test of technical knowledge of isotopic techniques studied earlier, on the need for an axiomatic, form-invariant theory to unify the unit of the base field with that of the envelope.

physical implications are however rather serious, such as the impossibility of performing a partial wave-analysis and the like.

5) General loss of Einstein's axioms. As well known (but not fully identified in the literature), all q-deformations imply a structural departure from *all* basic axioms of the special (and general) relativity, as established by the noncanonicity of the commutation rules, or the nonunitary character of the time evolution, or the deformation of the structure constants of the Poincaré symmetry, etc.

Again, this occurrence can be mathematically intriguing, but it carrier rather serious physical problems in the compliance with physical reality which must be addressed prior to any physical application.

Hadronic mechanics offers realistic possibilities of resolving all the above problematic aspects while leaving the results of q-deformations fundamentally unaffected, and this illustrates the relationship between hadronic mechanics and generalized theories.

In fact, hadronic mechanics does not require any change of the assumed structural lines of q-deformations (such as the explicit form of q, f(q,...) or F_{ij}^{k}(q, ...)), but only their reformulation in the axiomatically correct form which is invariant under the time evolution of the theory.

The hadronic reformulation of q-deformations is so simple as to appear trivial. For Type I it merely requires the joint lifting of the associative product *and* of the basic unit

$$AB \Rightarrow A*B = A\,q\,B, \qquad I \Rightarrow \hat{1} = q^{-1}, \tag{7.A.11}$$

with consequential reformulation of the theory with respect to isofield, isospaces, isotransformations, etc.

The reformulation for Type II was first studied by Jannussis and his collaborators [28] on conventional fields. That on genofields requires the selection of one "time arrow" and then the interpretation of the function f(q,...) in rules (7.A.6) as the genounits for that direction. Jointly, the q-deformation of the second term in the l.h.s. is not axiomatic and must be lifted into the inverse of the selected genounit, resulting in the reformulation

$$r\,p - q\;p\,r = i\,f(q,...) \quad \Rightarrow \quad \left\{ \begin{array}{l} r < p - p > r = r\,R\,p - p\,S\,r = 1^{>}, \\ 1^{>} = f(q, ...)\,/\,q, \quad S = q\,/\,f(q, ...), \quad R = f(q,...) \\ \text{or} \\ r < p - p > r = r\,R\,p - p\,S\,r = {}^{<}1\,, \\ {}^{<}1 = f(q, ...)\,/\,q, \quad S = f(q, ...), \quad R = q\,/\,f(q,...) \end{array} \right. \tag{7.A.12}$$

The entire theory must then be reformulated on genofields, genospaces, genotransformations, etc. of the selected direction of time.

The hadronic reformulation of q–deformations of Type III is more complex owing to their general character. The procedures has however been studied in detail in this volume and it is applicable to each case considered. The end–result is that, to achieve an axiomatic formulation for given deformed structure constants $F_{ij}{}^k(q, \ldots)$, one must identify an isotopic element $T(q, \ldots)$ such that the original Lie–deformation is turned into a Lie–isotopic algebra with $F_{ij}{}^k$ as structure functions

$$X_i\,X_j - X_j\,X_i \;=\; F_{ij}{}^k(t, q, \ldots)\,X_k \;\Rightarrow\; X_i\,T\,X_j \;-\; X_j\,T\,X_i \;=\; F_{ij}{}^k(t, q, \ldots)\,X_k. \quad (7.A.13)$$

The axiomatic reformulation of other q–deformations can be done with one or the other methods studied in this volume.

The researcher in q–deformations is urged to prove the form–invariance of the above isotopic reformulations under the time evolution of the theory. Equivalently, to understand the relationship between q–deformations and hadronic mechanics, one should study the image of all commutators under nonunitary time evolutions, e.g.,

$$r\,p \;-\; q\,p\,r \;=\; i\,f(q,\ldots) \;\Rightarrow\; r'\,R\,p' \;-\; p'\,(\,q\,R)\,r \;=\; i\,\hat{1}\,, \qquad (7.A.14a)$$

$$\hat{1} \;=\; f(q, \ldots)\,U\,U^{\dagger}, \qquad R \;=\; \hat{1}^{-1}. \qquad (7.A.14b)$$

As a result, starting from the (0,q)–number deformation (7.A.6) at a fixed value of timer, one reaches at arbitrary times the general (P,Q)–deformations, that is, the Lie–admissible equations at the foundations of hadronic mechanics. This shows the inevitability of the hadronic reformulation even when not desired.

It is equally instructive for the researcher in q–deformation to see that the above isotopic reformulations resolve *all* the problematic aspects indicated earlier. To begin, hadronic mechanics has been built from the beginning (Sect. I.1.1) under the condition of possessing a generalized, but well defined left and right unit $\hat{1}$. As now familiar, this implies a corresponding compatible isotopy of the base fields and Hilbert space, thus ensuring the Hermiticity/observability of the Hamiltonian and other operators at all times (Sect. I.6.3).

The basic assumptions of hadronic mechanics are centered in fundamental condition I.4.4.1 that the enveloping algebra (of both the Lie–isotopic and Lie–admissible branches) must have a well defined left and right unit. This implies the existence of a generalized Poincaré–Birkhoff–Witt theorem (Sect. I.4.3). The applicability of the measurement theory is proved in Vol. II by showing that the correct isoexpectation values of the isounit $\hat{1}$ turns out to be the conventional Planck value,

$$<\hat{1}> \;=\; \hbar \;=\; 1. \qquad (7.A.15)$$

As a result, the measurement theory of hadronic mechanics is the *conventional* one, as necessary for physical consistency and applicability to actual experiments, evidently because measures are conducted in our classical frame and, as such, cannot be modified by theoretical deformations introduced in the microworld.

The Lie–isotopic theory is also based on the existence of a unique infinite-dimensional basis which implies the uniqueness of the exponentiation in hadronic mechanics with consequential uniqueness of the physical laws defined on them.

The applicability of the isospecial functions at all times is evident from the studies of Ch. I.6, because they are constructed for an arbitrary integrodifferential operator T admitting of T^{-1} as the correct unit, rather than with respect to a q–number without a unit.

Finally, the most important objective of all the isotopic techniques is the preservation of Einstein's axioms under T–integral–operator–deformations and only their *realizations* in a nonlinear–nonlocal–noncanonical form as needed for interior problems. The important point stressed throughout our analysis is that both the exterior and interior problems are characterized by a unique set of algebraic–geometric–dynamical axioms.

This is stressed by the local isomorphism between Minkowski and isominkowski spaces, or the Poincaré and isopoincaré symmetries

$$M(x,\eta,R) \approx \hat{M}_Q(x,\hat{\eta},\hat{R}), \qquad P(3.1) \approx \hat{P}_Q(3.1), \tag{7.A.16}$$

which should be compared the corresponding lack of isomorphisms for conventional q–deformations

$$M(x,\eta,R() \not\approx M_q(x,\eta,R(, \qquad P(3.1) \not\approx P_q(3.1). \tag{7.A.17}$$

As a final note we should indicate that, even after reaching a fully axiomatic formulation of the Lie–admissible type (7.A.12), there is one additional problematic aspect requiring consideration. It deals with the relationship between the R– and S–operators which should be be restricted to verify the conjugation

$$R = S^{\dagger}, \tag{5.A.18}$$

in which case one has a direct applicability to all possible nonconservative systems. Physical applications for $R \neq R^{\dagger}$, even though evidently possible, are unknown at this writing, to our best knowledge.

Perhaps the best way to see the relationship between q–deformations and hadronic mechanics is to inspect Vol. II on the applications to specific physical problems and Vol. III on the experimental verification. It is at that stage where the researchers in q–deformation can see the inevitability of an axiomatic reformulation in order to reach a form acceptable for experimental verifications.

7.A.2: Hadronic mechanics and nonlinear theories. As well known, *nonlinear generalizations of Schrödinger's equations,* here referred to those nonlinear in the wavefunctions (only), have been proposed since the early stages of quantum mechanics, such as the nonlinear equation proposed by E. Fermi [29] back in 1927

$$i \frac{\partial}{\partial t} \psi_k = [- \frac{1}{2m} \Delta + V(r) + \bar{V}(\psi\bar{\psi})] \psi_k \quad , \qquad (7.A.19)$$

Since that time, the generalizations have been studied by a considerable number of authors and constitute today a new segment of theoretical physics. These studies are evidently valuable because they focus the attention on one of the expected limitations of quantum mechanics for interior dynamical problems (Sect. I.1.2), which is precisely the linearity in the wavefunctions.

The issue addressed by hadronic mechanics in Vol. II is the identification of *methods* appropriate for the elaboration of nonlinear *equations*, that is, methods verifying all the necessary principles, including the superposition principle and the conventional measurement theory.

More recently, a method for the study of the above type of nonlinear equations was proposed by S. Weinberg [30] in 1989 which is essentially characterized by an *enveloping algebra* U with product

$$U: \qquad A \hat{\times} B = \frac{\partial A}{\partial \psi_k} \frac{\partial B}{\partial \bar{\psi}_k}, \qquad (7.A.20)$$

"Heisenberg–type" equation for a physical quantity Q

$$i\, dQ / dt = Q \hat{\times} H - H \hat{\times} A , \qquad (7.A.21)$$

and "schrödinger's type" equation

$$i \frac{\partial}{\partial t} \psi_k = \frac{1}{2m} \Delta \psi_k + \frac{\partial H}{\partial \bar{\psi}_k}, \qquad (7.A.22)$$

where H is certain functional of ψ_k and $\bar{\psi}_k$, all equations being defined over a conventional Hilbert space $\mathcal{H}$ on a conventional field $F(a,+,\times)$.

Weinberg's nonlinear theory provides another illustration of the relationship between hadronic mechanics and generalized theories, this time, from a viewpoint different than that of the q–deformations.

In fact, the elegance of the theory and its independence from other methods are evident. Yet the theory is afflicted by a number of problematic aspects which are, again, of *physical* nature, as studied in detail by Jannussis, Mignani and Santilli [31].

The most dominant characteristic of Weinberg's nonlinear theory is that its envelope U is *a general, nonassociative, Lie-admissible algebra* (App. I.4.A and Sect. I.7.3). In fact, product (7.A.20) is *nonassociative* because

$$U: \quad A \hat{\times} (B \hat{\times}) C) \neq (A \hat{\times} B) \hat{\times} C; \tag{7.A.23}$$

it is *Lie-admissible* because the attached antisymmetric product is Lie

$$Q \hat{\times} H - H \hat{\times} Q = \text{Lie}, \tag{7.A.24}$$

and it is a *general* Lie-admissible algebra in the sense that it is characterized by the general law (7.3.1) without verifying simpler versions of the same, such as that of flexibility.

The immediate consequence is that *Weinberg's nonlinear theory does not admit a unit* (unless reduced to the trivial case of only one dimension). As a result, the theory suffers of a number of problematic aspects somewhat similar to those of q-deformations, such as [31]:

1) lack of existence of the measurement theory evidently because of the lack of existence of the unit

2) lack of well defined Casimir invariants, evidently because of the lack of the center of the envelope;

3) lack o the Poincaré-Birkhoff-Witt theorem for the basis of U[70];

4) lack of a consistent exponentiation, because of the lack of the needed infinite-dimensional basis;[71]

5) lack of a consistent formulation of space-time symmetries in their finite (exponentiated) form uniquely derivable from their Lie algebra;

6) lack of the general equivalence between the "Heisenberg-type" and the "Schrödinger-type" equations;[72]

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[70] As recalled in Sect. I.4.3, the largest nonassociative envelope admitting ordered monomials and a formulation of the Poincaré-Birkhoff-Witt theorem is given by the *flexible Lie-admissible algebras* while extreme technical problems exist in the formulation of the theorem for general Lie-admissible algebras.

[71] Note that, by comparison, exponentiations do exist for q-deformations, although they are not unique.

[72] This is a typical area of study of Vol. II. We here mention the origin of the problematic aspect which is due, on one side, to the *nonassociative* character of the envelope of the "Heisenberg-type" equations (i.e., the nonassociativity of the product $A \hat{\times} B$,), and the *associative* character of the modular structure of of the "Schrödinger-type" equations (i.e., the associativity of the action $\Delta \psi_k$, under which no equivalence is evidently possible. At the same time, a nonassociative reformulation of the modular action of the "Schrödinger's type" equation such as $H \hat{\times} \psi_k$ to achieve structural equivalence with the envelope of the the "Heisenberg-type" equation is confronted with large technical problems, because it would require a nonassociative generalization of Schrödinger's theory, i.e., one for which

7) lack of a well defined notion of particles because of the lack of well defined physical characteristics, such as spin, which evidently require a well defined Lie algebra, with an envelope possessing a well defined center, with a unique exponentiation to a well defined group, etc.

Again, the above occurrences do not prevent the theory from being *mathematically definable.* In fact, the occurrences have been called "intrinsic features" of the theory. The point is that they simply cannot be ignored for *physical applications.*

As it was the case for the q–deformations, *hadronic mechanics permit an axiomatic reformulation of Weinberg's nonlinear theory which, while leaving the physical content completely unchanged, avoids problematic aspects 1)–7) above.*

To understand the occurrence one must distinguish between the nonlinear *"equations"* represented by the theory, and Weinberg's nonlinear *"theory"* per se. Then, all possible Weinberg's nonlinear "equations" are an evident particular case of the isoschrödinger's equation of hadronic mechanics owing to its direct universality (Theorem 7.9.1)

$$i \frac{\partial}{\partial t} \psi_k = H(t, r, p)\, T(t, r, p, \dot{p}, \psi, \overline{\psi}, \partial\psi, \partial\overline{\psi}, \ldots\ldots)\, \psi_k . \qquad (7.A.25)$$

As a matter of fact, while Weinberg's "'theory" admits only one particular class of "equations" nonlinear in the wavefunctions, isoschrödinger's equations are much broader because they admit: 1) all possible nonlinear equations in the wavefunctions; 2) all possible equations nonlinear in the derivative of the wavefunctions; as well as 3) all possible equations which are nonlocal in the wavefunctions and their derivatives of arbitrary order.

The resolution of the problematic aspects in the treatment of the same "equations" then follows from their isotopic representation (7.A.25).

As an incidental note, one should be aware of the differences in the intended physical applicability of Weinberg's nonlinear theory and hadronic mechanics. In fact, the former has been formulated for what we essentially refer to as the *exterior dynamical problem in vacuum;* while the latter has been formulated for the *interior dynamical problem within physical media.*

This point is important to stress that the limitations emerged from experiments on Weinberg's theory [33] (essentially dealing with atomic structures), have no bearing of any nature for hadronic mechanics, evidently because they

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$$A \hat{\times} (B \hat{\times} \psi_k) \neq (A \hat{\times} B) \hat{\times} \psi_k .$$

In summary, the mathematical structures of the Heisenberg–type and Schrödinger–type equations are inequivalent in Weinberg's nonlinear theory, and the attempts at rendering them structurally equivalent are confronted with considerable technical problems which, at any rate, would leave the other problematic aspects completely unaffected. The above occurrence is rather synthetically expressed by the so–called *Okubo's No–Quantization Theorem* [32] studied in Vol. II.

are not applicable, say, to a proton in the core of a collapsing star. In fact, a primary nonlinearity of interior conditions is expected to be in the derivative of the wavefunctions which is absent in Weinberg's theory.

The different origins of the problematic aspects in q–deformations and in Weinberg's theory should be identified because instructive. All q–deformations possess a fully *associative* algebra, with consequential full capability to identify its correct left and right unit. By comparison, Weinberg's nonlinear theory is based on a *nonassociative* envelope with consequential *impossibility* to define the right and left unit.

Additional critical inspection of Weinberg's nonlinear theory can be found in ref.s [34]. An intriguing *reformulation* of Weinberg's theory which avoid some of the problematic aspects of the original formulation has been proposed by Jordan [35]. The identification of the algebraic origin of these resolutions is useful to cast additional light on the issues here considered.

Jordan [loc. cit.] introduces the following *generalization* of envelope (7.A.20)

$$U^*: \qquad A * B = \frac{\partial A}{\partial w_{jk}} w_{lk} \frac{\partial B}{\partial w_{lj}} . \tag{7.A.26}$$

The commutator $[A, B]_{U*} = A*B - B*A$ is Lie and, therefore U^* remains a general nonassociative Lie–admissible algebra as in Weinberg's case.

Jordan's reformulation does however allow the treatment of spin and other conventional quantum mechanical quantities. This is due to the fact that the space of functions A, B,... is restricted to those with the structure

$$A = w_{jk}\, a_{kj} \ , \quad B = w_{jk}\, b_{jk} , \tag{7.A.27}$$

where the terms in the r.h.s. are interpreted as matrix elements. The commutator $[A, B]_{U*}$ computed in the *nonassociative* envelope U^* is then turned into an equivalent commutator turned into an *associative* envelope,

$$[A,B]_{U^*} = a_{kj}\, w_{lk}\, b_{jl} - b_{kj}\, w_{lk}\, a_{jl} = [A, B]_{A^*} \tag{7.A.28}$$

the correct formulation of the Poincaré–Birkhoff–Witt theorem, space–time symmetries, exponentiation, Gaussian distribution, etc. is then consequential.

In fact, structure (7.A.28) is a realization of the Lie–isotopic product with an isoassociative envelope and isotopic element $T = (w_{ij})$ precisely of the type at the foundation of hadronic mechanics. More specifically, Jordan's transformation of Weinberg's nonassociative envelope into an equivalent isoassociative form is precisely a realization of Lemma I.4.A.1.

Jordan's reformulation itself is not immune of problematic aspects which are this time similar to those of the q–deformations (lack of joint isotopy of fields and Hilbert spaces, etc.).

The important information originating from these occurrences is that (Fundamental Condition I.4.4.1), according to current knowledge, physically

meaningful theories should be formulated with respect to an *associative* envelope with a well defined left and right *unit*, as it is the case for quantum mechanics and its hadronic covering.

There is little doubt that a next generation of theories will likely be based on *nonassociative* envelopes, precisely along Weinberg's lines [30]. The researchers interested in these latter lines should however be aware of the rather serious technical problems involved, both *mathematical* (e.g., the Poincaré–Birkhoff–Witt theorem) and *physical* (e.g., the equivalence of Heisenberg-type and Schrödinger-type equations for nonassociative moduli).

7.A.3: Hadronic mechanics and nonlocal theories. Deformations of quantum mechanics (Sect. 7.A.2) focus the attention on the relevance of *noncanonical* theories, while Weinberg's theory of the preceding section focuses the attention on the relevance of *nonlinear* theories. The next logical step along the lines of these volumes is to focus the attention on *nonlocal* theories.

We assume the reader is familiar with the variety of notions of nonlocality existing in the literature. Those particularly relevant for these volumes are the studies initiated by Russian physicists, such as Blochintsev [36] which have subsequently seen the most comprehensive development by Efimov and his associates (see monographs [37] and quoted literature).

Most significant for these volumes is the original motivation which stimulated the studies of nonlocal theories: *remove the divergencies which are inherent in the local in the local character of quantum field theories.*

Note that studies [36,37] deal with nonlocal formulations of *quantum field theory* while the studies of these volumes deal with nonlocal formulations of *quantum mechanics*. Despite that, the rather intriguing connections and possibilities for further advances are already identifiable.

Hadronic mechanics can be conceived as a generalization of quantum mechanics which can remove the singularity of Dirac's delta function *ab initio* precisely via a nonlocal formulation (Sect. I.6.6.4).

The field theoretical extension of the isodirac delta function has been preliminarily studied by Nishioka [38] and, as we shall see in Vol. II, it does indeed contains the necessary elements for the possible, future construction of a *nonlocal-isotopic field theory* which is also free of singularities *ab initio*.

Again, all results achieved in ref.s [36,.27] remain unchanged in their possible isotopic reformulation, which essentially provides mere alternative methods for their treatment.

One point appears to be certain: the conventional local-differential field theories have reached and surpassed the limits of their applicability. Irrespective of which theory will eventually result to be more viable, the need for nonlocal-integral theories is simply beyond credible doubts. We are not referring to ideal point-like particles moving in vacuum (exterior problem) in which the exact validity of local field theories is incontrovertible, but to extended wavepackets moving within those of other particles (interior problem).

At any rate, there exist physical systems simply beyond the descriptive capacities of local field theories, such as the *attractive* interaction of the *same* electrons of the Cooper pair in superconductivity, which can be quantitatively

interpreted via a suitable nonlocal representation of the overlapping of the wavepackets of the electrons (Vol. III). Similar needs for nonlocal theories exist in nuclear, particle and statistical physics, theoretical biophysics and other disciplines.

7.A.4: Hadronic mechanics and and discrete theories. Another field of research that is currently gaining momentum is at times known under the name of *discrete theories.* This area too is quite vast, by encompassing the use of discrete groups, lattices, discrete calculus, etc. We here focus the attention on only one aspect, the *discrete-time theories* which is sufficient to illustrate all other discrete theories.

Discrete time theories can be traced back to Caldirola's studies [39] of 1956. More recent studies have been conducted by Wolf [40] and others (see Vol. II).

These studies focus the attention on *the possibility that time has a discrete structure at a sufficiently small scale,* a possibility clearly deserving the proper attention in the mathematical, theoretical and experimental communities.

It was shown by Jannussis and his collaborators [41] that Caldirola's equations do have a structure precisely of the Lie-admissible type

$$i\hbar \frac{\rho(t) - \rho(t-\tau)}{\tau} = H R \rho(t) - \rho(t) S H, \qquad (7.A.29)$$

where τ, called *Caldirola's chronon,* is a measure the duration the interaction among particles. The full applicability of hadronic mechanics along universality Theorem I.7.9.2 is then completed by noting that the difference in the l.h.s. is a realization of the *isoderivative with discrete isounits* (Sect. I.6.7).

Thus, discrete time theories constitute an intriguing particular case of hadronic mechanics of Class V. Note that this interpretation permits an intriguing connection with q-deformations which does not appear to have been sufficiently identified in the literature.

By recalling that the basic axioms of quantum mechanics are preserved under isotopies, and only realizes in a more general way, the above hadronic reformulation is intriguing indeed because it shows that *discrete-time theories are admitted by the abstract axioms of quantum mechanics itself.*

The above unexpected property will be proved in Vol. II via the the isoexpectation value of the isounit $<\hat{1}> = 1$, which applies also for discrete isounits $\hat{1}$. To put it differently, a discrete structure of time emerges as admitted by the quantum mechanical axioms, evidently in a more general realization, when dealing with the microcosm. Nevertheless, when the theory is reduced to numbers suitable for macroscopic experiments via the isoexpectation values, such discreteness disappears. In fact, the future resolution of the possible discrete structure of time requires experiments specifically conceived for that purpose, whose study has been initiated by Wolf [40].

The current formulation of the discrete-time theories is also afflicted by

problematic aspects of *physical character* due to the fact that, on one side, they generalize the structure of quantum mechanics while, on the other side, they preserve conventional quantum mechanical formulations (conventional expectation values, conventional physical laws and principles, etc.) in the elaboration of the theories.

In effect, the transition from the continuous time of quantum mechanics to a time with a discrete structure implies a necessary *generalization of the underlying unit of time*, from the trivial unit 1 to a generalized unit of Kadeisvili's discrete Class V. In turn, this demands, for evident need of consistency, a step-by-step generalization of the entire quantum mechanics, including expectation values, physical laws and data elaboration needed for experiments.

7.A.5: Hadronic mechanics and other approaches. By no mean the preceding examples exhaust all possible connections between hadronic mechanics and ongoing research.

Among a number of additional aspects we shall study in Vol. II, it may be recommendable to indicate the following ones. Kadyshevsky and his associates [42] have constructed *a generalized quantum field theory with a fundamental length at small distances* which exhibits numerous intriguing connections to q-deformations, nonlocal field theories, etc. The re-inspection of the above theory with isotopic methods is significant because it can indicate that a fundamental length can be reconciled with the very axioms of quantum mechanics, evidently when realized in a sufficiently general way.

Another intriguing topic is the Lie-admissible re-interpretation of *conventional external electromagnetic interactions*, such as the studies by Studenikin, and others [43]. Even though these studies deal with purely quantum mechanical settings, their Lie-admissible reinterpretation may be intriguing and instructive for various reasons. After all, interactions with external fields imply the nonconservation of the energy or of some other physical quantity of the particle considered, thus implying the direct applicability of the Lie-admissible formulations.

Note that the reinterpretation identifies another hitherto unknown application of the q-deformations (the treatment of open systems due to external electromagnetic and other fields), when also treated with Lie-admissible techniques.

The implications of the reinterpretation are nontrivial. Recall that the electromagnetic interactions verify the Poincaré symmetry. Their reinterpretation as open systems and treatment via the Lie-admissible theory then permits the construction of the equivalent *Poincaré-admissible symmetry* (Sect. I.7.6). Once such genosymmetry has been established in the known grounds of electromagnetic interactions, its extension to more complex systems is then expected, such as to the characterization of a neutron in the core of a neutron

star.

The *Bogoliubov method of group variables* [44] is yet another field, as studied, e.g., by Khrustalev and his associates [45], which is particularly intriguing for hadronic mechanics. As well known, the method essentially consists of using collective group variables which greatly simplify complicated models in field theory and gravitation. But the method also has a nonlocal structure, and exhibits a clear connection with the Lie–isotopic branch of hadronic mechanics.

While the independent studies of Bogoliubov methods are evidently encouraged, their reinterpretation in terms of hadronic mechanics is also recommendable because of the predictable additional knowledge one can gain in the process for both approaches.

New bound states of hadrons are recently emerging such as the so–called *di–baryons* (see ref.s [46] and quoted literature). Such systems have a particular importance for hadronic mechanics because one of its primary objective is the study of the apparent cold fusion of massive particles into heavier particles (see ref.s [47] and Vol. III).

The entire field of *hidden variables* (see, e.g., ref. [48]) has a direct connection with hadronic mechanics. In fact, the isoeigenvalue equations

$$H * \psi = H T \psi = \hat{E}_T * \psi = E_T \psi \qquad (7.A.30)$$

is an explicit and concrete realization of the theory of hidden variables, which are actually turned into "hidden operators". This occurrence has rather deep implications studied in Vol. II, which lead to the reinterpretation of hadronic mechanics as a *completion of quantum mechanics* along the celebrated Einstein–Podolsky–Rosen argument [49].

Additional related studies of particular interest for hadronic mechanics are the novel studies on *hidden symmetries* initiated by Smorodinsky and Winternitz [50] and continued by Sissakian, Pogosyan and their associates [51]. These studies too are particularly significant for hadronic mechanics because they permit the identification of generalized bound states deeply linked to the hadronic bound states. In fact, the *isosymmetries of hadronic mechanics are hidden symmetries.*

Yet another topic of particular relevance is the *variational method to regain convergence in perturbative treatments* by Sissakian and his collaborators [52]. In fact, as indicated in Sect. I.6.2, one of the objectives of the isotopies of Hilbert spaces is precisely that of turning conventionally divergent series into isotopically convergent ones under the mere selection of isotopic elements such that $|T| \ll 1$. The above variational method can therefore be particularly useful for the isotopic achievement of convergent series.

The interested reader can find along similar lines the connection between hadronic mechanics and other topics, such as *Berry's phase, squeezed states,* and others.

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ABOUT THE BOOKS

These are the first books written on *Hadronic Mechanics*, which is an axiom-preserving generalization of quantum mechanics for the study of strongly interacting particles (called *hadrons*) with nonlinear, nonlocal, and nonpotential contributions due to the overlapping of wavepackets and charge distributions of the hyperdense hadrons at distances smaller than their size, as necessary to activate strong interactions. After being proposed by Santilli at Harvard University in 1978 under D.O.E. support, the new mechanics has been developed by numerous scholars, discussed at several international meetings and studied in numerous papers in physics and mathematics journals in various countries. The new mechanics is based on a generalization of the mathematical structure of quantum mechanics called of *isotopic type* in the sense of being axiom-preserving. Hadronic mechanics therefore provides a more general, nonlinear, nonlocal and noncanonical *realization* of conventional quantum mechanical axioms which can be interpreted as a form of *completion* of quantum mechanics as suggested by Einstein, Podolsky and Rosen. The unrestricted functional character of the isotopies renders the new mechanics "directly universal" for all interactions considered; the mathematical consistency of hadronic mechanics is assured by the preservation of the quantum axioms; and the significance for new applications is expressed by the broader representational capabilities, which are simply absent in quantum mechanics.

VOLUME I contains nonlinear, nonlocal and nonpotential isotopies of numbers, fields, spaces, algebras, groups, symmetries, geometries and functional analysis. VOLUME II presents a step-by-step isotopic generalization of quantum mechanical physical laws, Heisenberg's and Schroedinger's representations, Galilei's and Einstein's relativities, including the isotopies of perturbation theory, and of the scattering theory for inelastic collisions with nonlinear, nonlocal and nonhamiltonian internal effects due to mutual penetrations of hyperdense particles. VOLUME III presents applications and experimental verifications in nuclear physics, particle physics, statistical physics, astrophysics, gravitation, cosmology, antimatter physics, superconductivity and other fields such as theoretical biology and conchology. These books are recommended to all scientists.

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